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Master Analytic Representation: Reduction of $O(2,1)$ in an $O(1,1)$ Basis

JACOB G. KURIYAN*†
Institute for Advanced Study, Princeton, New Jersey

N. MUKUNDA‡§ AND E. C. G. SUDARSHAN
Syracuse University, Syracuse, New York

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We display the reduction of the pseudo-orthogonal group $O(2, 1)$ with respect to a noncompact $O(1, 1)$ basis. After the explicit solution is obtained, we rederive the results using the method of master analytic representations.

1. INTRODUCTION

Lie groups and Lie algebras have become increasingly familiar to particle physicists. Conservation laws and symmetries have been studied in terms of invariance and noninvariance groups. The central idea that is exploited in these applications is the assumption that the analytic properties of amplitudes have their counterpart in the analytic properties of the representations of Lie algebras.

We have studied the representation theory of Lie algebras in terms of analytic representations. Specifically, we wished to show that every linear representation of a (locally compact) Lie algebra is a special case of a master analytic representation; that the unitary representation of any of the Lie groups with this Lie algebra is a specialization of the master analytic representation (MAR).

The theory of the MAR synthesizes all Hermitian representations of the Lie algebra. It also brings out the relation between the representations of two different Lie algebras whose complex extensions are isomorphic. This is, therefore, an elegant and powerful method for finding the unitary representations of various noncompact groups.

Elsewhere,[1] we have illustrated the technique by finding the representations of some pseudo-orthogonal groups.

When a noncompact group is such that its maximal compact subgroup labels the states within a UIR uniquely, we believe that the MAR method is quite straightforward, and it is not too difficult to see why it works. We, however, believe that this method is quite general and fundamental and is applicable to many other groups as well. In particular, one could reduce UIR's of a noncompact group with respect to a noncompact subgroup. In this direction we have made a beginning by reducing representation of $O(2, 1)$ with respect to $O(1, 1)$. Throughout the paper we use, as far as possible, only infinitesimal-operator techniques. A difficult problem is to find out when a representation of the Lie algebra permits exponentiation to provide a representation of the group. We do

* Research supported by the National Science Foundation.
† Present address: UCLA, Los Angeles, California.
‡ Present address: Tata Institute of Fundamental Research, Bombay, India.
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[1] J. G. Kuriyan, N. Mukunda, and E. C. Sudarshan, Commun. Math. Phys. **8**, 204 (1968).

The Lie algebra of $O(2, 1)$ has three independent elements J_1, J_2, J_3 obeying the commutation rules

$$[J_1, J_2] = -iJ_3, \tag{3.1a}$$

$$[J_2, J_3] = iJ_1, \tag{3.1b}$$

$$[J_3, J_1] = iJ_2. \tag{3.1c}$$

A representation of the J_j by Hermitian operators would lead to a unitary representation of the group $O(2, 1)$. The operator Q given by

$$Q = J_3^2 - J_1^2 - J_2^2 \tag{3.2}$$

commutes with the J_j and so reduces to a real number in every Hermitian irreducible representation of the J_j.

We want to introduce a complete set of orthonormal eigenvectors for the operator J_2 in the space of a representation of the J_j. Naturally we used to know the nature of the eigenvalue spectrum of J_2. To this end, let us rewrite (3.1) in terms of the *Hermitian* operators $J_\pm$:

$$J_\pm = J_1 \pm J_3. \tag{3.3}$$

Then (3.1) reads

$$[J_2, J_\pm] = \pm iJ_\pm, \tag{3.4a}$$

$$[J_+, J_-] = 2iJ_2. \tag{3.4b}$$

The Hermitian operators J_2 and J_+ form a subalgebra of the $O(2, 1)$ Lie algebra. An irreducible Hermitian representation of all the J_j may be expected to be reducible with respect to the subalgebra generated by J_2 and J_+. Imagine this further reduction has been carried out. Within an irreducible representation of J_2 and J_+, what can be said about the spectrum of eigenvalues of J_2 and J_+? First we find easily that for all real α,

