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A NEW FORMULATION OF RELATIVISTIC QUANTUM FIELD THEORY*

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ABSTRACT

Quantum field theory is reformulated in such a manner that a complete set of ocillators for modes with both positive and negative energies are introduced. The theory leads to the proper connection between spin and statistics as in the standard formulation, but it implements the time reversal transformation and the TCP transformation as linear unitary transformations. Negative energy particles in the initial states are identified with antiparticles in the final state with reversed motion (and *vice versa*) as far as scattering amplitudes are concerned. A covariant perturbation theory is developed which yields scattering amplitudes which are essentially the same as in the usual theory.

I. INTRODUCTION

ACCORDING to the theory of relativity the energy and momentum of a free particle transform among themselves as the components of a four-vector under Lorentz transformations. The square of the energy is given by

$$E^2 = p^2 + m^2$$

where m^2 is a constant, positive zero or negative. For ordinary particles m is itself taken to be real so that m^2 is positive. The above equation yields two values for the energy

$$E = \pm \sqrt{p^2 + m^2} \tag{1}$$

It is customary to take only the positive square root and assume that we have always positive energy particles only. Such a point of view is consistent with relativistic invariance as long as m^2 is non-negative; for m^2 negative (faster-than-light particles) the distinction between positive and negative energy states is not relativistically invariant since a suitable Lorentz transformation can change the sign of the energy. In connection with this

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case we had studied the physical interpretation of negative energy particles and found them to be travelling backwards in time. This suggested an interesting physical reinterpretation postulate for negative energy particles.

Negative energy particles in classical physics.—Let us consider the nature of the negative energy particles for non-negative values of m^2. In this case (including $m^2 = 0$) the negative energy particle is a relativistically invariant concept. In classical relativistic physics the four-momentum vector and the space-time displacement vector are proportional since the world line of the particle is a straight line. If v is the velocity of the particle (with the speed of light equal to unity) the momentum is given by

$$p = m(1 - v^2)^{-\frac{1}{2}} \cdot v.$$

We shall take the displacement $\triangle x$ in the same direction (and sense) as the momentum. Then for positive (negative) energy of the particle, the time elapsed $\triangle t$ would be positive or negative. In other words, *negative energy particles travel backwards in time*. They arrive before they start and carry negative energy. Hence they are physically equivalent to ordinary positive energy particles going forward in time but in the opposite direction.

To see this more clearly we consider an apparatus A emitting a particle at time t_1 which is absorbed by another apparatus B at time t_2. For the case of a positive energy particle $t_2 > t_1$ and energy is transferred from A to B, with B gaining the energy last by A. In the case of a negative energy particle $t_1 > t_2$ and in the energy transfer between A and B, A gains (positive) energy and B loses it. In view of this it is more satisfactory physically to interpret the phenomenon as the apparatus B emitting a positive energy particle at time t_2 which is subsequently being absorbed by apparatus A at time t_1. With this reinterpretation only positive energy particles are involved in the ultimate physical interpretation and they all travel forward in time. This reinterpretation (and its quantum theory counterpart) are key concepts in the physics of faster-than-light particles; but we now see that they can equally well be considered for light-like or slower-than-light particles.

The replacement of the emission (absorption) of a negative energy particle by the absorption (emission) of the corresponding positive energy particle (but with reversed momenta) may be thought of as the classical equivalent of the principle of crossing symmetry in quantum mechanics.

It is known that classical particles furnish irreducible realizations of the Poincaré group; these realizations are not equivalent to their complex conjugates since such a transformation changes the sign of the energy. The

operation of space inversion is a canonical automorphism of the Poincaré group and is defined on the positive and negative energy particles separately. But the time inversion transformation cannot be defined on the Poincaré group as a canonical automorphism unless we include both positive and negative energy particles. If we have only positive energy particles we would have to deal with time reversal as an antiautomorphism. We shall not elaborate on these questions here, but only point to the related problem of time reversal invariance in quantum field theory discussed below.

Negative energy particles in quantum theory.—Let us now consider negative energy particles in quantum theory. By Ehrenfest's theorem we can still conclude that if we consider that on an average a particle must move in the direction of its momentum, then, negative energy particles travel backward in time. These two features of carrying negative energy and of travelling backward in time can be used to reinterpret the physics to say that the particles travel forward in time with positive energy but that emissions and absorptions must be interchanged. The reinterpretation of negative energy particles in quantum mechanics is obtained by virtue of "crossing".

It is necessary to note that *crossing is a property of transition amplitudes, not of quantum-mechanical states.* Each particle of negative energy in the *initial* state is associated with a positive energy particle in the *final* state; and *vice versa.* We must have both the initial and final states before we can make the transition amplitude.[1] And crossing is defined for the transition amplitudes only.

This implies in turn that while we would like to restrict attention to only those amplitudes which contain only positive energy particles both in the initial and final states as being *physical amplitudes*, we do not make a restriction on the states. In this sense the present method of dealing with negative energy particles is different from that of Dirac.[3] In Dirac's hole theory of the positron, the states themselves are given a new physical interpretation, but this necessitates a second quantized theory obeying Fermi Statistics. The present method is applicable equally to both Fermi and Bose systems.

To illustrate the content of the proposed reinterpretation and its enabling us to consider time reversal and space-inversion as both essentially geometric linear transformations consider transition amplitude $F(p_1q_1p_2q_2)$ for the following process:

$$\pi_1 + \eta_1 \rightarrow \pi_2 + \eta_2$$

where we denote the pion four-momenta by p_1, p_2 and the eta four-momenta by q_1 and q_2. Under a space-inversion we get the same process with the four-momenta changed as follows:

$$p' = \quad p \qquad p^{0\prime} = +p^0$$
$$q' = -q \qquad q^{0\prime} = +q^0.$$

In this spinless case the conservation of angular momentum in the collision automatically implies invariance, under space-inversion so that $F(p_1'q_1'p_2'q_2') = F(p_1q_1p_2q_2)$. All this is quite standard. If we now consider time inversion we have, in the present formalism, a very similar transformation: the four-momenta change according to:

$$p'' = +p \qquad p^{0\prime\prime} \qquad p^0$$
$$q'' = \quad q \qquad q^{0\prime\prime} \qquad q^0$$

This is a purely geometric transformation and we get the transformed amplitude:

$$F'' = F(p_1''q_1''p_2''q_2'')$$

Again in this spinless case we have

$$F'' = F.$$

But F'' is a transition amplitude with negative energy particles in the initial and final states. Hence by the reinterpretation postulate we should identify $F(p_1''q_1''p_2''q_2'')$ with the amplitude for the crossed process: $F(-p_2'' - q_2'' - p_1'' - q_1'')$. But this is the same as the reverse process

$$\pi_2 + \eta_2 \rightarrow \pi_1 \quad \eta_1$$

with all the momenta reversed. This result coincides with the standard (Wigner) prescription for transition amplitudes.[4] We have thus demonstrated the equivalence of our formulation with that of the usual theory, though for transformations reversing the direction of time the behaviour of the states is quite different in the two cases. We shall encounter this circumstance in our discussion of quantum field theory below.

