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Spin-s Spherical Harmonics and ð

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Recent work on the Bondi–Metzner–Sachs group introduced a class of functions ${}_sY_{lm}(\theta, \phi)$ defined on the sphere and a related differential operator ð. In this paper the ${}_sY_{lm}$ are related to the representation matrices of the rotation group R_3 and the properties of ð are derived from its relationship to an angular-momentum raising operator. The relationship of the ${}_sT_{lm}(\theta, \phi)$ to the spherical harmonics of R_4 is also indicated. Finally using the relationship of the Lorentz group to the conformal group of the sphere, the behavior of the ${}_sT_{lm}$ under this latter group is shown to realize a representation of the Lorentz group.

1. INTRODUCTION

A RECENT paper by Newman and Penrose on the Bondi–Metzner–Sachs group[1] features a new differential operator,[2] symbolized by ð ("edth," the phonetic symbol for the hard "th"), and a related class of functions ${}_sY_{lm}(\theta, \phi)$, all defined on a sphere, in a central formal role. It is the purpose of the present paper to study ð and these generalized spherical functions and to relate them to more familiar structures.

In Sec. 2, we review previous work and give some further geometrical interpretation of thop as well as an illustration of the suitability of ð and the ${}_sY_{lm}(\theta, \phi)$, $s = 1,\ 0,\ -1$, in the manipulation of Maxwell's equations. In Sec. 3, we introduce and develop the formalism which allows one to view ð as a thinly disguised angular-momentum lowering operator and to relate the ${}_sY_{lm}(\theta, \phi)$ to the elements of the representation matrices of the rotation group R_3. This work was on the one hand motivated by inspection of the results reviewed in Sec. 2 and on the other hand allows a simple rederivation and ready extensions of such results. As an adjunct to this section, the relationship of ${}_sY_{lm}(\theta, \phi)$ to the spherical harmonics of R_4, i.e., those functions which carry the representations of R_4 defined on the unit sphere in four dimensions, is briefly indicated. In Sec. 4, we discuss the well-known relationship of the Lorentz group to the conformal group of the sphere and determine the behavior of the ${}_sY_{lm}$ under the conformal group, thereby realizing a representation of the Lorentz group of somewhat unusual appearance.

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[1] E. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).

[2] The operator symbolized by ð has been referred to colloquially as "thop."

2. SUMMARY OF PREVIOUS WORK

In this section we discuss some of the previous work[1] on the differential operator ð and the spin-s spherical harmonics ${}_sY_{lm}$.

In three-dimensional Euclidean space with polar coordinates r, θ, ϕ, we introduce an orthonormal triad **a**, **b**, and **c** of vector fields. The vectors **a** and **b** are tangent to the sphere of radius r at each of its points while **c** is in the direction of the radius vector **r**. Of course **a** and **b** are only defined up to a rotation of angle ψ about **c**. It is very convenient to introduce in place of **a** and **b** the complex vector **m** and its complex conjugate $\bar{\mathbf{m}}$ by means of

$$\sqrt{2}\mathbf{m} = \mathbf{a} + i\mathbf{b}; \tag{2.1}$$

then **m** is defined up to a phase factor, i.e., $\mathbf{m}' = e^{i\psi}\mathbf{m}$. A quantity η is now said to be of (integral) spin-weight s if, under (2.1), it transforms according to

$$\eta' = e^{is\psi}\eta. \tag{2.2}$$

Examples of quantities of spin weights $s = 1, 0, -1$, respectively, are

$$\mathbf{A}\cdot\mathbf{m},\ \mathbf{A}\cdot\mathbf{c},\ \mathbf{A}\cdot\bar{\mathbf{m}},$$

where **A** is any vector. More generally, examples of quantities of spin-weight s are furnished by three-dimensional tensors of rank n contracted k_1, k_2, and k_3 times with **m**, **c**, and $\bar{\mathbf{m}}$, respectively, where $k_1 - k_3 = s$, $k_1 + k_2 + k_3 = n$. We adopt the convention that the real and imaginary parts of **m** point along the coordinate lines and hence transform according to (2.2) under coordinate transformations.

The differential operator ð, acting on a quantity η of spin-weight s, is defined by

$$ð\eta = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi}\right](\sin\theta)^{-s}\eta. \tag{2.3}$$