$$\exp(-i\alpha J_2) J_+ \exp(i\alpha J_2) = e^\alpha J_+, \tag{3.5}$$

so that the eigenvalue spectrum of J_+ consists of all real positive or of all real negative numbers. One can then consider a Hermitian operator $\ln J_+$ *or* $\ln(-J_+)$, depending on whether J_+ is positive- or negative-semidefinite. Then assuming $\ln J_+$ to be Hermitian, say, we find

$$\exp(i\alpha \ln J_+) J_2 \exp(-i\alpha \ln J_+) = J_2 + \alpha. \tag{3.6}$$

The spectrum of J_2 then consists of all real numbers from $-\infty$ to $+\infty$. This then is the situation within a subspace irreducible under J_2 and J_+ alone.

It is natural then to introduce a basis of eigenvectors of J_2 as follows:

$$J_2 |\lambda; r\rangle = \lambda |\lambda; r\rangle; \quad \langle \lambda'; r' \mid \lambda; r\rangle = \delta_{r'r}\delta(\lambda' - \lambda);$$
$$-\infty < \lambda, \quad \lambda' < \infty. \tag{3.7}$$

What we have to discover is how often a given eigenvalue λ appears, or how many irreducible representations of J_2 and J_+ are needed to synthesize our irreducible representation of J_2, J_+, and J_-. The label r corresponds to this "multiplicity." It is clear though that the range of values of r is independent of the particular eigenvalue λ.

At this point we must comment on the structure of the commutation rules (3.4a). Taken literally, they seem to say, for example, that the state

$$J_+ |\lambda; r\rangle \tag{3.8}$$

is an eigenstate of J_2 with eigenvalue $\lambda + i$. This is impossible since J_2 is a Hermitian operator. We infer that it is not possible to apply the operators $J_\pm$ to the vectors $|\lambda; r\rangle$. The solution to this problem is the following. We must remember that in any case the states $|\lambda; r\rangle$ are "ideal" vectors, which do not represent normalizable vectors in Hilbert space. Omitting for the moment the index r, a normalizable vector $|\phi\rangle$ is really a linear combination of the form

$$|\phi\rangle = \int_{-\infty}^{\infty} d\lambda\, \phi(\lambda) |\lambda\rangle. \tag{3.9}$$

The wavefunction $\phi(\lambda)$ is normalizable in the sense

$$\|\phi\|^2 \equiv \int_{-\infty}^{\infty} d\lambda\, |\phi(\lambda)|^2 < \infty \tag{3.10}$$

and the total Hilbert space is made up of *all* vectors $|\phi\rangle$ with (Lebesgue) square-integrable wavefunctions $\phi(\lambda)$. Now the generators J_2, $J_\pm$ are, in general, unbounded operators and each one has a corresponding domain of vectors $|\phi\rangle$ on which it is defined. For example, J_2 can only act on a vector $|\phi\rangle$ if, in addition to $\phi(\lambda)$, even $\lambda\phi(\lambda)$ is square-integrable. [In this sense, (3.7) is quite formal.] Among all wavefunctions $\phi(\lambda)$, those that J_+ can act upon are characterized as follows: $\phi(\lambda)$ should be the boundary value of an analytic function of λ, such that $f(\lambda)\phi(\lambda - i)$ is also a square-integrable wavefunction:

$$\int_{-\infty}^{\infty} |\phi(\lambda - i)|^2 |f(\lambda)|^2 d\lambda < \infty. \tag{3.11}$$

Here, $f(\lambda)$ is a function to be determined, and which plays the role of the matrix element of J_+. Thus for a vector in the domain of J_+, the wavefunction $\phi(\lambda)$ determines, via analytic continuation, a unique new wavefunction $f(\lambda)\phi(\lambda - i)$, and

$$J_+ |\phi\rangle = J_+ \int_{-\infty}^{\infty} d\lambda\, \phi(\lambda) |\lambda\rangle = \int_{-\infty}^{\infty} d\lambda\, f(\lambda)\phi(\lambda - i) |\lambda\rangle. \tag{3.12}$$