II. SECOND QUANTIZATION OF BOSE FIELDS

Beginning with the discovery of radiation oscillators by Planck[5] and the statistics of photons by Bose[6] it has been gradually accepted that the proper relativistic description of quantum-mechanical systems was by a

"second-quantized" theory. The ideas of Planck and Bose found their logical completion in the work of Fock who gave an operator formalism which described a collection of symmetrized many-particle states with a variable number of particles. The method of Fock[7] could be extended to the description of particles obeying Fermi statistics and for relativistic theories. In the case of the second-quantized Schrodinger field we proceed as follows:

Let $u_n(x)$ be a complete orthonormal set of one particle wave functions satisfying the relations

$$\sum_n u_n(x)\, u_n^*(y) = \delta(x-y)$$
$$\int u_n^*(x)\, u_{n'}(x)\, d^3x = \delta_{nn'}. \tag{2}$$

Let $a_n, a_n^\dagger$ be a set of annihilation and creation operators satisfying the commutation (or anticommutation) relations

$$[a_n, a_{n'}]_\pm = [a_n^\dagger, a_{n'}^\dagger]_\pm = 0$$
$$[a_n, a_{n'}^\dagger]_\pm = \delta_{nn'}. \tag{3}$$

Then the quantized fields $\psi(x)$, $\psi^\dagger(x)$ are defined according to

$$\psi(x) = \sum_n a_n u_n(x)$$
$$\psi^\dagger(x) = \sum_n a_n^\dagger u_n^*(x) \tag{4}$$

and satisfy the commutation (or anticommutation) relations

$$[\psi(x), \psi^\dagger(y)]_\pm = \delta(x-y). \tag{5}$$

For wave functions with spin an obvious extension of this formalism is needed. When the basis functions are replaced by another orthonormal set the creation and annihilation operators undergo a canonical transformation so that the field commutation (or anticommutation) relations are preserved.

When one proceeds to relativistic theory the standard method of second quantization makes use of a complete set of *positive energy solutions* of the spin 0 field. The field is then broken up into positive and negative frequency parts which are respectively associated with annihilation and creation operators:

$$\phi^{(+)}(x,t) = \Sigma a_n f_n(x,t)$$
$$\phi^{(-)}(x,t) = \Sigma a_n^\dagger f_n^*(x,t)$$

where the functions $f_n(x, t)$ are positive energy solutions of the Klein-Gorden equation which are orthonormal in the appropriate scalar product:

$$2 \quad \{f_n^* \overset{\circ}{f}_{n'} - \overset{\circ}{f}_n^* f_{n'}\} d^3x = \delta_{nn'} \tag{7}$$

$$[a_n, a_{n'}] = [a_n^\dagger, a_{n'}^\dagger] = 0$$
$$[a_n, a_{n'}^\dagger] = \delta_{nn'} \tag{8}$$

Then

$$[\phi^{(+)}(x, t), \phi^{(-)}(x', t')] = \frac{1}{(2\pi)^3} \int e^{ik.(x-x')} e^{-i\omega(t-t')} \frac{d^3k}{2\omega} \tag{9}$$

where

$$+\sqrt{m^2 + k^2}$$

and m is the mass of the scalar particles. From this relation we obtain

$$[\phi^{(+)}(x, t), \phi^{(-)}(x', t)] \quad [\phi^{(} \quad x, t), \phi^{(} \quad (x', t)] = - \; i\delta(x - x'). \tag{10}$$

On the basis of this formalism we can establish the theory of the free (neutral) scalar field and demonstrate the equivalence of this theory with the canonically quantized Klein-Gorden field $\phi(x, t)$ with the Lagrangian density

$$\mathbf{L} \quad \tfrac{1}{2} \{\partial^\mu \phi \partial_\mu \phi \quad m^2 \phi^2\}. \tag{11}$$

With the associated commutation relations

$$[\phi(x, t), \dot{\phi}(y, t)] \quad i\delta(x \quad y) \tag{12}$$

Despite the relativistic invariance of the theory so obtained, the decomposition into positive and negative frequency parts is not local.

We propose a new formalism in which the complete set of solutions (of positive as well as of negative energy) is associated with annihilation operators just as in the case of the second quantized Schrodinger field; accordingly we write

$$\phi(x) \quad \frac{1}{(2\pi)^{3/2}} \int \{a(\omega\, k)\, e^{-i\omega x_0 + ik.x} + a \quad \omega - k)\, e^{i\omega x_0 - ik.x}\} \frac{d^3k}{2\omega}$$

(13)

The conjugate field is defined by

$$\phi^{\dagger}(x) = \frac{1}{(2\pi)^{3/2}} \int \{a^{\dagger}(\omega k)\, e^{i\omega x_0 - ik.x}$$

$$+ a^{\dagger}(-\omega - k)\, e^{-i\omega x_0 + ik.x}\} \frac{d^3k}{2\sqrt{2.\omega}}. \tag{14}$$

The commutation relations are

$$[a(\pm\omega, k), a^{\dagger}(\pm\omega' k')] = \pm 2\omega\delta(k - k')$$

$$[a(\omega k), a(-\omega' k')] = [a(\omega k), a^{\dagger}(-\omega' k')] = 0. \tag{15}$$

With this choice of the commutation relations we obtain

$$[\phi(x), \phi(y)] = 0$$

$$[\phi(x), \phi^{\dagger}(y)] = i\triangle(x - y). \tag{16}$$

$$\triangle(x - y) = -\frac{1}{(2\pi)^3} \int e^{ik.(x-y)} \sin\omega(x^0 - y^0) \frac{d^3k}{\omega}. \tag{17}$$

These immediately lead to the equal time commutation relations

$$\delta(x^0 - y^0)\,[\phi(x), \phi^{\dagger}(y)] = 0$$

$$\delta(x^0 - y^0)\,[\phi(x), \phi^{\dagger}(y)] = i\delta(x - y) \tag{18}$$

which are the familiar canonical commutation relations.

We could deduce these commutation relations and equations of motion starting with the Action Principle[8] and the Lagrange density

$$\mathrm{L} = \tfrac{1}{2}\{\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - m^2\phi^{\dagger}\phi\} \tag{19}$$

The Action Principle states

$$i\delta\phi(x) = [\phi(x), \delta\mathrm{A}]; \quad \mathrm{A} = \int \mathrm{L}(x)\, d^4x. \tag{20}$$

Considering variations which vanish at the end points we obtain the equations of motion

$$(\Box^2 + m^2)\,\phi = (\Box^2 + m^2)\,\phi^{+} = 0. \tag{21}$$

Considering variations which do not vanish at end points we get the commutation relations

$$\delta(x^0 \quad y^0)\,[\phi(x),\ \partial^0\phi^\dagger\ (y)] = i\delta(x-y)$$

$$\delta(x^0 - y^0)\,[\phi(x),\ \partial^0\phi(y)] = 0$$

Finally, considering variations of the upper limits of space and time integration we get the expressions for the generators of space and time displacements, that is the four-momentum.

