K_{l4} Decay in an Algebraic Model for Mesons*

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The associative algebraic model $\mathcal{B}$ which had previously been applied to the meson-spectrum problem and the two- and three-body leptonic decays of mesons has been used to calculate the absolute rates for K_{l4} decays. It predicts $\Gamma(K^+{}_{l4})=3.47\cdot 10^3\ \mathrm{sec}^{-1}$ and $\Gamma(K_{l4}{}^0)=0$, in agreement with the experimental data.

In a series of papers[1] we have introduced an associative algebra $\mathcal{B}$ as a model of hadrons and applied it to the calculation of leptonic decays of pseudoscalar mesons. It is a characteristic feature of the algebra $\mathcal{B}$ that the generators $E_{\pm 2}$, $E_{\pm 3}$, G_i, and $F_{\pm\alpha}$ of the internal-spectrum-generating noninvariance group $SL(3,c)_{G_i,F_\alpha}\supset SU(3)_{H_i,E_\alpha}$ do not commute with the momenta P_μ but obey the relations (5) of II. This lack of commutation implies a nontrivial mass spectrum and corresponds, in our approach, to generalized wave equations. The strength of this noncommutativity is controlled by a single parameter g with the dimensions of a squared mass, and a value given by

$$2/g=(0.36\times 10^{-13}\ \mathrm{cm})^2.$$

The predicted mass spectrum is discrete and in good agreement with the experimental boson spectrum. In III we had postulated that the leptonic interactions of the mesons is described by a transition operator

$$T=L^\lambda H_\lambda+\mathrm{H.c.},$$

where the hadronic operator H_λ is an element of $\mathcal{B}$:

$$H^\lambda=G\sum_\alpha\{P^\lambda, E_\alpha+F_\alpha\},$$

$$\alpha=\pm 1, \pm 2.$$

The matrix element of the leptonic part L^λ vanishes unless all the strong quantum numbers of the initial and final hadron states are the same. For the process

$$\gamma\to\alpha_1+\alpha_2+\cdots+e+\bar{\nu},$$

it is given by

$$\begin{aligned}\langle\bar{\nu}ep_\nu p_e\alpha_1\alpha_2\cdots p_{\alpha_1}p_{\alpha_2}\cdots|L^\lambda|p_\beta=(m_\beta/m_\gamma)p_\gamma,\beta\rangle\\ =(ab)^{N+1}\delta^3(p_\nu+p_e+p_{\alpha_1}+p_{\alpha_2}+\cdots-p_\gamma)\\ \times\bar{u}^{(e)}\gamma^\lambda(1+\gamma_5)u^{(\nu)},\end{aligned}$$

where

$$b=c_{\alpha_1}{}^{1/2}/2m_{\alpha_1}=c_{\alpha_2}{}^{1/2}/2m_{\alpha_2}=\cdots$$

is a $\mathcal{B}$-invariant normalization factor which cancels out in the final result. The value of the parameter aG was determined from the experimental value of the decay $\pi\to\mu\nu$ to be

$$aG=1.32\times 10^{-7}\pm(5\%).$$

The constant a was calculated from the branching ratio $(\pi\to\pi e\nu)/(\pi\to\mu\nu)$ to be

$$a=0.12\pm(15\%).$$

The relation of the above ansatz for the leptonic factor L^λ to the conventional formalism was discussed in IV. Also we have shown there that the suppression factors for the K_{l2} and K_{l3} decays as well as the spectrum shape are predicted correctly by the algebraic model independently of the numerical values of G and a. We have thus a *dynamical determination of the Cabibbo angles independent of any new parameters*.

We wish to study the predictions of $\mathcal{B}$ for the reaction

$$K\to\pi\pi e\nu,$$

about which a fair amount is known experimentally. We shall use the notations and conventions of II to calculate the quantity

$$\begin{aligned}\dot{P}=\sum_{\mathrm{pol}}\int\frac{d^3p_e}{2E_e}\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_+}{2E_+c_\pi}\frac{d^3p_-}{2E_-c_\pi}\\ \times\delta(E_++E_-+E_\nu+E_e-E_K)\\ \times\left|\left\{\int\frac{d^3p_K}{2c_KE_K(p_K)}\phi_K(p_K)\right.\right.\\ \left.\left.\times\langle e\nu\pi^+\pi^-p_ep_\nu p_+p_-|T|p_KK^+\rangle\right\}\right|^2.\end{aligned}$$