Since one has

$$(ð\eta)' = e^{i(s+1)\psi}(ð\eta), \tag{2.4}$$

it is seen that ð has the important property of raising the spin weight by 1. Similarly if one defines $\bar{ð}$ by

$$\bar{ð}\eta = -(\sin\theta)^{-s}\left[\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\phi}\right](\sin\theta)^{s}\eta \tag{2.3a}$$

with η here also a quantity of spin-weight s, one can see that $\bar{ð}$ lowers the spin weight by 1. Also one has

$$(\bar{ð}ð - ð\bar{ð})\eta = 2s\eta.$$

Of importance too is the effect of ð on ordinary spherical harmonics:

$$Y_{lm}(\theta,\phi),\quad -l \le m \le l,\quad l = 0, 1, 2, \cdots.$$

Indeed we can define spin-s spherical harmonics ${}_sY_{lm}$ for integral s, l, and m by

$$\begin{aligned}{}_sY_{lm}(\theta,\phi) &= [(l-s)!/(l+s)!]^{\frac12}ð^sY_{lm}(\theta,\phi), \\ &\qquad 0 \le s \le l, \\ &= [(l+s)!/(l-s)!]^{\frac12}(-)^s\bar{ð}^{-s}Y_{lm}(\theta,\phi), \\ &\qquad -l \le s \le 0.\end{aligned} \tag{2.5}$$

The ${}_sY_{lm}$ (which are not defined for $|s| > l$) form a complete orthonormal set for each value of s; i.e., any spin-weight s function can be expanded in a series in ${}_sY_{lm}$. The spin-s spherical harmonics have the further properties:

(i) $${}_s\bar{Y}_{lm} = (-)^{m+s}{}_{-s}Y_{lm}, \tag{2.6}$$

(ii) $$ð\,{}_sY_{lm} = [(l-s)(l+s+1)]^{\frac12}{}_{s+1}Y_{lm}, \tag{2.7a}$$

(iii) $$\bar{ð}\,{}_sY_{lm} = -[(l+s)(l-s+1)]^{\frac12}{}_{s-1}Y_{lm}, \tag{2.7b}$$

(iv) $$\bar{ð}ð\,{}_sY_{lm} = -(l-s)(l+s+1)\,{}_sY_{lm}. \tag{2.8}$$

Thus ð and $\bar{ð}$ act as raising and lowering operators on the "quantum number" s, and the ${}_sY_{lm}$ are eigenfunctions of $\bar{ð}ð$.

For many computations, a more convenient coordinate system for the sphere is the set of complex stereographic coordinates $(\zeta, \bar{\zeta})$ which are introduced by

$$\zeta = e^{i\phi}\cot\tfrac12\theta. \tag{2.9}$$

ð and $\bar{ð}$ become

$$\begin{aligned} ð\eta &= 2P^{1-s}[\partial(P^s\eta)/\partial\zeta], \\ \bar{ð}\eta &= 2P^{1+s}[\partial(P^{-s}\eta)/\partial\bar{\zeta}], \end{aligned} \tag{2.10}$$

with $P = \frac12(1 + \zeta\bar{\zeta})$. In the $(\zeta, \bar{\zeta})$ system, the spin-s spherical harmonics take the form

$$\begin{aligned} {}_sY_{lm} &= \frac{a_{lm}}{[(l-s)!\,(l+s)!]^{\frac12}}(1+\zeta\bar{\zeta})^{-l} \\ &\quad\times\sum_p\binom{l-s}{p}\binom{l+s}{p+s-m}\zeta^p(-\bar{\zeta})^{p+s-m}, \end{aligned} \tag{2.11}$$

with

$$a_{lm} = (-)^{l-m}[(l+m)!\,(l-m)!\,(2l+1)/4\pi]^{\frac12}. \tag{2.12}$$

Expression (2.11) applies also to "spinor harmonics" for which l, m, and s are all half-odd integers.

ð can be related to covariant differentiation in the following manner: using coordinates on the sphere such that the metric takes the form[3]

$$ds^2 = P^{-2}\,d\zeta\,d\bar{\zeta},$$

we introduce two complex vectors $m^\alpha = \sqrt{2}P\delta_\zeta^\alpha$, $\bar{m}^\alpha = \sqrt{2}\delta_{\bar{\zeta}}^\alpha$, $\alpha = \zeta, \bar{\zeta}$. From a spin-weight s quantity η, we can define a totally symmetric and trace-free

[3] The function P and the coordinates ζ need not be the ones used in Eq. (2.10); as a matter of fact the surface need not even be a sphere.

tensor of rank s,

$$\eta_{(\alpha\cdots\beta)} = \eta\bar{m}_\alpha\cdots\bar{m}_\beta + \bar{\eta}m_\alpha\cdots m_\beta,$$

with the inverse relations

$$\eta = \eta_{(\alpha\cdots\beta)}m^\alpha\cdots m^\beta;\ \bar{\eta} = \eta_{(\alpha\cdots\beta)}\bar{m}^\alpha\cdots\bar{m}^\beta.$$

It is now easy to prove

$$ð\eta = \sqrt{2}\eta_{(\alpha\cdots\beta);\gamma}m^\alpha\cdots m^\beta m^\gamma. \tag{2.13}$$

As a simple example illustrating the use of ð and the ${}_sY_{lm}$, we consider the Maxwell equations

$$\begin{aligned}\nabla\cdot(\mathbf{E}+i\mathbf{B}) &= 0,\\ \nabla\wedge(\mathbf{E}+i\mathbf{B}) - i(\partial/\partial t)(\mathbf{E}+i\mathbf{B}) &= 0.\end{aligned} \tag{2.14}$$