In particular the energy is given by

$$H = P^0 \quad \tfrac{1}{2}\ \{\dot{\phi}^\dagger\dot{\phi} + \nabla\phi^\dagger \quad \phi \quad m^2\phi^\dagger\phi\}\,d^3x \tag{22}$$

which is the familiar expression for a Klein-Gorden field. If we rewrite it in terms of the creation and annihilation operators we get

$$H = \int \omega\,\{a^\dagger(\omega k)\,a(\omega k) + a^\dagger(\quad \omega k)\,a \quad \omega k)\}\,\frac{d^3k}{4\omega}.$$

This expression, together with the commutations relations (15) for $a(\pm\omega k)$, enable us to interpret H as the energy of a collection of positive and negative energy particles, provided the invariant vacuum state $|0\rangle$ is defined by

$$a(\omega k)\ |0\rangle = a(-\omega k)\,|0\rangle = 0. \tag{24}$$

The quantized field is thus equivalent to a collection of particles with either sign of the energy.

In dealing with the scalar field we have made use of quantization according to Bose statistics. This necessary relation between spin and statistics is a consequence of an invariance requirement on the Action function called the S-principle. This matter is discussed in some detail in a later section.

III. Interacting Fields and the Physical Reinterpretation Postulate

As an illustration of the construction of the theory of interacting fields, let us consider the coupling of a neutral scalar field with another scalar field ψ bilinearly. We write

$$L_{int} = g(\psi^\dagger + \psi)^2(\phi^\dagger + \phi).$$

Assuming the mass m of the ϕ field to be less than twice the mass M of the ψ field there are no processes to first order in the interaction (25). To second order we have the following processes:

$$\begin{aligned} &M + m \to M + m \\ &M + M \to M + M \end{aligned} \tag{26}$$

The first one is elastic scattering of a ϕ-quantum by a ψ-quantum. The second one is the scattering of two ψ quanta with the exchange of a virtual ϕ-quantum. To calculate the ψ-ϕ scattering we consider the equation of motion of the ψ field:

$$(\Box^2 + M^2)\,\psi = g\,(\psi^\dagger + \psi)\,(\phi^\dagger + \phi) \tag{27}$$

so that

$$\psi(x) = \psi_0(x) + g \int G(x - y)\,\big(\phi^\dagger(y) + \phi(y)\big)\,\big(\psi^\dagger(y) + \psi(y)\big)\,d^4y \tag{28}$$

where $G(x - y)$ is a Green's function satisfying

$$(\Box^2 + M^2)\,G(x - y) = \delta(x - y) \tag{29}$$

and $\psi_0(x)$ is a solution of the homogeneous equation. Hence the effective interaction leading to $\phi\psi$ scattering is

$$\tfrac{1}{2}g^2 \int\int \psi^\dagger(x)\,\phi^\dagger(x)\,G(x - y)\,\psi(y)\,\phi(y)\,d^4x d^4y \tag{30}$$

where the factor $\frac{1}{2}$ has been added to compensate for the double counting. If we denote the ψ and ϕ momenta by p and q, the effective transition amplitude to second order is given by

$$\tfrac{1}{2}g^2\big(\{(p + q)^2 - M^2\}^{-1} + \{(p - q^1)^2 - M^2\}^{-1}\big). \tag{31}$$

Since the denominators never vanish, we do not have to specify the Green's function any further.

To calculate the effective $\psi\psi$ scattering we solve the equation of motion of the ϕ field to obtain

$$(\Box^2 + m^2)\,\big(\phi^\dagger(x) + \phi(x)\big) = g\,\big(\psi^\dagger(x) + \psi(x)\big)^2 \tag{32}$$

to obtain the contribution of the source $g\,(\psi^\dagger + \psi)^2$ at the point y to the ϕ field at the point x to be equal to

$$g \int G(x - y;\ m)\,\big(\psi^\dagger(y) + \psi(y)^2\big)\,d^4y.$$

Consequently the effective $\psi\psi$ interaction is

$$-g^2 \quad (\psi^\dagger(x) \quad \psi(x))^2 \, G(x \quad y; \quad m) \, (\psi^\dagger(y) \quad \psi(y))^2 \, d^4x d^4y \quad (33)$$

If we denote the momenta in the initial state by p_1, p_2 and in the final state by p_1', p_2' the scattering amplitude, in the second order approximation, is:

$$:g^2 \left\{ \{(p_1 - p_1 \quad M^2\} \quad \{(p_1 \quad p_2')^2 - M^2\}^{-1}. \right) \quad (34)$$

Both the $\psi\phi$ (" Compton ") scattering amplitude and the $\psi\psi$ (" Moller ") scattering amplitudes have, to the second order in the coupling constant g, exactly the same expression, as we would have obtained from the standard theory. It would be necessary to note that the present theory not only furnishes the amplitudes for the mutual scattering of positive energy particles but also of negative energy particles. But in this case we make use of the physical reinterpretation postulate.[9] *A negative energy particle in the initial (final) state of a transition is to be identified with a positive energy (anti) particle with opposite linear (and angular) momentum in the final (initial) state.* We find that the scattering amplitudes that we have computed above in the second-order approximation do have this feature. We shall henceforth implicitly assume the reinterpretation postulate in discussing transition amplitude: without loss of any generality we could restrict attention to positive energy particles insofar as transition amplitudes are concerned. In a real sense, *only positive energy particles are physical.*

We have seen that the second-order scattering amplitudes could be computed without specifying the precise choice of the Green's function, since their Fourier transforms never vanished in the physical domain for the initial and final particles. This is a property of more general class of processes represented by "tree diagrams". Of course these processes share with the simpler second order processes the defect that they do not yield unitary scattering amplitudes.

Before proceeding to the systematic computation of the higher order effects it is desirable to discuss the question of the freedom in the choice of the Green's function. We have seen that for the equation of motion

$$(\Box^2 + m^2) \, \phi(x) \quad \xi(x) \quad (35)$$

the general solution is

$$\phi(x) = \phi_0(x) \quad G(x - y) \, \xi(y) \, d^4y \quad (36)$$

where $\phi_0(x)$ is any solution of the homogeneous equation and $G(x-y)$ is any Green's function. For the same Heisenberg field $\phi(x)$ and the same equation of motion, the change in the Green's function entails a change in the "free field" $\phi_0(x)$. The change from $G(x-y)$ to $G_1(x-y)$ without change of $\phi(x)$ implies that $\phi_0(x)$ be replaced by $\phi_0(x) + \int \{G(x-y) - G_1(x-y)\} \xi(y)\, d^4y$. In particular, the change from the half retarded-half advanced Green's functions

$$\bar{\triangle}(x \quad y) \quad \frac{1}{(2\pi)^4} \mathrm{P} \int \frac{e^{ik(x-y)}}{k^2 - m^2}\, d^4k$$

to the causal Green's function

$$\triangle_c(x \quad y) = \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}\, d^4k$$

is equivalent to the augmenting of the free field by the terms

$$\frac{i}{2} \int \triangle^{(1)}(x-y)\, \xi(y)\, d^4y$$

where $\triangle^{(1)}$ is the symmetric invariant function

$$\triangle^{(1)}(x) = \frac{1}{(2\pi)^3} \int e^{ikx}\, \delta(k^2 - m^2)\, d^4k.$$

Conversely, the change in the Green's function is entailed by a change in the asymptotic field.