Here ϕ_K represents the physical state of the decaying K^+ meson and $(E_\pm,p_\pm)$ represent the energy momentum of the $\pi^\pm$ mesons in the final state. We calculate the transition matrix element

$$A=\langle\nu e\pi^+\pi^-p_\nu p_ep_+p_-|T|p_KK\rangle$$

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[1] A. Böhm, Phys. Rev. **158**, 1408 (1967), hereafter referred to as II; A. Böhm and E. C. G. Sudarshan, *ibid.* **178**, 2264 (1969), hereafter referred to as III; A. Böhm and E. C. G. Sudarshan, Nuovo Cimento **60**, 395 (1969), hereafter referred to as IV.

by the same method as in III:

$$A=\sum_{\alpha}\int\frac{d^3p_\alpha}{2c_\alpha E_\alpha(p_\alpha)}\langle\nu e\pi^+\pi^-|L^\lambda|p_\alpha\alpha\rangle\langle\alpha p_\alpha|H_\lambda|p_KK\rangle,$$

where the summation runs over all basis vectors, labeled by I, I_3, Y, and λ, of the irreducible representation space $H(\zeta, s=0, m_3=0, \rho_3=2b=0)$ of $\mathfrak{B}$.

Since the two-pion state has hadronic quantum numbers $Y=0$, $I=0$, 1, 2, $I_3=0$, the matrix element vanishes unless the state labeled by α has $Y=0$, $I=0$, 1, 2, $I_3=0$. In addition, for $\lambda>2$ (greater than the 27-plet) the matrix elements vanish by virtue of the $SL(3,c)$ properties of E_α, F_α. For $\lambda=2$, $I=2$ the matrix elements vanish by virtue of the $I=\frac{1}{2}$ nature of the relevant generator.[2] It follows that the two-pion state should be assigned to an $I=0, 1$; $Y=0$ state. The states $(\lambda=I=Y=0)$, $(\lambda=1, I=Y=0)$, and $(\lambda=I=1, Y=0)$ are already assigned for the σ, η, and π respectively. Hence the only possible choices for the dipion are $(\lambda=2, I=0, Y=0)$ and $(\lambda=2, I=1, Y=0)$. But since the two-pion system should be in a spin-0 configuration ($\mathfrak{B}$ does *not* include transitions between different spins), the space wave function is symmetric. Since the pions are bosons, it follows that $I=1$ is forbidden for this dipion. We shall denote this state $(\lambda=2, I=0, Y=0)$ by S; and it should have even parity, since in the representation for bosons the boson parity is given by the simple formula

$$\Pi=(-)^\lambda.$$

The mass formula (II.37) predicts[3,4] a mass of 1030 MeV for the S. Experimentally an even-parity spin-0 $I=Y=0$ dipion resonance has been found at about 1070 MeV. We now obtain

$$A=G\int\frac{d^3p_S}{2c_SE_S}\langle\nu e\pi^-\pi^+|L^\mu|p_SS\rangle$$
$$=\langle Sp_S|\{P_\mu F_{-2}\}|p_KK\rangle.$$

The calculation of the hadronic matrix element is performed in a manner similar to III:

$$\langle Sp_S|\{P_\mu,F_{-2}\}|p_KK^+\rangle =C_{SK}\left(1+\frac{m_S}{m_K}\right)p_\mu{}^{(K)}\left\langle Sp_S\middle|\frac{m_S}{m_K}p_K,S\right\rangle$$

where C_{SK} denotes the $SL(3,C)$ matrix element:

$$C_{SK}=\langle\lambda=2, I=0, I_3=0, Y=0|\hat{F}_{-2}|I_3=\tfrac{1}{2}, I=\tfrac{1}{2}, Y=1, \lambda=1\rangle$$
$$=\langle\lambda=2\|N\|\lambda=1\rangle C(\tfrac{1}{2}\,\tfrac{1}{2}\,0;+\tfrac{1}{2}-\tfrac{1}{2}0)\times U(1,1,2;\tfrac{1}{2},1,\tfrac{1}{2},-1,0,0).$$