The quantities[4]

$$\begin{aligned}G_+ &= (\mathbf{E}+i\mathbf{B})\cdot\mathbf{m},\\ G_0 &= (\mathbf{E}+i\mathbf{B})\cdot\mathbf{c},\\ G_- &= (\mathbf{E}+i\mathbf{B})\cdot\bar{\mathbf{m}}\end{aligned} \tag{2.15}$$

of spin weight 1, 0, and -1, respectively, can be shown from (2.14) to satisfy the equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)rG_+ - ð\bar{ð}_rG_+ = 0, \tag{2.16a}$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)r^2G_0 - \bar{ð}ð_rG_0 = 0, \tag{2.16b}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)rG_- - ðð_rG_- = 0, \tag{2.16c}$$

in which the quantities G_+, G_0, G_- are already uncoupled. If we assume solutions of these equations of the form

$$\begin{aligned}rG_+ &= F_+(\mathbf{r},t)\,{}_1Y_{lm}(\theta\phi),\\ r^2G_0 &= F_0(\mathbf{r},t)\,{}_0Y_{lm}(\theta\phi),\\ rG_- &= F_-(\mathbf{r},t)\,{}_{-1}Y_{lm}(\theta\phi),\end{aligned} \tag{2.17}$$

it is seen from Eqs. (2.7) and (2.8) that

$$\begin{gathered}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)F_+ + (l-1)(l+2)F_+ = 0,\\ \left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)F_0 + \frac{1}{r^2}\,l(l+1)F_0 = 0,\\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)F_- + (l-1)(l+2)F_- = 0,\end{gathered} \tag{2.18}$$

the dependence on angular variables having canceled out. These latter equations can be solved by a variety of standard techniques, though it is not our purpose to go into this question here.

[4] G_+, G_0, G_- have been referred to elsewhere as ϕ_0, ϕ_1, ϕ_2. See, e.g., E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

The main point to be made is that Maxwell's equations or more generally vector equations can be simply solved in terms of the ${}_sY_{lm}$ instead of the cumbersome apparatus of the vector spherical harmonics.[5]

3. RELATIONSHIP TO R_3 AND R_4

In this section, we identify the functions ${}_sY_{lm}$ with the elements of the matrices of the representation D^l of the ordinary rotation group R_3, and relate ð to an ordinary angular-momentum raising operator. We thereby obtain the principal properties of the ${}_sY_{lm}$ and ð as transcriptions of results familiar in the theory of angular momentum.

We proceed first to the above mentioned identification of the ${}_sY_{lm}$. For our purpose it is convenient to have an explicit definition of ${}_sY_{lm}(\theta,\phi)$ rather than the expression in terms of stereographic coordinates given in Eq. (2.11). By direct substitution of (2.9) we obtain[6]

$$\begin{aligned}{}_sY_{lm}(\theta,\phi) &= \left[\frac{(l+m)!\,(l-m)!}{(l+s)!\,(l-s)!}\,\frac{(2l+1)}{4\pi}\right]^{\frac{1}{2}}(\sin\theta/2)^{2l}\\ &\times\sum_r\binom{l-s}{r}\binom{l+s}{r+s-m}(-)^{l-r-s}e^{im\phi}(\cot\theta/2)^{2r+s-m}.\end{aligned} \tag{3.1}$$

Now we give[7] careful definitions of and appropriate explicit formulas for the elements of the matrix D^l, of the representation of R_3 associated with total angular momentum l. If a spatial rotation R of angle ω about a unit vector $\mathbf{n}$ is given by

$$\begin{gathered}x^k \to x'^k = R^{kl}x^l,\\ R^{kl} = \delta^{kl}\cos\omega + n^kn^l(1-\cos\omega) - \epsilon^{klm}\sin\omega,\end{gathered} \tag{3.2}$$

then the matrix D^l may be defined by its action on spherical harmonics

$$\begin{aligned}Y_{lm}(\hat{\mathbf{x}}) &= \langle\hat{\mathbf{x}}\mid lm\rangle \to Y_{lm}(\hat{\mathbf{x}}'),\\ \hat{\mathbf{x}} &= (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\\ Y_{lm}(\hat{\mathbf{x}}') &= \sum_{m'}Y_{lm'}(\mathbf{x})D^l_{m'm}(R^{-1}).\end{aligned} \tag{3.3}$$

[5] It has recently been pointed out to us that the functions ${}_sY_{lm}$ have already been introduced, though by very different techniques. For this alternate method, and its detailed application to Maxwell theory, see I. M. Gel'fand, R. A. Minlos, and Z. Ya Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (The Macmillan Company, New York, 1963).

[6] In this passage from Eq. (2.11) defining ${}_sY_{lm}(\zeta,\bar{\zeta})$ to Eq. (3.1) one should not only insert the definition (2.9) but also introduce an additional phase factor $e^{is\phi}$ to account for the rotation of the vectors associated with the change of coordinates $(\zeta,\bar{\zeta})$ to (θ,ϕ).

