An Associative Algebraic Model for Weak Meson Decays (*)

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Summary. — An algebraic model describing symmetry breaking is applied to the weak meson decay. The suppression of the strangeness-changing weak decay is obtained as a consequence of symmetry breaking.

In two previous papers (1, 2) we have introduced a finitely generated associative algebra $\mathcal{A}$ as a model for hadrons and obtained a meson mass spectrum in good agreement with the observed masses. The characteristic property of the algebra is that the generators $H_i, E_{\pm a}, G_i, F_{\pm a}$ of the internal group $SL_{a,e}$ do not commute with the generators $P_\mu$ of the Poincaré group $\mathcal{P}$ but they do commute with the four-velocity operators $M^{-1}P_\mu$. We are interested in extending the model to describe weak meson decays.

For this purpose we consider the possibility of factorizing the interaction in terms of a four-vector operator $H_\lambda$ (hadronic part), which causes transitions between different components of the $SL_{a,e}$ multiplet and another four-vector $L^\lambda$ which describes the leptonic part of the interaction but acts as an invariant with respect to the hadronic $SL_{a,e}$. The natural candidate for the operator $H_\lambda$ is the operator

$$H_\lambda = \sum_{a=\pm 1,\pm 2} \{P_\mu, E_a + F_a \}.$$  

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This operator is bilinear in the generators and is an element of the associate algebra \( \mathfrak{B} \).

This ansatz is suggested by the fact that the generators of \( SU_3 \) or \( SU_3 \otimes SU_3 \), \( F_a \) and \( F_b \), correspond to the integrals over the weak hadron currents. For the matrix element of \( L^3 \) we require that it describes the properties contained in conventional leptonic current. Then we consider writing an effective transition operator of the form

\[
T = g L^3 H_4
\]

(where \( g \) is a constant characterizing the strength of the interaction) to describe the weak decay \((\beta)\).

However the ansatz for the matrix element of \( L^3 \) has an unconventional form \((\beta')\) due to the following. It is in the spirit of this approach that \( H_4 \) changes the energy-momentum of the states. Therefore—as the operator \( T \) is to conserve energy-momentum—this is to be compensated in the leptonic part such that it results in over-all energy-momentum conservation. As (2) is not in the usual form of a space-time integral of a suitable density this compensation requirement leads to the unconventional form for the matrix element of \( L^3 \).

To obtain our results in a more familiar way, we replace (2) by

\[
T = g \int d^4x L^3(x) H_4(x),
\]

where \( L^4(x) \) and \( H_4(x) \) are suitable field operators corresponding to \( L^4 \) and \( H_4 \) respectively.

We want to consider the (semi-)leptonic decays

\[
\beta \rightarrow \alpha_1 + \alpha_2 + \ldots + \alpha_N + l + \nu
\]

where \( \beta \) and \( \alpha_1, \ldots, \alpha_N \) \((N = 0, 1)\) are the various mesons and \( l + \nu \) is the lepton pair. Consequently we have to calculate

\[
\mathcal{A} = \int d^4x \langle \nu \alpha_1 \ldots \alpha_N p, p_\nu \rangle |L^4(x)H_4(x)|\beta p_f\rangle =
\]

\[
= \sum_p \int \frac{dq^\gamma}{2c_d E_d(q)} \int d^4x \langle \nu \alpha_1 \ldots \alpha_N |L^4(x)|p, q, \gamma \rangle \langle \gamma q \mid H_4(x) |\beta p_f\rangle .
\]

The hadronic matrix element \( \langle \gamma | H_4(x) | \beta \rangle \) can be calculated in the frame of

our algebraic model. If we define the local operator $H_\lambda(x)$ by

\begin{equation}
H_\lambda(x) = \exp[-iP_\mu x^\mu]H_\lambda \exp[iP_\mu x^\mu]
\end{equation}

we obtain (2)

\begin{equation}
\begin{aligned}
&\langle \gamma q_\alpha | H_\lambda(x) | \beta p_\beta \rangle = \exp[i(p - q) \cdot x](p_\lambda + q_\lambda)C_{\gamma \beta} \langle \gamma', q'_\beta | m_{\gamma'} | m_{\beta'} \rangle \\
&= \exp[i(p - q) \cdot x](p_\lambda + q_\lambda)C_{\gamma \beta}2c_\gamma E_\gamma(q)\delta^4 \left( q - \frac{m_\gamma}{m_\beta} p \right) ;
\end{aligned}
\end{equation}

\begin{equation}
E_\gamma(q) = \sqrt{m_\gamma^2 + q^2}.
\end{equation}

The quantities $C_{\gamma \beta}$ are $SU_3$, (or $SU_3 \times SU_3$) Clebsch-Gordan coefficients, depending upon the position of $\gamma$ and $\beta$ in the $SU_3$, (or $SU_3 \times SU_3$) multiplet (Appendix A of ref. (2)); the quantities $C_\gamma$ are normalization constants due to our normalization of the hadron states with respect to the invariant measure of $\mathcal{S}$ (Appendix B of ref. (2)).