Assuming that a wavefunction $\phi(\lambda)$ is such that both $J_2 J_+$ and $J_+ J_2$ may be applied to it, one can explicitly verify the validity of (3.4a). A similar situation exists related to λ, via a Fourier transformation:

$$\psi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ix\lambda} \phi(\lambda)\, d\lambda,$$
$$\| \psi(x) \|^2 = \int_{-\infty}^{\infty} |\psi(x)|^2\, dx. \tag{3.25}$$

In the x language we have

$$J_2 = -i \frac{\partial}{\partial x},$$
$$J_+ = e^{-x}\left(b + \tfrac{1}{2}i - i\frac{\partial}{\partial x}\right), \tag{3.26}$$
$$J_- = e^{x}\left(b + \tfrac{1}{2}i + i\frac{\partial}{\partial x}\right).$$

Let us look for eigenfunctions of $J_3 = \frac{1}{2}(J_+ - J_-)$ for an eigenvalue m:

$$-i \cosh x \frac{\partial}{\partial x} \psi_m(x) - (b + \tfrac{1}{2}i) \sinh x \psi_m(x) = m\psi_m(x); \tag{3.27}$$

the solution turns out to be

$$\psi_m(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\frac{1 + ie^x}{1 - ie^x}\right)^m (\cosh x)^{-\frac{1}{2}+ib}. \tag{3.28}$$

Restricting ourselves now to single-valued representations of $O(2, 1)$, m is an integer, positive, negative, or zero. The question now is this: how often must each eigenvalue λ of J_2 appear in order that J_3 have one eigenvector for each integer m as eigenvalue, and such that eigenvectors of J_3 for distinct eigenvalues be orthogonal? We can explicitly compute the scalar product of two wavefunctions $\psi_m(x)$ and $\psi_{m'}(x)$:

$$\int_{-\infty}^{\infty} dx\, \psi_m^*(x)\psi_{m'}(x), \tag{3.29}$$

and we find that this expression is of the form $\delta_{m,m'}$ only if m and m' are both even integers or both odd integers! Thus the set of functions

$$\psi_{2n}(x), \quad n = 0, \pm 1, \pm 2, \cdots, \tag{3.30}$$

by itself forms a complete orthonormal basis for the Hilbert space of square-integrable functions of n; and the same is true for the set of functions

$$\psi_{2n+1}(x), \quad n = 0, \pm 1, \pm 2, \cdots. \tag{3.31}$$

This shows that if we assume that every eigenvalue λ of J_2 occurs only once [in a representation of the continuous nonexceptional series of $O(2, 1)$], we have a contradiction since we end up with the wrong spectrum of eigenvalues for the compact generator J_3. But it is quite clear that this situation can be remedied as follows. We define the eigenfunctions of J_3 to be two-rowed column vectors, each element being made up of a function of x:

$$\Psi_m = \begin{pmatrix} \psi_m(x) \\ (-1)^m \psi_m(x) \end{pmatrix}; \quad m = 0, \pm 1, \pm 2, \cdots. \tag{3.32}$$

By definition the Ψ_m are to be a basis for the Hilbert space of a representation of J_1, J_2, J_3. Since each of the sets of wavefunctions (3.30) and (3.31) forms a complete orthonormal system for the space of square-integrable functions of x, it is clear that every column vector of the form

$$\Phi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix},$$
$$\| \Phi \|^2 \equiv \int_{-\infty}^{\infty} dx (|\phi_1(x)|^2 + |\phi_2(x)|^2) < \infty, \tag{3.33}$$

where $\phi_1(x)$ and $\phi_2(x)$ are chosen quite independently of one another, can be expanded as a linear combination of the Ψ_m. And one can see that one now has

$$(\Psi_m, \Psi_{m'}) = \delta_{mm'}; \quad m, m' = 0, \pm 1, \pm 2, \cdots. \tag{3.34}$$