[7] The necessity for the detail of the discussion here stems from the fact that we could not simply refer to one of the few completely consistent treatments of the theory of the rotation group available in the literature, without extensive modification of the notation employed in Ref. 4 and related papers.

If we define a rotation $R(\alpha, \beta, \gamma)$ of Euler angles α, β, γ as being composed of[8] γ about OZ followed by β about OY and then α about OZ we have

$$\begin{aligned} D^l_{m'm}(\alpha\beta\gamma) &\equiv D^l_{m'm}(R(\alpha\beta\gamma)^{-1}) \\ &= e^{im'\gamma} d^l_{m'm}(\beta) e^{im\alpha}. \end{aligned} \tag{3.4}$$

Following Wigner,[9] in principle if not in detail, we employ the relationship of R_3 to SU_2 in order to give an explicit formula for $D^l_{m'm}(\alpha\beta\gamma)$. If the element A of SU_2 acts on a two-component spinor $w = \binom{u}{v}$, where

$$u = e^{i\phi/2} \cos \tfrac{1}{2}\theta, \quad v = e^{-i\phi/2} \sin \tfrac{1}{2}\theta, \tag{3.5}$$

so that $u/v = \zeta$, according to $w \to w' = Aw$, then the correspondence of $A \in SU_2$ to $R \in R_3$ can be given in the form[10]

$$\begin{aligned} R^{kl} &= \tfrac{1}{2} \operatorname{Tr} (\sigma^k A \sigma^l A^\dagger), \\ A &= \pm(1 + \sigma^k \sigma^l R^{kl}) / [4(1 + \operatorname{Tr} R)]^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

which allows us to obtain the image $A(\alpha\beta\gamma)$ of $R(\alpha\beta\gamma)$ in the form

$$A(\alpha\beta\gamma) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a = e^{-\frac{1}{2}i(\alpha+\gamma)} \cos \tfrac{1}{2}\beta, \quad b = e^{-\frac{1}{2}i(\alpha-\gamma)} \sin \tfrac{1}{2}\beta. \tag{3.7}$$

Now defining

$$\phi_{jm}(u, v) = \frac{u^{j+m} v^{j-m}}{[(j+m)!\,(j-m)!]^{\frac{1}{2}}}$$

as usual we can, in agreement with Eq. (3.5), write

$$\begin{aligned} &\phi_{jm}(u, v) \to \phi_{jm}(u', v'), \\ &\phi_{jm}(u', v') = \sum_{m'} \phi_{jm'}(u, v) D^j_{m'm}(A(\alpha\beta\gamma)^{-1}), \end{aligned} \tag{3.8}$$

and with a little algebra obtain

$$\begin{aligned} &D^j_{m'm}(\alpha\beta\gamma) \\ &\equiv D^j_{m'm}(A(\alpha\beta\gamma)^{-1}) \\ &= \left[\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}\right]^{\frac{1}{2}} \sum_r \binom{j+m'}{r} \binom{j-m'}{r-m-m'} \\ &\qquad \times \bar{a}^r \bar{b}^{j+m-r} (-b)^{j+m'-r} a^{r-m-m'} \\ &= \left[\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}\right]^{\frac{1}{2}} (\sin \tfrac{1}{2}\beta)^{2j} \\ &\quad \times \sum_r \binom{j+m'}{r} \binom{j-m'}{r-m-m'} (-)^{j+m'-r} \\ &\quad \times e^{im\alpha} (\cot \tfrac{1}{2}\beta)^{2r-m-m'} e^{im'\gamma}. \end{aligned} \tag{3.9}$$

We may now insert $\alpha = \phi$, $\beta = \theta$, $j = l$, $m' = -s$ into Eq. (3.9) and, by comparison with Eq. (3.3) obtain

$${}_sY_{lm}(\theta\phi) e^{-is\gamma} = [(2l+1)/4\pi] D^l_{-sm}(\phi\theta\gamma), \tag{3.10}$$

so that for $\gamma = 0$ we can make the promised identification

$${}_sY_{lm}(\theta\phi) = [(2l+1)/4\pi]^{\frac{1}{2}} D^l_{-sm}(\phi\theta 0). \tag{3.11}$$

Note that for $s = 0$, we have

$${}_0Y_{lm}(\theta\phi) = [(2l+1)/4\pi]^{\frac{1}{2}} D^l_{0m}(\phi\theta 0) = Y_{lm}(\theta\phi),$$

so that the spin-s spherical harmonics with spin weight $s = 0$ are exactly the ordinary spherical harmonics. It may also be noted that our procedure extends the definition of spin-s spherical harmonics to the case of s half-integral.