IV. Second Quantization of the Dirac Field

The free Dirac field obeys the equation

$$(i\gamma^\mu \partial_\mu \quad \mathrm{M})\, \psi(x) = 0.$$

This equation can be solved to obtain plane wave solutions of the form

$$\psi(x) = u_r(x) e^{ikx}$$

where the four-momentum k satisfies

$$k^0 = \pm \sqrt{\mathrm{M}^2 + k^2} \quad \pm \omega$$

and $u_r(k)$ satisfies the equations

$$(\gamma^\mu k_\mu - \mathrm{M})\, u_r(k) \quad 0.$$

The two values of r are related to the two possible values of the helicity:

$$\sigma \cdot k u_r(k) = (-)^{r-1} \mid k \mid u_r(k). \tag{44}$$

For the same value of the spatial momentum k there are four solutions, two solutions with positive energy and two with negative energy.

According to our general principle we define for each value of the spatial momentum four annihilation operators, two for positive energy and two for negative energy. We write, accordingly, for the second quantized Dirac field

$$\psi(x) = \int \{a_r(\omega k)\, u_r(\omega\, k)\, e^{-ikx} + a_r(-\omega - k)\, u_r(-\omega - k)\, e^{ikx}\} \frac{d^3k}{2\sqrt{2\omega}} \tag{45}$$

with the conj gate field

$$\psi^\dagger(x) = \int \{a_r^\dagger(\omega k)\, u_r^\dagger(\omega k)\, e^{ikx} + a_r^\dagger(-\omega - k)\, u_r^\dagger(-\omega - k)\, e^{-ikx}\} \frac{d^3k}{2\sqrt{2\omega}}. \tag{46}$$

The creation and destruction operators are chosen to satisfy the anticommutation relations

$$\{a_r(\pm\omega k), a_s^\dagger(\pm\omega' k')\} = 2\omega\delta(k - k')\,\delta_{rs}$$
$$\{a_r(+\omega k), a_s(\pm\omega' k')\} = \{a_r(\omega k), a_s^\dagger(-\omega' k')\} = 0 \tag{47}$$

It then follows that

$$\delta(x^0 - y^0)\{\psi(x), \psi^\dagger(y)\} = \delta(x - y) \tag{48}$$

$$\delta(x^0 - y^0)\{\psi(x), \psi(y)\} = 0 \tag{49}$$

provided we normalize the solutions $u_r(\pm\omega k)$ by

$$u_r^\dagger(\pm\omega k)\, u_r(\pm\omega k) = 2\omega. \tag{50}$$

In writing down the second step of (48) we have made use of the completeness of the four solutions $u_r(\pm\omega k)$ for a fixed value of the spatial momentum k. Note the fact that the anticommutators of the annihilation and creation

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with the conjugate field

$$\psi^\dagger(x) = \int \{a_r^\dagger(\omega k)\, u_r^\dagger(\omega k)\, e^{ikx} + a_r^\dagger(-\omega - k)\, u_r^\dagger(-\omega - k)\, e^{-ikx}\} \frac{d^3k}{2\sqrt{2\omega}}. \tag{46}$$

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In writing down the second step of (48) we have made use of the completeness of the four solutions $u_r(\pm\omega k)$ for a fixed value of the spatial momentum k. Note the fact that the anticommutators of the annihilation and creation

operators are positive for both positive and negative energy states. This is to be contrasted with the commutators of the annihilation and creation operators for the spin 0 field which change sign with the sign of the energy. This difference is to be traced to the difference in the nature of the scalar product: for the spin 0 case the scalar product changed sign with the sign of the energy, while the spin $\frac{1}{2}$ scalar product (50) does not change sign with the sign of the energy.

The general anticommutation relations can also be written down from the basic anticommutation relations for the annihilation and creation operators. We get

$$\{\psi(x),\ \psi(y)\} \quad 0$$

$$\{\psi(x),\ \psi^{\dagger}(y)\} \quad \frac{1}{(2\pi)^3}\int\{e^{-ik(x-y)}\,u_r(\omega k)\,u_r^{\dagger}(\omega k)$$

$$e^{ik(x-y)}\,u_r(\omega \quad k)\,u^{\dagger}(\cdot\ \omega-k)\}\,\frac{d^3k}{4\omega}.$$

But

$$\Sigma\, u_r(\omega k)\,u_r^{\dagger}(\omega k) = \gamma^{\circ}(\gamma \quad k + \mathrm{M}),\ k^{\circ} \quad + \sqrt{k^2 + \mathrm{M}^2}$$

$$\Sigma\, u_r(\cdot \quad \cdot k)\,u_r^{\dagger}(\ \omega-k) = \gamma^{\circ}(\gamma \quad k - \mathrm{M}).$$

Hence

$$\{\psi(x),\ \bar{\psi}(y)\} \quad \left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \mathrm{M}\right) i\triangle(x-y)$$

$$\mathrm{S}(x \quad y) \tag{51}$$

where $\triangle(x-y)$ is the invariant commutator function for the scalar field. The function $\mathrm{S}(x-y)$ has the properties

$$\delta(x^0 \quad y^0)\,\mathrm{S}(x \quad y) \qquad \delta(x \quad y)$$

$$\mathrm{S}(y \quad x) \quad + \mathrm{S}(x \quad y)$$

$$\left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} \quad \mathrm{M}\right)\mathrm{S}(x-y) = 0, \tag{52}$$

The fundamental equal-time anticommutation relations as well as the equations of motion can be deduced from the Action Principle starting with the Lagrangian density

$$L(x) = \frac{i}{2}\left\{\bar{\psi}\gamma^\mu \frac{\partial}{\partial x^\mu}\psi - \frac{\partial \bar{\psi}}{\partial x^\mu}\gamma^\mu \psi\right\} \quad M\bar{\psi}\psi$$

and considering anticommuting variations.

The general equation of motion for the Dirac field is of the form

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} \quad M\right)\psi(x) = \eta(x)$$

the general solution

$$\psi(x) = \psi_0(x) + \int \mathcal{G}(x-y)\,\eta(y)\,d^4y \tag{53}$$

where $G(x-y)$ is any Green's function satisfying the inhomogeneous equation

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} \quad M\right)\psi_0(x) \quad \delta(x). \tag{54}$$

For the same field $\psi(x)$, if we change the Green's function $\mathcal{G}(x-y)$ we must change the free field $\psi_0(x)$. The two special choices for $\mathcal{G}(x-y)$ are the half retarded-half advanced Green's function

$$\bar{\mathcal{G}}(x \quad y) = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right)\bar{\Delta}(x \quad y) \tag{55}$$

the causal Green's function

$$\bar{\mathcal{G}}_c(x-y) \quad \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} \quad M\right)\Delta_c(x-y). \tag{56}$$

The change from $\mathcal{G}$ to $\mathcal{G}_c$ for the same Heisenberg field $\psi(x)$ can be generated by a change in the free field from $\psi_0(x)$ to

$$\psi_0(x) + \frac{i}{2}\int S^{(1)}(x-y)\,\eta(y)\,d^4y$$

where

$$S^{(1)}(x) = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right)\Delta^{(1)}(x) \tag{57}$$

The freedom in the choice of the free field $\psi_0(x)$ is thus equivalent to the freedom in the choice of the Green's function.