The requirement that J_3 have the right spectrum of eigenvalues led to the fact that we have to consider a Hilbert space of wavefunctions of the type (3.33). The variable x is related by Fourier transformation to λ, which is the eigenvalue of J_2. It follows that in representations of J_1, J_2, and J_3 corresponding to the continuous nonexceptional series, every eigenvalue λ of J_2 appears twice; the multiplicity index r has two values. This is in agreement with the observation of Bargmann.[10] Corresponding to the two values of the multiplicity index r, the expressions (3.26) have to be modified by writing the generators as two-dimensional matrices in addition to being linear differential operators in x. The appropriate expressions have been derived elsewhere,[11] and here we quote the results:

$$J_2 = -i \frac{d}{dx} \otimes \sigma_3,$$
$$J_1 = \left[i \sinh x \frac{d}{dx} + i(\tfrac{1}{2} - ib) \cosh x \right] \otimes \sigma_3, \tag{3.35}$$
$$J_3 = \left[-i \cosh x \frac{d}{dx} - i(\tfrac{1}{2} - ib) \sinh x \right] \otimes \mathbf{1}.$$

To summarize the above discussion, the factorization of Eq. (3.18) so as to yield the simplest possible expressions for the functions $f(\lambda)$ and $g(\lambda)$ led to operators $J_\pm$ which were Hermitian only when the parameter b was real. This corresponded exactly

[11] See N. Mukunda, Ref. 6.

are separately Hermitian operators. If, analogous to the nonexceptional continuous representation treated in Sec. 3, we attempt to express $f(\lambda)$ and $g(\lambda)$ as quantities linear in λ by writing, say,

$$f(\lambda) = \lambda - \tfrac{1}{2}i + ik, \quad g(\lambda) = -(\lambda - \tfrac{1}{2}i - ik), \tag{3.51}$$

then clearly the Hermiticity of J_+ and J_- is violated. It is desirable, nonetheless, to have both f and g linear in λ. We can reconcile these two requirements by working with an *indefinite-metric* space: we introduce a triplet of Pauli matrices τ_α, $\alpha = 1, 2, 3$, we double the spectrum of J_2 and write

$$\begin{aligned} f(\lambda) &= \lambda - \tfrac{1}{2}i + ik\tau_1, \\ g(\lambda) &= -(\lambda - \tfrac{1}{2}i) + ik\tau_1, \\ J_2\,|\lambda; a\rangle &= \lambda\,|\lambda; a\rangle, \\ \langle\lambda'; a' \mid \lambda; a\rangle &= \delta(\lambda' - \lambda)(\tau_3)_{a'a}. \end{aligned} \tag{3.52}$$

Then J_+ and J_- are given by

$$\begin{aligned} J_+ &= \exp\left(-i\frac{\partial}{\partial\lambda}\right) f(\lambda + i) \\ &= e^{-x}\left[-i\frac{\partial}{\partial x} + \frac{i}{2} + ik\tau_1\right], \\ J_- &= \exp\left(i\frac{\partial}{\partial\lambda}\right) g(\lambda) = e^{x}\left[i\frac{\partial}{\partial x} + \frac{i}{2} + ik\tau_1\right]. \end{aligned} \tag{3.53}$$

[Subscripts a, b, c', b', $\cdots$ will be used to denote rows and columns associated with the matrices τ_α.] Because of the indefinite metric introduced by the matrix τ_3 above, the operators J_+ and J_- are Hermitian with respect to this metric. [We should call them pseudo-Hermitian.] It should be emphasized that the doubling of the spectrum of J_2 introduced above is not the same as the possible need for doubling the spectrum of J_2 *within* a UIR of $O(2, 1)$ belonging to the continuous exceptional family. Whether or not this latter doubling is called for has to be investigated. The doubling introduced above is just so that J_+ and J_- may be represented by linear differential operators in x, and so that at the same time they may be (pseudo) Hermitian with respect to the appropriate metric. If the spectrum of J_2 within a UIR of $O(2, 1)$ is covered twice, this will certainly have nothing to do with an indefinite metric.