Now the functions $D^l_{m'm}(\alpha\beta\gamma)$ provide[11] a complete orthonormal basis for functions defined on R_3, so that orthogonality and completeness relations for ${}_sY_{lm}(\theta, \phi)$ follow easily. The orthogonality relations

$$\begin{aligned} \int_0^{2\pi} d\alpha \int_{-l}^{1} d\cos\beta \int_0^{2\pi} d\gamma\, \bar{D}^l_{-sm}(\alpha\beta\gamma) D^{l'}_{-s'm'}(\alpha\beta\gamma) \\ = [8\pi^2/(2l+1)] \delta_{ll'} \delta_{mm'} \delta_{ss'} \end{aligned} \tag{3.12}$$

translate, by use of Eq. (3.10) and relabeling, into

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, {}_s\bar{Y}_{lm}(\theta\phi)\, {}_sY_{l'm'}(\theta\phi) = \delta_{ll'} \delta_{mm'}. \tag{3.13}$$

It is noteworthy that we obtain in this way only an orthogonality relation involving spin-s spherical harmonics of the same spin weight. Orthogonality of the D^l_{-sm} with respect to s in Eq. (3.12) is of course associated with the variable γ which is absent in Eq. (3.13). Also from the completeness relation

$$\begin{aligned} &\sum_{lmm'} \bar{D}^l_{m'm}(\alpha\beta\gamma) D^l_{m'm}(\alpha'\beta'\gamma') \\ &\quad = 8\pi^2/(2l+1) \delta(\alpha - \alpha') \delta(\cos\beta - \cos\beta') \delta(\gamma - \gamma'), \end{aligned} \tag{3.14}$$

we can prove, by evaluating

$$\int_0^{2\pi} d\gamma e^{-is\gamma} (\cdots)$$

(where s is any integer) on both sides, that we have a completeness relation

$$\sum_{lm} {}_s\bar{Y}_{lm}(\theta\phi)\, {}_sY_{lm}(\theta'\phi') = \delta(\phi - \phi') \delta(\cos\theta - \cos\theta'), \tag{3.15}$$

for each integral value of s. Thus for each integral s the function ${}_sY_{lm}(\theta\phi)$ form according to Eqs. (3.13) and (3.15) a complete orthonormal set of functions on the unit sphere with respect to which any function of spin-weight s defined on the unit sphere can be expanded.

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[8] This procedure is clearly equivalent to the more usual one of a rotation α around OZ, followed by β around OY' and finally γ around OZ''.

[9] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

[10] It follows now that under $w \to w' = \bar{A}w$, $W = \tilde{w}\sigma\bar{w}\ (= \hat{\mathbf{x}})$ has the transformation law $W^k = W'^k = R^{kl}W^l$ as consistency of course requires.

[11] R. Penrose, Proc. Cambridge Phil. Soc. **55**, 137 (1959).

We turn now to the use of Eqs. (3.10) and (3.11) to motivate the association of ð with an angular-momentum raising operator. We set out from the observation, familiar from the theory of the symmetric top, that if one defines operators L_z, $L_\pm$,

$$L_z = -i\frac{\partial}{\partial\alpha},$$
$$L_\pm = \pm e^{\pm i\alpha}\left(\frac{\partial}{\partial\beta} \pm i\cot\beta\frac{\partial}{\partial\alpha} \pm i\csc\beta\frac{\partial}{\partial\gamma}\right) \quad (3.16)$$

which obey the commutation relations

$$[L_z, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_z$$

of angular momentum, then for each allowed value of s, $D^l_{-sm}(\alpha\beta\gamma)$ behaves like an eigenvector $|lm\rangle$, i.e.,

$$\mathbf{L}^2 D^l_{-sm} = l(l+1)D^l_{-sm},$$
$$L_z D^l_{-sm} = mD^l_{-sm},$$
$$L_\pm D^l_{-sm} = [(l\mp m)(l\pm m+1)]\ D^l_{-sm\pm 1}. \quad (3.17)$$

We do not relate L_+ to ð, of course, but instead define a second angular-momentum operator $\mathbf{K}$, which commutes with $\mathbf{L}$, and with respect to which D^l_{-sm} behaves like an eigenvector $|ls\rangle$ for each allowed value of m. The way to define $\mathbf{K}$ follows easily from the symmetry of $D^l_{-sm}(\alpha, \beta, \gamma)$ with respect to m, α on the one hand and $s - \gamma$, on the other. Thus, we define

$$K_z = i\frac{\partial}{\partial\gamma},$$
$$K_\pm = \pm e^{\pm i\gamma}\left(\frac{\partial}{\partial\beta} \pm i\cot\beta\frac{\partial}{\partial\gamma} \pm i\csc\beta\frac{\partial}{\partial\alpha}\right) \quad (3.18)$$

and deduce

$$[K_z, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = 2K_z,$$
$$[\mathbf{L}, \mathbf{K}] = 0,$$

and

$$\mathbf{K}^2 D^l_{-sm} = l(l+1)\ D^{-sm}_l,$$
$$K_z D^l_{-sm} = sD^l_{-sm},$$
$$K_\pm D^l_{-sm} = [(l\mp s)(l\pm s+1)]^{\frac{1}{2}} D^l_{-(s\pm 1)m}. \quad (3.19)$$