The reduced $SL(3,C)$ matrix element is obtained from (III.A8):

$$\langle 2\|N\|1\rangle=-i\tfrac{1}{3}\sqrt{5}.$$

Using the tabulated isoscalar factors[5] we obtain

$$C_{SK}=i/\sqrt{24},$$

while from Appendix B of III we have

$$\left\langle Sp_S\middle|\frac{m_S}{m_K}p_KS\right\rangle=2c_SE_S(p_S)\delta^3\left(p_S-\frac{m_S}{m_K}p_K\right).$$

Thus

$$A=GC_{SK}p_\lambda{}^{(K)}\left(1+\frac{m_S}{m_K}\right)\times a^3b^3\delta^3(p_\nu+p_e+p_++p_--p_K)\bar{u}^{(e)}\gamma^\lambda(1+\gamma_5)u^{(\nu)}.$$

For the case of the electronic decay mode, we could simplify this expression by neglecting the electron mass. We write the polarization sum in the form

$$\sum_{\text{pol}}|\bar{u}^{(\nu)}\gamma^\lambda(1+\gamma_5)p_\lambda u^{(e)}|^2$$
$$=16\{(p_K\cdot p_\nu)[(p_K\cdot p_e)+m_e{}^2]-\tfrac{1}{2}(p_K{}^2-m_e{}^2)p_e\cdot p_\nu\}$$
$$\simeq16(Q\cdot p_\nu Q\cdot p_e-\tfrac{1}{2}Q^2p_e\cdot p_\nu),$$

where

$$Q=p_++p_-.$$

The decay width is

$$\Gamma=\left|\frac{GC_{SK}(1+m_S/m_K)a^3}{8m_\pi{}^2m_K}\right|^2\times\frac{1}{2^5}\times\frac{16}{m_K}\times\int\frac{d^3p_e}{E_e}\frac{d^3p_\nu}{E_\nu}\frac{d^3p_+}{E_+}\frac{d^3p_-}{E_-}\delta^4(p_K-p_e-p_\nu-p_+-p_-)\times(Q\cdot p_\nu Q\cdot p_e-\tfrac{1}{2}Q^2p_e\cdot p_\nu).$$

This expression leads to a dipion mass spectrum in agreement with the available experimental data.[6] The

[2] Note that the $\Delta Q=\Delta S$ rule for leptonic decays is a consequence of the $I=\frac{1}{2}$ current rule for the operators E_α, F_α: S. Okubo, R. E. Marshak, E. C. G. Sudarshan, W. B. Teutsch, and S. Weinberg, Phys. Rev. **112**, 665 (1958).

[3] As already remarked in the Introduction of III, the λ dependence of (II.37) might not be correct. The present experimental data show some preference for the mass formula $m^2=\text{const}+g[\frac{1}{4}Y^2-I(I+1)+2\lambda^2]$ over (II.37). If this is confirmed, the defining relations of $\mathfrak{B}$, Eqs. (II.5h) and (II.5i) ("wave equations"), which determine the λ dependence of m^2, would suffer minor alterations. Therefore, we prefer to use in the present calculation the experimental value $m_s=1069\pm20$ MeV for $m_{(2,0,0)}$ instead of the value predicted by our mass formula. This difference in the predicted decay rate coming from the change in the mass of S is small compared with the uncertainty originating in the errors on the values of a^2 and aG.

[4] The predicted mass for the $(\lambda=2, I=1, Y=0)$ 0^+ meson is between 920 and 860 MeV. A resonance with such quantum numbers at 946 MeV has been reported by B. D. Hyams *et al.*, Nucl. Phys. **B7**, 1 (1968).

[5] J. G. Kuriyan, D. Lurié, and A. J. Macfarlane, J. Math. Phys. **6**, 722 (1965).