With the formalism as developed so far we can compute the lowest order (second order) predictions of the Yukawa interaction

$$L_{int} = g\Psi^\dagger \gamma^0 \Psi \Phi \tag{58}$$

for Fermion-Fermion and Boson-Fermion scattering. Here Ψ is the self-conjugate fermion field containing both creation and annihilation parts

$$\Psi_r(x) = \psi_r(x) + B_{rs}\psi_s^\dagger(x)$$

where B is a matrix so chosen that $B_{rs}\psi_s^\dagger$ transforms under Lorentz transformations like ψ_r. This matrix is dependent on the representation chosen for the Dirac matrices; in the Majorana representation it is the unit matrix. More generally it is defined by

$$B(\sigma^{\mu\nu})^* B^{-1} = -\sigma^{\mu\nu}$$

$$B(\gamma^\mu)^* B^{-1} \quad -\gamma^\mu$$

The second of these conditions, of course, implies the first one. There are no essential difference in this calculation from the corresponding calculation that we carried out in the last section for boson-boson scattering and we shall not reproduce the calculation here.

The Connection between Spin and Statistics

We would like to discuss the Action Principle and the relation between spin and statistics.[10] To provide for a uniform treatment we shall arrange to have the equations of motion to be of the first order so that the Lagrangram are linear in the " velocities". We denote the field variables by $\phi_r(x)$ where r runs over the many components of the field ϕ and write the Lagrangian density in the form

$$L(x) \quad \tfrac{1}{2}(\Gamma^\mu)_{rs}\{\phi_r^\dagger \partial_\mu \phi_s \quad \partial_\mu \phi_r^\dagger \phi_s\} \quad H(x)$$

where $H(x)$ contains no dependence on the gradients of the field components. In particular it includes the bilinear mass terms. The matrices Γ^μ are so chosen that the Lagrangian density is invariant under Lorentz transformations. This Lagrangian density has the shortcoming that it treats ϕ and $\phi^\dagger$ in an unsymmetric fashion.

Since ϕ furnishes a representation of the complete Lorentz group (which is in general reducible and not necessarily unitary) $\phi^\dagger$ also furnishes a representation of the Lorentz group. For any collection of finite dimensional representations, $\phi^\dagger$ is equivalent to the representation furnished by ϕ. We shall consider finite dimensional representations only here. If the representation furnished by ϕ is denoted by $D(\wedge)$:

$$\phi_r(x) \to D_{rs}(\wedge)\,\phi_s(\wedge^{-1}x) \tag{61}$$

then the representation by $\phi^\dagger$ is of the form

$$\phi_r^\dagger(x) \to D_{rs}^*(\wedge)\,\phi_s^\dagger(\wedge^{-1}x).$$

The said equivalence implies

$$E_{rn}D_{ns}(\wedge) = D_{rn}^*(\wedge)\,E_{ns}. \tag{62}$$

If we therefore write

$$\psi_r = (E^{-1})_{rs}\phi_s^\dagger$$

then ψ transforms like ϕ and we may rewrite the above Lagrangian density in the form

$$L(x) = \tfrac{1}{2}(E\Gamma^\mu)_{rs}(\psi_r \cdot \partial_\mu\phi_s - \partial_\mu\psi_r \,.\, \phi_s) - H(x).$$

In this form we can implement a suitable symmetrization of L under the interchange of the creation and annihilation operators that is of the fields ϕ, ψ. We shall require that the *Lagrangian density and, hence, the Action be invariant under the replacement*[10]

$$\begin{aligned}\phi(x) &\to \psi(-x)\\ \psi(x) &\to \phi(-x).\end{aligned} \tag{63}$$

We shall refer to this requirement as the S-principle. The above Lagrangian density does not satisfy this property. We can, however, symmetrize it so as to satisfy the S-principle, to obtain:

$$L(x) = \tfrac{1}{4}(E\Gamma^\mu)_{rs}\{\psi_r\partial_\mu\phi_s - \partial_\mu\psi_r\phi_s - \phi_r\partial_\mu\phi_s + \partial_\mu\phi_r\psi_s\} - H(x). \tag{64}$$

Henceforth we shall consider (64) as the Lagrangian density from which the Action function would be constructed.

The Action Principle now states that all variations in the dynamical variables are generated by the changes in the Action:

$$\mathrm{A} = \int \mathrm{L}(x)\, d^4x$$

so that

$$\begin{aligned} i\delta\phi_r(x) &= [\phi_r(x), \delta\mathrm{A}] \\ i\delta\psi_r(x) &= [\psi_r(x), \delta\mathrm{A}] \end{aligned} \tag{65}$$

We shall consider two possibilities for the field variations: either that they shall commute with all the field variables or that they shall anticommute with all field variables. The first possibility leads to commutation relations and the second to anticommutation relations. The Lagrangian density (64) may be rewritten in the form

$$\begin{aligned} \mathrm{L}(x) = {} & \tfrac{1}{4}(\mathrm{E}\Gamma^\mu)_{rs}\{\psi_r \partial_\mu \phi_s - \partial_\mu \psi_r \phi_s\}\, \partial \\ & + \tfrac{1}{4}(\mathrm{E}\Gamma^\mu)_{sr}\{\partial_\mu \phi_s \psi_r - \phi_s \partial_\mu \psi_r\} - \mathrm{H}(x). \end{aligned}$$

Considering variations which vanish at end points we get the equations of motion. By considering end point variations alone we get, according to (65),

$$\begin{aligned} i\delta\phi_r(x) = {} & \tfrac{1}{4}(\mathrm{E}\Gamma^0)_{ns} \int [\phi_r(x), \psi_n(y)\, \delta\phi_s(y)]\, \delta(x^0 - y^0)\, d^4y \\ & + \tfrac{1}{4}(\mathrm{E}\Gamma^0)_{sn} \int [\phi_r(x), \delta\phi_s(y)\, \psi_n(y)]\, \delta(x^0 - y^0)\, d^4y \\ i\delta\psi_r(x) = {} & -\tfrac{1}{4}(\mathrm{E}\Gamma^0)_{ns} \int [\psi_r(x), \delta\psi_r(y)\, \phi_s(y)]\, \delta(x^0 - y^0)\, d^4y \\ & - \tfrac{1}{4}(\mathrm{E}\Gamma^0)_{sn} \int [\psi_r(x), \phi_s(y)\, \delta\psi_n(y)]\, \delta(x^0 - y^0)\, d^4y. \end{aligned} \tag{66}$$

For variations commuting with the field variables we get from either of the above relations:

$$\tfrac{1}{2}\delta(x^0 - y^0)\{(\mathrm{E}\Gamma^0)_{ns} + (\mathrm{E}\Gamma^0)_{sn}\}[\phi_r(x), \psi_n(y)] = \delta_{rs}\delta(x - y). \tag{67}$$

If on the other hand we had considered variations which anticommuted with the fields we would instead get the anticommutation relations

$$\tfrac{1}{2}\delta(x^0 - y^0)\{(\mathrm{E}\Gamma^0)_{ns} - (\mathrm{E}\Gamma^0)_{sn}\}\{\phi_r(x), \psi_n(y)\} = \delta_{rs}\delta(x - y). \tag{68}$$

The relations (67) would become inconsistent if $\mathrm{E}\Gamma^0$ is antisymmetric; and (68) would become inconsistent if $\mathrm{E}\Gamma^0$ is symmetric. Hence the commuting

field should have $E\Gamma^0$ symmetric; and could then be subjected to Bose statistics. While the anticommuting field should have $E\Gamma^0$ antisymmetric.