For the leptonic matrix element we make—in accordance with the usual assumptions about the leptonic current—the following ansatz

\begin{equation}
\langle \bar{\nu} \alpha_1 ... \alpha_\lambda | p_{v1} p_{v2} ... p_{v \lambda} | L^\mu(x) | \gamma q_\gamma \rangle = \frac{\sqrt{C_\gamma}}{2m_\gamma} \frac{\sqrt{C_{\alpha_1}} ... \sqrt{C_{\alpha_\lambda}}}{2m_{\alpha_1} ... 2m_{\alpha_\lambda}}.
\end{equation}

\begin{equation}
1 \frac{1}{(2\pi)^4} \exp[-i(p_\nu + p_t + p_s + ... + p_{s\gamma} - q_\gamma)^2] \bar{w} (p_\gamma) \gamma^\mu (1 + \gamma_5) w'(p_t).
\end{equation}

Inserting (6) and (7) into (4) we obtain

\begin{equation}
\mathcal{A} = \sum_\gamma G_{\gamma \beta} \frac{m_\gamma}{2m_\beta} \frac{\sqrt{C_\gamma}}{2m_\gamma} \frac{\sqrt{C_{\alpha_1}} ... \sqrt{C_{\alpha_\lambda}}}{2m_{\alpha_1} ... 2m_{\alpha_\lambda}} \delta^4(p_\beta - p_\nu - p_t - p_s - ... - p_{s\gamma}) \bar{w}(p_\gamma) \gamma^\mu (1 + \gamma_5) w'(-p').
\end{equation}

Here $\sum_\gamma$ means the sum over all $SU_3$ states $\gamma$, which have the same internal $SU_3$ (or $SU_3 \times SU_3$) quantum numbers as $\alpha_1 ... \alpha_\lambda$. E.g. for the special case of (3): $K^+ \rightarrow \pi^0 + e^+ \nu$, $\gamma$ can be only $\pi^+$; for $K^+ \rightarrow e^+ \nu$, $\gamma$ can be only $\sigma$, the state with the quantum numbers of the vacuum $I = 0, I_3 = 0, Y = 0 (y_3 = Y) = 0$ (for which we assume $m_\sigma \approx 0$); for $K^+ \rightarrow \pi^+ \pi^- e^+ \nu$, $\gamma$ could be a resonance of the 27-plet with the quantum number $Y = 1, I_3 = 0, I = 0, 1, 2$.

Let us now calculate the transition rates for the leptonic decays of the kaon and the pion. Consider the two-body decay $K^+ \rightarrow \mu^+ \nu$. Then, from (8), we get

\begin{equation}
\langle \bar{\nu}_{\mu} p_{\mu} | p_{v1} p_{v2} ... p_{v \lambda} | \gamma \rangle = \frac{\sqrt{C_\gamma}}{m_\gamma} \delta^4(p_\nu + p_{v1} - p_\mu) \bar{w} (p_\mu) \gamma^\mu (1 + \gamma_5) w'(-p_\mu).
\end{equation}
This together with (8) implies

$$\langle \bar{p}_\mu p_\nu T | p_\eta K \rangle = G C_{\text{cc}} \bar{p}_\nu \frac{1}{2m_K} \left( 1 + \frac{m_\sigma}{m_K} \right) \sqrt{C_\text{cc}} \cdot \delta(p_\nu + p_\eta - p_K) \cdot \bar{u}^{\text{out}}(p_\nu) \gamma_\mu (1 + \gamma_5) u^{\text{in}}(-p_\nu).$$

Making use of the relation

$$\sum_{\text{pol}} \bar{u}^{\text{out}}(p_\nu) \gamma^\mu (1 + \gamma_5) (p_\eta^\omega + p_\eta^{\alpha \beta}) u^{\text{in}}(-p_\nu) = 8m_\nu^2 \bar{p}_\nu^{\mu \nu} \cdot p^{\text{out}}$$

the time rate of change of the kaon wave function is

$$\left\{ G C_{\text{cc}} \frac{1}{2m_K} \left( 1 + \frac{m_\sigma}{m_K} \right) \right\} \frac{8m_\nu^2}{2E_\eta(q)} \int \frac{d^3p_\nu}{2E_\mu} \frac{d^3p_\eta}{2E_\eta} \delta^4(q - p_\eta^\omega - p_\eta^{\alpha \beta}) p_\mu^{\text{out}}(p_\mu, p_\eta^\nu).$$

The partial-decay width is then deduced to be

$$\Gamma(K \rightarrow \eta \nu) = \frac{\pi}{4} \left( \frac{G}{m_K} \right)^2 C_{\text{cc}} \left( 1 + \frac{m_\sigma}{m_K} \right)^2 m_\nu m_\eta \left( 1 - \frac{m_\nu^2}{m_\eta^2} \right)^2.$$