[6] The histogram of the dipion masses is given in R. W. Birge *et al.*, Phys. Rev. **139**, B1600 (1965); the theoretical curve is presented in R. H. Dalitz, in *Proceedings of the International School of Physics "Enrico Fermi,"* edited by T. D. Lee (Academic Press Inc., New York, 1964).

integrals for the total decay rate are among those computed numerically by Okun' and Shabalin[7] and have the value

$$\frac{m_K{}^8 \times 2 \times (2\pi)^8}{2^{10}\pi^5 \times 360} \times 4 \times 0.0296\,,$$

which yields

$$\Gamma_{\text{theor}} = 2.28 \times 10^{-18}\,\text{MeV} \pm (70\%)$$
$$= 3.47 \times 10^3\,\text{sec}^{-1} \pm (70\%)\,.$$

This is in good agreement with the experimental value[8]

$$\Gamma_{\text{expt}}(K^+ \to \pi^+\pi^- e^+ \nu) = (2.9 \pm 0.6) \times 10^3\,\text{sec}^{-1}.$$

Since the predicted value of Γ is very sensitive to the value of a, the agreement of the theoretical and experimental widths constitutes a very good check of the value of a and the theoretical framework developed in III.

We should also point out that the decay

$$K^0 \to \pi^0 \pi^- e^+ \nu$$

should vanish. If we carried out the calculation for the $K_{e4}{}^0$ decay along the same lines, it is impossible to have a final state of zero isospin since one of the two pions is charged. Since $I=1$ and $I=2$ are forbidden, it follows that this decay is forbidden.

Finally, the ratio of $K_{\mu 4}$ and K_{e4} decays can be immediately deduced. Our model predicts

$$\Gamma(K_{\mu 4})/\Gamma(K_{e4}) = 0.17\,,$$

which is consistent with the experimental branching ratio[9]

$$\frac{\Gamma(K_{\mu 4})}{\Gamma(K_{e4})} = \frac{(1.4 \pm 0.9) \times 10^{-5}}{(3.6 \pm 0.8) \times 10^{-5}} = 0.39 \pm 0.26\,.$$

[7] L. B. Okun' and E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **37**, 1775 (1959) [English transl.: Soviet Phys.—JETP **10**, 1252 (1960)]; E. P. Shabalin, *ibid.* **39**, 345 (1960) [*ibid.* **12**, 245 (1961)].

[8] Birge *et al.* (Ref. 6).

[9] V. Bisi, R. Cester, A. M. Chiesa, and M. Vigone, Phys. Letters **25B**, 572 (1967).

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Some Comments on Unitarity Corrections to Current-Algebra Results

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Some further comments on unitarity corrections to current-algebra results are made.

RECENTLY, Bhargava, Biswas, Gupta, and Datta[1] proposed a method of unitarizing the results of a soft-pion current-algebra calculation of pion-nucleon scattering in the context of the N/D formalism. Though this method gives reasonably good values for the threshold parameters, it is found to give unsatisfactory results for the scattering parameters away from the threshold. This is perhaps understandable because in the above method we use current algebra (CA) in the soft-pion limit and hence it is difficult to justify the use of CA except to evaluate the amplitude at a point. Later extensions of the above method to π-π scattering by Datta, Gupta, and Varma[2] and to K^+p scattering by this author[3] suggest the possibility of using the CA amplitude in the off-mass-shell limit ($q_\pi{}^2 \to 0$) as input to obtain parameters away from the threshold.

In this note, in addition to unitarized threshold parameters for π-K and π-Ξ scattering, we obtain s-wave π-N phase shifts by unitarizing the CA amplitude for π-N scattering in the off-mass-shell limit. The present calculations are done purely in the spirit of our earlier work[1,3] and the main aim here is to show that CA coupled with unitarity can give parameters away from the threshold.

Thus if we assume that our off-mass-shell amplitude is an analytic function of s in the complex s-plane, with cuts analogous to those imposed on the physical amplitude by unitarity and crossing, we can write

$$g_0(s) = \frac{N(s)}{D(s)} = N(s) \bigg/ \left(1 - \frac{(s-s_0)}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \times \frac{N(s')\,|\mathbf{q}'|}{(\sqrt{s'})(s'-s)(s'-s_0)}\right), \quad (1)$$

where the phase-space factor $\sqrt{s}$ has been introduced because of our definition

$$g_0(s) = (\sqrt{s}) \frac{e^{i\delta_0} \sin\delta_0}{|\mathbf{q}|}\,. \quad (2)$$

[1] S. C. Bhargava, S. N. Biswas, K. C. Gupta, and K. Datta, Phys. Rev. Letters **20**, 558 (1968); hereafter BBGD.

[2] K. Datta, K. C. Gupta, and V. S. Varma, Phys. Rev. **173**, 1547 (1968).

[3] S. C. Bhargava, Phys. Rev. **177**, 2205 (1969).