The symmetry of the matrix $E\Gamma^0$ is now to be studied. The transformation of "strong reflection" in which the space time co-ordinates one all reversed is a proper Lorentz transformation and can be thought of as a rotation through π around any space axis and a complex pure Lorentz transformation equivalent to a "rotation" through $i\pi$. Hence the transformation of a Dirac field is of the form

$$\psi(x) \rightarrow i\gamma_5\psi(-x). \tag{69}$$

On the other hand, the matrix Γ^μ is given by $\gamma^0\gamma^\mu$ so that

$$E\Gamma^\mu = i\gamma^0\gamma^\mu\gamma_5$$

which is antisymmetric (independent of the representation of the Dirac matrices). It follows that a Dirac field ought to be quantized according to Fermi statistics. For a Klein-Gorden field the matrix is again simply given because it leaves scalars (and second rank tensors unchanged but changes the sign of vectors. On the other hand the matrix connects the scalar with the vector and is antisymmetric. Hence $E\Gamma^\mu$ connects scalars with vectors and is symmetric, so that a spin 0 field should obey Bose statistics.

More generally, since all finite dimensional representations of the Lorentz group can be obtained as totally symmetric multispinors it is possible to show that $E\Gamma^\mu$ is symmetric for tensor representations and antisymmetric for spinor representations. Hence the basic commutation relations (67) and (68) assert that tensor fields should be quantized according to Bose statistics and spinor fields according to Fermion statistics. This is the fundamental theorem on the relation between spin and statistics.[10] As deduced here it is purely a consequence of the Action Principle, and the S-principle.

Since no reference was made to the sign of the energy density or the charge density these considerations apply equally well to our formulation of quantization or to the more conventional method quantization.

VI. Reduction of the S-Matrix

To be able to treat the interaction to higher approximations it is necessary to develop a systematic perturbation theory and to provide a scheme for the reduction of the S-matrix. A heuristic method of developing this is to proceed to the interaction picture and consider the expression for the S-matrix:

$$S = T\{\exp. i \int W(x)\, d^4x\} \tag{71}$$

where W (x) is the interaction in the interaction picture. If we rewrite this expression in a normal ordered fashion the coefficients of the various terms are the (unrenormalized) expressions for the S-matrix elements for the various processes. This reduction is facilitated by the use of Wick's theorem and the use of the contraction functions. But before doing that we recall that there is still some freedom in the definition of the asymptotic "*in*" fields in terms of which the particles are introduced into the field theory. We make the choice[11]

$$\phi_{in}(x) = \phi_0(x) \quad \frac{1}{4}\int \triangle^{(1)}(x \quad y)\ \xi(y) d^4y$$

$$\psi_{in}(x) = \psi_0(x) - \frac{1}{4}\int S^{(1)}(x-y)\ \eta(y)\, d^4y$$

where ϕ_0 and ψ_0 are (the annihilation part of) the Boson and Fermion fields and $\xi(y)$, $\eta(y)$ the sources of the Heisenberg fields.

$$(\Box^2 + m^2)\,\phi(x) = \xi(x)$$

$$(i\gamma^\mu \partial_\mu - \mathrm{M})\,\psi(x) = \eta(x).$$

This choice of the asymptotic field is equivalent to an apparent change of the interaction. For example, for the Yukawa interaction

$$\mathrm{L}_{int}(x) = g\Psi^\dagger(x)\,\gamma^0\Psi(x)\,\Phi(x)$$

$$\xi(x) = g\Psi^\dagger(x)\,\gamma^0\,\Psi(x)$$

$$\eta(x) = g\Phi(x)\,\Psi(x)$$

we get the effective interaction

$$\mathrm{W}_1(x) = g\Psi^\dagger(x)\,\gamma^0\Psi(x)\,\Phi(x)$$

$$+ \frac{g^2}{4}\Psi^\dagger(x)\,\gamma^0\Psi(x)\int \triangle^{(1)}(x \quad y)\,\Psi^\dagger(y)\,\gamma^0\Psi(y)\,d^4y$$

$$+ \frac{g^2}{4}\Psi^\dagger(x)\,\gamma^0\Phi(x)\int S^{(1)}(x \quad y)\,\Psi(y)\,\Phi(y)\,d^4y.$$

Let us now compute the contraction functions. The Boson fields in the interaction (25) occurs only in the combination

$$\Phi(x) = \phi^\dagger(x) + \phi(x)$$

Then

$$[\Phi(x), \Phi(y)] \quad [\phi^{\dagger}(x), \phi(y)] + [\phi(x), \phi^{\dagger}(y)]$$
$$= i\Delta(x \quad y).$$

the contraction function

$$\tau_0(x, y) = \langle 0 \mid \mathrm{T}\left(\Phi(x)\Phi(y)\right) \mid 0 \rangle$$
$$= \frac{i}{2}\epsilon(x^0 \quad y^0)\,\Delta(x \quad v)$$
$$\frac{i}{(2\pi)^4}\,\mathrm{P}\int \frac{e^{ik(x-y)}}{k^2 - m^2}\,d^4k \tag{74}$$

coincides with the half retarded-half advanced Green's function. In contrast, in the standard formalism the contraction function is equal to the causal Green's function. Similarly for the Fermion field we have the contraction function

$$\sigma_0(x, y) = \langle 0 \mid \mathrm{T}\left(\Psi(x)\Psi(y)\right) \mid 0$$
$$= \tfrac{1}{2}\epsilon(x^0 - y^0)\,\mathrm{S}(x - y) \tag{75}$$

which is again the time symmetric Green's function.

We have already seen that the freedom in the choice of the asymptotic field, entails a freedom in the choice of the Green's function. In particular the choice of the asymptotic fields ϕ_{in}, ψ_{in} according to (72) yields effective contraction functions

$$\tau(x, y) = \tau_0(x, y) + \frac{i}{2}\Delta^{(1)}(x \quad y) = \quad {}_{\mathrm{c}}(x - y)$$
$$\sigma(x, y) \quad \sigma_0(x, y \quad \frac{i}{2}\mathrm{S}^{(1)}(x \quad y) \quad \mathrm{S_c}(x \quad v \tag{76}$$

and the effective interaction

$$\mathrm{W}(x) = g\Psi^{\dagger}(x)\gamma^0\Psi(x)\,\Phi(x). \tag{77}$$

This may be verified by direct calculation for the second and fourth order matrix elements. A combinatorial argument can be constructed to show the validity of this effective contraction functions to all orders in perturbation theory.

We have thus recovered, in the present theory, the (unrenormalized) expansion of the scattering amplitude as a power series in the coupling

constant.[11, 12] The result so obtained contains all the familiar infinities of perturbation theory and is therefore without precise mathematical content. But the heuristic method of renormalization of the perturbation expansion developed within the standard formalism can be transplanted into the present formalism to provide a renormalized perturbation expansion in which each term is finite.

The renormalized perturbation series yields an amplitude which is unitary to the order of approximation desired. But the essential point is that *the unitarity relation is true when only the positive energy particles are included in the intermediate states.* This remarkable result is consistent with our assertion that only positive energy particles are physical.

VII. Discrete Transformations and the TCP Theorem

The formulation of quantum field theory given above is invariant under relativistic transformations belonging to the proper orthochronous group. We now wish to consider the discrete transformations of space inversion (P), time inversion (T) and charge conjugation (C) and the invariance of the theory under such transformations.