We are now in a position to make explicit the relationship of K_+ to ð. When acting on D^l_{-sm} the operator K_+ can be written in the form

$$K_+ = e^{-i\gamma}\left(\frac{\partial}{\partial\beta} - is\cot\beta + i\csc\beta\frac{\partial}{\partial\alpha}\right)$$
$$= e^{-i\gamma}(\sin\beta)^s\left(\frac{\partial}{\partial\beta} + i\csc\beta\frac{\partial}{\partial\alpha}\right)(\sin\beta)^{-s}, \quad (3.20)$$

so that

$$[K_+ D^l_{-sm}]_{\alpha=\phi,\beta=\theta,\gamma=0} = ðD^l_{-sm}\,|\phi\theta 0) \quad (3.21)$$

follows in accordance with Eq. (3.1). Thus K_+ is the differential operator to which the operator ð is more closely related. The reason that ð is not defined as a differential operator by Newman and Penrose stems from the fact that they work only with ${}_sY_{lm}(\theta\phi) \sim D^l_{-sm}(\phi\theta 0)$ rather than $D^l_{-sm}(\phi\theta\gamma)$, i.e., from the nonappearance of the variable γ. Of course this in turn results from the fact that such a variable is not needed by them on physical grounds. However, the properties of ð follow very easily from its relation to K_+. For example, from (19), (21), and (10) we get directly

$$ð_s Y_{lm}(\theta\phi) = [(l-s)(l+s+1)]^{\frac{1}{2}}{}_{s+1}Y_{lm}(\theta\phi), \quad (3.22)$$

which is Eq. (3.22) of the paper by Newman and Penrose. Of course, it was results like this one which initially suggested the relationship of s to a magnetic quantum number and motivated the identifications of ð with an angular-momentum operator.

Finally, it may be worthwhile here to point out the relationship of ð to representations of R_4 defined on the unit 4-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. It is well known that the generators of infinitesimal rotations of R_4 can be defined according to

$$M_{kl} = -i(x_k\partial_l - x_l\partial_k), \quad 1 \le k, \quad l \le 4,$$

and replaced by a pair of commuting angular momentum operators $\mathfrak{L}$, $\mathfrak{K}$:

$$\mathfrak{L}_1 = M_{23} + M_{14}, \quad \mathfrak{L}_2 = M_{31} + M_{24},$$
$$\mathfrak{L}_3 = M_{12} + M_{34},$$
$$\mathfrak{K}_1 = M_{23} - M_{14}, \quad \mathfrak{K}_2 = M_{31} - M_{24},$$
$$\mathfrak{K}_3 = M_{12} - M_{34}.$$

Now, in view of the consequence $\mathfrak{L}^2 = \mathfrak{K}^2$ of these definitions, only the subset (l, k) of representations of R_4 with $l = k = 0, \frac{1}{2}, 1, \cdots$ can be defined on the unit 4-sphere with these "standard" definitions of the six infinitesimal generators. However, only these representations arise in the previous discussion. We can explicitly make contact with the formalism of the previous paragraph by introducing polar coordinates according to

$$x_1 = \sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}(\alpha-\gamma),$$
$$x_2 = \cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}(\alpha+\gamma),$$
$$x_3 = \sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}(\alpha-\gamma),$$
$$x_4 = \cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}(\alpha+\gamma), \quad (3.23)$$

for then $\mathfrak{L} = \mathbf{L}$, $\mathfrak{K} = \mathbf{K}$ follow. Alternatively we could remark that the $D^j_{m'm}(A^{-1})$ form a complete orthonormal basis for functions defined 'on $SU(2)$.' Explicitly this latter term refers to functions of a, b such that

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

belongs to SU(2), or simply such that $|a|^2 + |b|^2 = 1$. Now from (7) and (23), we have

$$a = x_2 - ix_4, \quad b = x_1 - ix_3,$$

so that functions of a, b such that

$$|a|^2+|b|^2=x_1^2+x_2^2+x_3^2+x_4^2=1$$

can be read as functions defined on the unit 4-sphere. This remark is of course what underlies the identification of the $D^j_{m'm}$ with the basis of the representation (j,j) or R_4.

It is perhaps worth emphasizing that the ${}_sY_{lm}(\theta\phi)$ or the D^l_{-sm} play two very different roles being on the one hand closely related to matrix elements of the representation matrices of O_3 and on the other hand closely related to bases functions of certain representations of O_4.