Again we compute the eigenfunctions of J_3, and see whether we can find an orthonormal family of such eigenfunctions, with the eigenvalues being all integers, positive, negative, and zero. Since on the one hand the spectrum of J_3 within a UIR of $O(2, 1)$ is *simple*, and on the other hand we have explicitly introduced a doubling of states via the indefinite metric above, we would expect to find two eigenvectors of J_3 for each eigenvalue m:

$$J_3\Psi_{m,a} = m\Psi_{m,a}; \quad m = 0, \pm 1, \pm 2, \cdots, \quad a = 1, 2 \tag{3.54}$$

with the property

$$(\Psi_{m',a'}, \Psi_{m,a}) = \delta_{m'm}(\tau_3)_{a'a}. \tag{3.55}$$

We first compute the eigenfunctions of J_3, taking J_3 from (3.53),

$$J_3\psi_m \equiv -i\left[\cosh x\frac{\partial}{\partial x} + \tfrac{1}{2}\sinh x + k\tau_1 \sinh x\right]\psi_m = m\psi_m.$$

We find indeed two independent solutions which we choose to be

$$\begin{aligned} \psi_{m,1} &= [\cosh x]^{-\frac{1}{2}}\left(\frac{1 + i\sinh x}{\cosh x}\right)^m \times \begin{pmatrix}(\cosh x)^{-k} + (\cosh x)^{+k} \\ (\cosh x)^{-k} - (\cosh x)^{k}\end{pmatrix}, \\ \psi_{m,2} &= [\cosh x]^{-\frac{1}{2}}\left(\frac{1 + i\sinh x}{\cosh x}\right)^m \times \begin{pmatrix}(\cosh x)^{-k} - (\cosh x)^{k} \\ (\cosh x)^{-k} + (\cosh x)^{k}\end{pmatrix}. \end{aligned} \tag{3.56}$$

The column vectors appearing in these wavefunctions are vectors in the space of the τ_α matrices. When we compute the inner products

$$(\psi_{m',a'}, \psi_{m,a}) = \int_{-\infty}^{\infty} dx\, \psi^\dagger_{m',a'}(x)\tau_3\psi_{m,a}(x)$$

of these wavefunctions, however, we find

$$\begin{aligned} (\psi_{m',1}, \psi_{m,2}) &= (\psi_{m',2}, \psi_{m,1}) = 0; \\ (\psi_{m',1}, \psi_{m,1}) &= -(\psi_{m',2}, \psi_{m,2}) \\ &= \begin{cases} 4\pi, & \text{if } m = m', \\ \dfrac{4e^{i(m-m')\pi/2}}{i(m - m')}[1 - (-1)^{m+m'}], & \text{if } m \neq m'. \end{cases} \end{aligned} \tag{3.57}$$

Thus the expected relations (3.55) hold only for odd values of m, or only for even values of m, but not jointly for both. This is exactly the situation encountered in our analysis of the continuous nonexceptional UIR's. Again we introduce a second doubling of the spectrum of J_2. We use the Pauli matrices σ_a to describe representations of $O(2, 1)$ in an $O(2)$ basis—and thus obtain Eq. (3.18) which was explicitly derived from first principles in the main body of the text. More specifically, we shall use the method of MAR and Eq. (2.3) to obtain Eq. (3.18).

We define

$$\begin{aligned} N_1 &\equiv J_1', \\ N_2 &\equiv iJ_3', \\ N_3 &\equiv iJ_2', \end{aligned} \tag{4.1}$$

and observe that N_1, N_2, N_3 generate an $O(2, 1)$ group which leaves $-N_1^2 - N_2^2 + N_3^2$ invariant. Our aim is to diagonalize N_2.