Space inversion.—The space inversion transformation should be viewed as a purely geometric transformation in which momenta and co-ordinates change sign but energies and angular momenta remain the same. For the scalar field creation operators this implies

$$a(\omega, k) \rightarrow a(\omega, -k)$$
$$a(-\omega, k) \rightarrow a(-\omega, -k)$$

independent of whether the corresponding particles are self-conjugate or not. This implies, for the scalar field

$$\phi(x, t) \rightarrow \phi(-x, t).$$

For a pseudoscalar field there is an additional phase factor of -1. To include this also, we may write this more generally in the form

$$\phi(x, t) \rightarrow \eta_p \phi(-x, t). \quad \eta_p = \pm 1. \tag{78}$$

In the case of a vector field there is the additional geometric feature distinguishing the space and time components of the four-vector.

$$A^0(x, t) \rightarrow \eta_p A^0(-x, t)$$
$$A(x, t) \rightarrow -\eta_p A(-x, t) \tag{79}$$

Finally for a spinor

$$\psi(x, t) \to \eta_p \gamma^0 \psi(-x, t). \tag{80}$$

To complete the definition we must exhibit a unitary operator U (P) which implements these transformations on the field operators and leaves the vacuum unchanged so that

$$U(P) \mid 0 \rangle = \mid 0 \rangle$$

$$U(P)\, \phi(x, t)\, U^{-1}(P) \quad \eta_p \phi \quad x, t), \text{ etc.}$$

A formal construction of such an operator U (P) can be carried out easily we note, for example, that the operator

$$U_1 \quad \exp. \frac{i\pi}{2} \Big(\{a^\dagger(\omega \quad k) \quad \eta_p\, a^\dagger(\omega \quad k)\} \{a(\omega k) - \eta_p\, a(\omega, -k)\}$$

$$\{a^\dagger(\cdot \quad \omega k) - \eta_p\, a^\dagger(\cdot \quad \omega - k)\} \{a(-\omega k) \quad \eta_p\, a \quad \omega - k)\} \Big)$$

has the property

$$U_1 a(\pm \omega, k)\, U_1^{-1} = \eta_p\, a(\pm \omega \quad k)$$

$$U_1 \mid 0 \rangle = \mid 0 \rangle.$$

A product of such factors would serve to define U (P). Since the behaviour of the fields under space inversion are the same as in the standard formulation, we shall not enter into a detailed discussion of the space inversion invariance of various interactions.

Particle conjugation.—The operation of charge conjugation implies the replacement of particle creation (annihilation) operators by antiparticle creation (annihilation) operators. When the particles are self-conjugate (*i.e.*, the antiparticles are the same as the particles) the charge conjugation transformation reduces to a real phase factor $\eta_c = \pm 1$. For a general field we have

$$a_r(\pm \omega, k) \rightleftarrows b_r(\pm \omega, k)$$

for the non-self conjugate case; and

$$a_r(\pm \omega, k) \to \eta_c a_r(\pm \omega, k)$$

or the self-conjugate case. No loss of generality is entailed by not including a phase factor in the non-self conjugate case. We should now construct a

unitary charge conjugation operator U (C) which would implement transformations unitarily and leave the vacuum invariant:

$$U(C)\, a_r(\pm\omega, k)\, U^{-1}(C) = b_r(\pm\omega, k)$$
$$U(C)\,|0\rangle = |0\rangle.$$

Here again the transformations have the same structure as in the standard formulation.

It is worthwhile to point out the elementary fact that any given non-self-conjugate field can be expressed as a definite linear combination of self-conjugate fields with opposite charge conjugation phases. We write

$$a_1(\pm\omega, k) = \frac{1}{\sqrt{2}}\{a(\pm\omega, k) + b(\pm\omega, k)\}$$

$$a_2(\quad\omega, k) \quad \frac{1}{i\sqrt{2}}\{a(\quad\omega\ \ k) \quad b(\pm\omega, k)\}.$$

Then

$$U(C)\, a_1 U^{-1}(C) = +\, a_1$$
$$U(C)\, a_2 U^{-1}(C) = \quad a_2$$

illustrating the statement.

In terms of the self-conjugate fields we may write

$$U(C)\,\phi(x, t)\, U^{-1}(C) \quad \eta_c \phi(x, t)$$

for boson fields and fermion fields alike. We could construct non-Hermitious fields

$$\phi(x) = \frac{1}{\sqrt{2}}\{\phi^{(1)}(x) + i\phi^{(2)}(x)\}$$

$$\psi(x) = \frac{1}{\sqrt{2}}\{\phi^{(1)}(x) - i\phi^{(2)}(x)\} \tag{84}$$

which would transform according to

$$U(C)\,\phi(x)\, U^{-1}(C) = \psi(x)$$
$$U(C)\,\psi(x)\, U^{-1}(C) = \phi(x).$$

So far the fields ϕ, ψ contain only annihilation operators. Their Hermitian adjoints which contain the creation parts transform according to

$$U(C)\,\phi^{\dagger}(x)\,U^{-1}(C) = \psi^{\dagger}(x)$$
$$U(C)\,\psi^{\dagger}(x)\,U^{-1}(C) = \phi^{\dagger}(x)$$

with

$$\phi^{\dagger}(x) = \frac{1}{\sqrt{2}}\left(\phi^{(1)\dagger}(x) - i\phi^{(2)\dagger}(x)\right)$$

$$\psi^{\dagger}(x) = \frac{1}{\sqrt{2}}\left(\phi^{(1)\dagger}(x) + i\phi^{(2)\dagger}(x)\right).$$

For a Boson field, the crossing symmetric combination which enters the interactions is

$$\Phi(x) = \psi^{\dagger}(x) + \phi(x)$$

while for the Dirac field it is

$$\Phi_r(x) = \phi_r(x) + B_{rs}\psi_s^{\dagger}(x).$$

These are the generalizations to non-self conjugate fields of the self-conjugate expressions used in (59) previously. Now

$$\Phi(x) = \Phi^{(1)}(x) + i\Phi^{(2)}(x)$$
$$\Phi_r(x) = \Phi_r^{(1)}(x) + i\Phi_r^{(2)}(x) \tag{86}$$

where all the fields which appear on the right-hand side are self-conjugate fields. The charge conjugation transformation yields

$$U\Phi(x)\,U^{-1} = \Phi^{\dagger}(x)$$
$$U\Phi_r(x)\,U^{-1} = B_{rs}\,\Phi_s^{\dagger}(x)$$
$$U\Phi^{(n)}(x)\,U^{-1} = (-)^{n-1}\,\Phi^{(n)}(x) \tag{87}$$

Using these expressions we may study the invariance of interactions involving conjugate or non-self-conjugate fields under charge conjugation. As an example, the interaction (25) that we have considered in an earlier section is charge conjugation invariant provided $\phi(x)$ is even under charge conjugation.