In a similar manner we can compute the width for the corresponding decay of the pion

$$\Gamma(\pi \rightarrow \mu \nu) = \frac{\pi}{4} \left( \frac{G}{m_\pi} \right)^2 C_{\text{cc}} \left( 1 + \frac{m_\sigma}{m_\pi} \right)^2 m_\pi m_\mu \left( 1 - \frac{m_\pi^2}{m_\mu^2} \right)^2.$$

The C.G. coefficient $C_{\text{cc}}$ replaces $C_{\text{cc}}$ since the transition is caused by the strangeness-conserving operator $F_0$, rather than by the strangeness-changing operator $F_1$.

The $SL_{2,c}$ C.G. coefficients $C_{\text{cc}}, C_{\text{cc}}$ are

$$C_{\text{cc}} = C_{\text{cc}} = \frac{1}{6 \sqrt{2}} \sqrt{q - b^2},$$

where $b$ is a parameter characterizing the $SL_{2,c}$ representation chosen.

From (9) and (10) we obtain the suppression of the strangeness-changing decay

$$S_\nu^\eta = \frac{\Gamma(K \rightarrow \mu \nu)}{\Gamma(\pi \rightarrow \mu \nu)} = \frac{m_\eta}{m_\pi} \left( 1 - \frac{m_\nu^2}{m_\eta^2} \right)^2.$$
If we express this suppression by an (axial vector) Cabibbo factor \(^{(\ast)}\) we get

\[
\tan \theta_A = S_A = \frac{m_\pi}{m_K} \left( \frac{1 + m_\pi/m_K}{1 + m_\pi/m_\pi} \right) = 0.28,
\]

which is to be compared with the value 0.27 deduced experimentally \(^{(\ast\ast)}\).

Let us now calculate the three-body leptonic decay rates. For the process \(K^+ \rightarrow \pi^+ e^- \nu\) we get

\[
\langle e^+ \pi^0 p^+ p^- \mu^- T | p K \rangle = \int \frac{d^3 q}{2E_q} \langle e^+ \pi^0 p^+ p^- | L^1 | \pi K \rangle \langle \pi K | H_1 | p K \rangle =
\]

\[
= G C_{p^0}(1 + \frac{m_\pi}{m_\pi}) \langle e^+ \pi^0 p^+ p^- | L^1 | \pi K \rangle \frac{m_\pi}{m_\pi} \langle \pi K | \rangle
\]

\[
= G C_{p^0}(1 + \frac{m_\pi}{m_\pi}) \sqrt{C_\pi} \sqrt{C_K} \delta^4(p_\pi + p_K + p_\pi - p_K) \bar{u}(p_\pi) v(l + \gamma_4 u^\alpha) u^\alpha,
\]

where \(C\) is a suitable linear combination of C.G. coefficients of \(SL_\alpha, \gamma\). It is conventional to write the invariant matrix element for this process in the form \(^{(4)}\)

\[
G_C \left((f_+ + f_-) p_k^{(K)} + (f_+ - f_-) p_l^{(K)}\right) \bar{u} \gamma^\nu(1 + \gamma_4) u^\alpha,
\]

where \(f_+\) and \(f_-\) are suitable functions of the invariant momentum transfer to the leptons. Comparing with our result (12), we see that our model predicts a constant form factor with both \(f_+\) and \(f_-\) equal to each other. Our constant form factor is to be compared with the experimental determination \(^{(\ast\ast\ast)}\)

\[
f_+(q^2) + f_-(0) \left(1 + \frac{q^2}{m_\pi^2}\right),
\]

\[\lambda = 0.02 \pm 0.01.\]

For \(\xi = f_-/f_+\) we predict the value unity which is not inconsistent with the experimental determination. \(^{(\ast)}\)


\(^{(\ast\ast)}\) N. Brene, M. Roos and A. Sirlin: CERN preprint TH. 872 (1968).


We can also calculate the matrix element for the decay $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ by the same method. The quantity of interest is the suppression factor

\[
S_t = \frac{m_{\pi^+}}{m_{\pi^0}} \frac{1 + m_e/m_{\pi^0}}{1 + m_e/m_{\pi^+}} \simeq 0.18,
\]

which is to be compared with the experimental value (**) 0.21.

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**RIASSUNTO (**)**

Si applica al decadimento debole del mesone un modello algebrico che descrive la rottura della simmetria. La soppressione del decadimento debole che cambia la stranezza è conseguenza della rottura della simmetria.

(**) Traduzione a cura della Redazione.

Ассоциативная алгебраическая модель для распадов слабых мезонов.

Резюме (**). — Алгебраическая модель, описывающая нарушение симметрии, применяется к распаду слабых мезонов, как следствие получается нарушение симметрии.

(**) Переведено редакцией.