4. THE LORENTZ TRANSFORMATION AND SPIN-s SPHERICAL HARMONICS

A. Conformal Mappings

Up to this time the discussion of the spin-s spherical harmonics has been based on their relationship to the rotation group. The rigid rotations are a three-parameter group of isometric mappings of the unit sphere onto itself. Thus

$$ds^2=d\theta^2+\sin^2\theta\,d\phi^2=d\theta'^2+\sin^2\theta'\,d\phi'^2 \quad (4.1)$$

if the mapping $\{\theta,\phi\}\to\{\theta',\phi'\}$ is a rigid rotation. In order to relate the spin-s spherical functions to the Lorentz group it is necessary to enlarge this group of homeomorphic mappings of the 2-sphere. The mapping $\{\theta,\phi\}\to\{\theta',\phi'\}$ is conformal if

$$\begin{aligned}ds^2&=d\theta^2+\sin^2\theta\,d\phi^2\\&=K^2(\theta',\phi')(d\theta'^2+\sin^2\theta'\,d\phi'^2). \quad (4.2)\end{aligned}$$

Clearly the rigid rotations form that subgroup of the conformal transformations for which the conformal factor $K^2=1$. The conformal group, which preserves the angle between two curves and its direction, can be shown to be a six-parameter Lie group which is isomorphic to the proper homogeneous Lorentz group.[11,12] The result can be easily derived and as it introduces the notation we wish to use in our discussion of spin-s spherical functions, we give the proof here.

In terms of the stereographic coordinates $\zeta=e^{i\phi}\cot\theta/2$ which were introduced in Sec. 2, the metric on the unit sphere has the form

$$ds^2=4(1+\zeta\bar\zeta)^{-2}\,d\zeta\,d\bar\zeta. \quad (4.3)$$

The complex coordinate ζ defines a point in the complex plane. Therefore, the conformal transformations of the complex plane will induce the conformal transformations of the unit sphere onto itself. The only transformations with a simple pole and a simple zero at the new north and south poles, respectively, are given by the Möbius transformation

$$\zeta'=(\alpha\zeta+\beta)/(\gamma\zeta+\delta);\quad \alpha\delta-\beta\gamma=1. \quad (4.4)$$

Applying this transformation to Eq. (4.3) we find

$$ds^2=K^2=K^2[4(1+\zeta'\bar\zeta')^{-2}]\,d\zeta'\,d\bar\zeta', \quad (4.5)$$

$$K=\frac{(\alpha\zeta+\beta)(\bar\alpha\bar\zeta+\bar\beta)+(\gamma\zeta+\delta)(\bar\gamma\bar\zeta+\bar\delta)}{1+\zeta\bar\zeta}. \quad (4.6)$$

The complex constants α, β, γ, and δ together with the restriction indicated in Eq. (4.4) represent six real parameters.

To show the isomorphism of Eq. (4.3) with the proper homogeneous Lorentz group, we introduce a two-dimensional complex linear vector space. Let u_1 and u_2 be the components of a vector in this space. To each transformation (4) there corresponds a transformation of $SL(2)$ as follows:

$$u_1'=\alpha u_1+\beta u_2,\quad u_2'=\gamma u_1+\delta u_2, \quad (4.7)$$

as can be seen by the identification $\zeta=u_1/u_2$. $SL(2)$ furnishes a double covering of the conformal transformation exactly as it furnishes a double covering of the proper homogeneous Lorentz group. Thus the required isomorphism is established.

B. The Irreducible Representations $\mathfrak{D}^{(j_1)(j_2)}$

If ξ and η are two independent basis vectors in the two-dimensional spinor space [the space of vectors (u_1,u_2) which satisfy the transformation law (4.7)], then a basis for the linear vector space defining the irreducible representation of the Lorentz group denoted by[12] $\mathfrak{D}^{(j_1)(j_2)}$ is given by

$$\begin{gathered}(\xi^{2j_1-m_1}\eta^{m_1})(\bar\xi^{2j_2-m}\bar\eta^{2m_2}),\\ 0\le m_1\le 2j_1,\quad 0\le m_2\le 2j_2. \quad (4.8)\end{gathered}$$

The parentheses indicate complete symmetrization of the factors. This linear vector space is $(2j_1+1)$ $(2j_2+1)$ dimensional. Therefore, an arbitrary vector in this space is determined by $(2j_1+1)(2j_2+1)\times$ numbers a_{m_1,m_2}. The transformation (4) which maps (u_1,u_2) into (u_1',u_2') induces a corresponding mapping of the components a_{m_1,m_2} into components a'_{m_1,m_2}. By considering the transformation of the quantities

$$\begin{aligned}(u_1')^{2j_1-m_1}&(u_2')^{m_1}(\bar u_1')^{2j_2-m_2}(\bar u_2')^{m_2}\\&=(\alpha u_1+\beta u_2)^{2j_1-m_1}(\gamma u_1+\delta u_2)^{m_1}\\&\quad(\bar\alpha\bar u_1+\bar\beta\bar u_2)^{2j_2-m_2}(\bar\gamma u_1+\bar\delta\bar u_2)^{m_2}\\&=\sum_{n_1=0}^{2j_1}\sum_{n_2=0}^{2j_2}A^{(j_1)(j_2)}_{m_1m_2;n_1n_2}u_1^{2j_1-n_1}u_2^{n_1}\bar u_1^{2j_2-n_2}\bar u_2^{n_2}, \quad (4.9)\end{aligned}$$

we establish the transformation

$$a'_{m_1m_2}=\sum_{n_1=0}^{2j_1}\sum_{n_2=0}^{2j_2}A^{(j_1)(j_2)}_{m_1m_2;n_1n_2}a_{n_1n_2}. \quad (4.10)$$

[12] R. K. Sachs, Phys. Rev. **128**, 385 (1962).