Time inversion.—The discrete transformations like time inversion (T) and strong reflection (TCP) are treated as antilinear transformations in the

standard formulations of quantum field theory. The reason for this is that if these transformations are treated purely geometrically two difficulties are encountered. The first one is that under a geometric transformation involving time inversion the energies, being fourth components of the energy-momentum four-vectors, should change sign while the momentum would not. However, the notion of viewing the time evolution of a physical system in reverse would lead us to expect the momenta to reverse while the energies retain their sign. The second difficulty is that for finite-component Boson fields the canonical commutation relations change sign under such a geometric transformation. To avoid these difficulties time reversal transformation is defined as an antiunitary transformation in the standard formulation of quantum field theory. The transformation has the property of interchanging initial and final states, reversing all momenta, but preserving all helicities and energies and interchanging " in " and " out " states. In particular, on a single particle state the behaviour of the time reversal transformation has the action

$$a^{\dagger}(\omega, k)\,|\,0\,\rangle \rightarrow \langle\,0\,|\,a(\omega, \quad k). \tag{88}$$

No generality is lost in not including a phase factor.

In the present formalism none of these difficulties are encountered. The energy spectrum contains both positive and negative energies and the geometric transformations do not introduce any contradiction with the energy spectrum. In view of the nature of the commutation relations

$$[a(\pm\,\omega, k), a^{\dagger}(\pm\,\omega', k')] = 2\omega\delta(k \quad k')$$

which change sign with the sign of the energy, the geometric transformation

$$\phi(x, t) \rightarrow \phi(x, -\,t)$$

for the Boson field preserves the commutation relations. As long as we deal with self-conjugate fields it appears that the geometric transformations .

$$\phi(x, t) \rightarrow \eta\phi(x, \quad t)$$

$$x, t) \rightarrow i\eta\gamma^{0}\gamma_{5}\,\psi(x, -\,t)$$

$$\mathrm{A}^{0}(x, t) \rightarrow \quad \eta\mathrm{A}^{0}(x, -\,t)$$

$$\mathrm{A}(x, t) \rightarrow \eta\mathrm{A}(x,$$

or something fairly close to it can be defined to represent time inversion.

Closer study shows that the transformation so defined with $\eta = \pm 1$ are applicable to self-conjugate fields which are even and odd under charge conjugation. In other words, the transformation that we have defined is not T but TC. This can be seen by considering the transformation (89) for a non-self-conjugate field, say the scalar field as transcribed in terms of the annihilation and creation operators:

$$\begin{aligned} a(\pm\omega, k) &\rightarrow a(\mp\omega, k) \\ a^\dagger(\pm\omega, k) &\rightarrow a^\dagger(\mp\omega, k). \end{aligned} \tag{91}$$

Hence the transformation of the one particle state is according to:

$$a^\dagger(\omega, k) \mid 0\rangle \rightarrow a^\dagger(-\omega, k) \mid 0\rangle.$$

In other words a positive energy particle in the (initial) state is transformed into a negative energy particle in the (initial) state. By the physical reinterpretation postulate this is equivalent to a positive energy antiparticle in the final state. In other words the net result is as if the transformation is:

$$a^\dagger(\omega k) \mid 0\rangle \rightarrow \langle 0 \mid b(\omega, -k). \tag{92}$$

Comparing this with the behaviour (88) of the time reversal transformation in the standard formulation we see that the transformation is really TC and not T. Since C has already been defined, if we so choose, we can define T by considering

$$\mathrm{T} = (\mathrm{TC}) \,.\, \mathrm{C}. \tag{93}$$

The time reversal transformation so defined is unitary in contrast to the antiunitary nature of the transformations in the standard formulation of field theory.

We note that while a time reversal transformation T can be defined by (93), it is the TC transformation rather than T that has a simple geometric behaviour. For self-conjugate fields the two differ by a real phase factor. This is relevant in view of the geometric nature of TCP transformation and the TCP theorem.[13]

Strong reflection and the TCP theorem.—Let us now consider the geometric operation

$$\begin{aligned} \phi(x) &\rightarrow \phi(-x) \\ \psi(x) &\rightarrow i\gamma_5\psi(-x) \\ \mathrm{A}^\mu(x) &\rightarrow -\mathrm{A}^\mu(-x) \end{aligned} \tag{94}$$

which corresponds to reflection of all the space time axes. It could be seen to be an element of the complex Lorentz group. *Any theory involving finite component fields which is manifestly invariant under the real Lorentz group is also invariant under the complex Lorentz group.*[13] Under such a transformation any manifestly covariant local Lagrangian density transforms according to

$$L(x) \rightarrow L(-x). \tag{95}$$

Hence the Action function is unchanged. This is the TCP Theorem in our formulation of quantum field theory.

To see the physical implications of this geometric transformation we study the special case of the scalar field

$$\phi(x) \rightarrow \phi(-x). \tag{96}$$

We shall take this field to be non-self-conjugate. Then transcribed in terms of annihilation operators we get

$$a(\omega, k) \rightarrow a(-\omega, -k) \tag{97}$$

so that the one-particle states transform according to

$$a^{\dagger}(\omega, k) \mid 0 \rangle \rightarrow a^{\dagger}(-\omega, -k) \mid 0 \rangle. \tag{98}$$

By the physical interpretation postulate we get the negative energy particle (in the initial state) is to be identified with a positive energy antiparticle with opposite momentum (in the final state). The net result is then as if we defined the transformation as

$$a^{\dagger}(\omega, k) \mid 0 \rangle \rightarrow \langle 0 \mid b(\omega, k) \tag{99}$$

which is the conventional TCP operation.

Just as in the case of the time reversal operation, the TCP operation is represented by a unitary operator within the present version of quantum field theory.

VIII. Concluding Remarks

In the preceding sections we have presented a reformulation of quantum theory of fields which has several simplifying features as compared with the standard formulation. The basic idea is to use Fock's method of associating the entire field with the wave functions of one-particle states with both

positive and negative energy states. The conjugate field is now a field containing only creation operators with both positive and negative energy states. The theoretical framework so obtained is in accordance with canonical commutation relations and the Action principle.

The negative energy particles that are described by the field demand a physical interpretation as positive energy antiparticles travelling in the opposite direction. This interpretation is suggested by the study of the physics of classical negative energy particles. The physical reinterpretation postulate is of fundamental importance in the present version of quantum field theory.

The present theory can represent time inversion and TCP as linear (unitary) operators. The physical implications of the invariance under the various discrete transformations as far the transition amplitudes are concerned is the same as in the usual formulation.

It may be a small advantage, but the present formulism can also be employed for the quantum field theory of faster-than-light particles.

By studying a simple Yukawa interaction we have endeavoured to show how a covariant perturbation theory can be constructed. The perturbation expansion so obtained contains divergent quantities and has to be renormalized, or regularized using an indefinite metric to obtain finite and meaningful answers.[14]

The usual relation between spin and statistics can be derived in this version of quantum theory of fields.

Because of the negative energies occurring in this theory the spectral postulate of axiomatic field theory[15] is violated. As a consequence the standard results of axiomatic field theory no longer hold. As an example we see that local fields annihilating the vacuum can be constructed. Despite this the essential result of TCP invariance can be deduced within the present framework. The theory has local fields satisfying local commutation relations.

The point of view that only the positive energy particles need be considered physical receives mathematical confirmation from the unitarity relation satisfied by the perturbation theory scattering amplitude. In the unitarity sum-over-intermediate-states only those states containing exclusively positive energy states need be included.

Since the perturbation theory amplitude in the present theory is identical with that of the usual theory, dispersion relations of the usual kind are satisfied by the scattering amplitude. In other words while the Wightman functions in the present theory do not have the " future-tube " analyticity,[16] the scattering amplitudes may be expected to have the usual analytic properties.

IX. References

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