The eigenstate $|m\rangle$ of J_3' with eigenvalue m is now an eigenstate of N_2, which we call $|\lambda\rangle'$, with eigenvalues $\lambda \equiv im$. That is

$$N_2 |\lambda\rangle' \equiv iJ_3' |m\rangle = im |m\rangle \equiv \lambda |\lambda\rangle', \tag{4.2}$$

where we define

$$\lambda \equiv im$$

and

$$|\lambda\rangle' \equiv |m\rangle. \tag{4.3}$$

The raising and lowering operators defined by

$$N_\pm \equiv J_1' \pm iJ_2' \equiv J_\pm' \tag{4.4}$$

change the (eigen-) state of J_2' with eigenvalue $m = -i\lambda$ to a state of eigenvalue $m \pm 1 = -i(\lambda \pm i)$

Therefore we can use the notation of Sec. 3, and proceed in a cavalier fashion, to define $f(\lambda)$ and $g(\lambda)$:

$$\begin{aligned} N_+ |\lambda\rangle' &= f(\lambda + i) |\lambda + i\rangle', \\ N_- |\lambda\rangle' &= g(\lambda) |\lambda - i\rangle', \end{aligned} \tag{4.5}$$

such that

$$N_+N_- |\lambda\rangle' = f(\lambda)g(\lambda) |\lambda\rangle'. \tag{4.6}$$

The lhs of Eq. (4.6) is thus

$$\begin{aligned} N_+N_- |\lambda\rangle' &= J_+'J_-' |m\rangle && \text{(4.7a)} \\ &= (m - \tfrac{1}{2})^2 - (j + \tfrac{1}{2})^2 |m\rangle && \text{(4.7b)} \\ &= (-i\lambda - \tfrac{1}{2})^2 - (j + \tfrac{1}{2})^2 |\lambda\rangle' && \text{(4.7c)} \\ &= -(\lambda + \tfrac{1}{2}i)^2 - (j + \tfrac{1}{2})^2 |\lambda\rangle', && \text{(4.7d)} \end{aligned}$$

where to obtain (4.7a) and (4.7c) we have used Eqs. (4.3) and (4.4) and to obtain (4.7b) we have used Eq. (4.3).

We have, on comparing (4.7) with Eq. (4.6), $(f\lambda)g(\lambda) = -(\lambda + \frac{1}{2}i)^2 - (j + \frac{1}{2})^2$ which is Eq. (3.18). This serves to illustrate the power of the method of MAR and renders the claims of general validity of this principle more plausible—at least to the discerning reader.

5. DISCUSSION

We had asserted that the method of MAR is not only useful for the purpose of reducing noncompact groups with respect to its maximal compact subgroups, but also to reduce noncompact groups with respect to its noncompact subgroups. To render this assertion plausible, we reduced $O(2, 1)$ with respect to an $O(1, 1)$ subgroup. Since this problem at the time of writing this paper had been handled only with global techniques, we analyzed this problem in great detail (Sec. 3)—and later (in Sec. 4) obtained the same results in a MAR.

We have not attempted to formulate the prescription in those cases where a state labeling problem exists. We hope that when the state labeling problem is solved one could guarantee the method of MAR to those cases as well.

The crucial fact that is exploited in the whole approach is that there exists a master analytic function which describes the representations of groups that have the same complex extension—and once this is determined, the representations are obtained after some algebraic manipulations. The implication is that a student, armed with the matrix elements of the generators of the group $SU(n)$ and $SO(n)$ that are tabulated in Gel'fand and Tseitlin,[13] can obtain after some trivial manipulations the matrix elements of the generators of the groups such as $SU(n - 1, 1)$, $SO(n - 1, 1)$[14] [and perhaps even $SU(n - 2, 2)$, $SO(n - 2, 2)$!]. Then these can be analyzed to obtain the representations of the group in question.

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[13] I. M. Gel'fand and M. L. Tseitlin, Dokl. Akad. Nauk SSSR 71, 1017 (1950).

[14] Degenerate representations of groups such as $O(p, q)$ reduced with respect to $O(p) \otimes O(q)$ and $U(p, q)$ reduced with respect to $U(p) \otimes U(q)$ have been obtained by the Trieste group, J. Fischer, J. Niederle, and R. Rączka, International Center for Theoretical Physics Preprint IC/65/63, 1965; R. Rączka, IC/65/80, 1965; and *Elementary Particle Theories* (Springer-Verlag, Berlin, 1966). MAR can be used directly to obtain these results as well.