[13] See, for example, P. Roman, *Theory of Elementary Particle* (North-Holland Publishing Company, Amsterdam, 1960).

C. The Transformation of the Spin-s Spherical Harmonics

Consider the set of functions

$$ {}_sZ^L_{m_1m_2} = (1+\zeta\bar\zeta)^{-L}\zeta^{L-s-m_1}\bar\zeta^{L+s-m_2}, $$
$$ |s| \le L, \quad 0 \le m_1 \le L-s, \quad 0 \le m_2 \le L+s. \tag{4.11} $$

Applying the transformation (4.4), we get for the transformed set

$$ {}_sZ'^L_{m_1m_2} = [e^{is\lambda}/K^L(1+\zeta\bar\zeta)^L]\{(\alpha\zeta+\beta)^{L-m_1}(\alpha\zeta+\delta)^{m_1} \times(\bar\alpha\bar\zeta+\bar\beta)^{L+s-m_2}(\bar\gamma\bar\zeta+\bar\delta)^{m_2}\} \tag{4.12} $$

with

$$ \frac{1}{1+\zeta'\bar\zeta'} = \frac{(\gamma\zeta+\delta)(\bar\gamma\bar\zeta+\bar\delta)}{K(1+\zeta\bar\zeta)}, $$
$$ e^{i\lambda} = \frac{\gamma\zeta+\delta}{\bar\gamma\bar\zeta+\bar\delta}, $$

and K given by Eq. (4.6). Comparing (4.12) with Eq. (4.9) and (4.10), we find that

$$ {}_sZ'^L_{m_1m_2} = K^{-L}e^{is\lambda}\sum_{n_1=0}^{L-s}\sum_{n_2=0}^{L+s} A^{[\frac12(L-s)][\frac12(L+s)]}_{m_1m_2;n_1n_2}\,{}_sZ^L_{n_1n_2}. $$

Therefore, up to the conformal factor $K^{-L}e^{is\lambda}$, the functions ${}_sZ^L_{m_1m_2}$ transform under the $\mathfrak{D}^{[\frac12(L-s)][\frac12(L+s)]}$ irreducible representation of the Lorentz group.

Clearly these functions do not form an orthonormal set of functions on the sphere for fixed s. Indeed, for all $L > |s|$ they form a redundant set of functions for definite spin-weight s. However, the spin-s spherical harmonics ${}_sY_{lm}$ do form an orthonormal set for fixed s. It is easy to show that for $l \le L$ the ${}_sY_{lm}$ are given uniquely by the ${}_sZ^L_{m'm}$:

$$ {}_sY_{lm} = \sum_{m_1=0}^{L-s}\sum_{m_2=0}^{L+s} {}_sB^{L\ m_1m_2}_{lm}\,{}_sZ^L_{m_1m_2}, \tag{4.13} $$
$$ s \le l \le L, \quad |m| \le l; $$

$$ {}_sB^{L\ m_1m_2}_{lm} = \frac{a_{lm}}{[(l-s)!\,(l+s)!]^{\frac12}}\sum_{p=0}^{p_m}(-1)^{p+s-m}\binom{l-s}{p} \times\binom{l+s}{p+s-m}\binom{L-l}{L-s-m_1-p} \times\delta_{m_2,m_1+s+m}, \tag{4.14} $$

$$ p_m = \min\{L-s-m_1,\, l-s,\, l+m\} \tag{4.14a} $$

and the a_{lm} are the constants defined in Eq. (2.11).

For fixed s and L the coefficients ${}_sB^{L\ m_1m_2}_{lm}$ form a nonsingular

$$ (L-s+1)(L+s+1)\times(L-s+1)(L+s+1) $$

matrix $[(l,m),(m_1,m_2)]$ connecting the ${}_sZ^L_{m_1m_2}$ to the ${}_sY_{lm}$. Since the ${}_sZ^L_{m_1m_2}$ transform under the

$$ \mathfrak{D}^{[\frac12(L-s)][\frac12(L+s)]} $$

representation of the Lorentz group up to the factor $K^{-L}e^{il\lambda}$, it follows that the ${}_sY_{lm}(|s| \le l \le L$ and $|m| \le l)$ transform under an equivalent representation up to the same factor.

The above results hold both for L and s integral, or half-integral.