On Kernels Realizing Associative Multiplications between Phase-Space Functions.

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Summary. — An analysis is shown of integral kernels realizing multiplication between phase-space functions, which are in a correspondence with quantum-mechanical variables of a nonrelativistic system. Kernels satisfying the associative property are found as solution of a functional equation.

1. – Recent works [1,2] have studied some general features of the possible correspondences between spaces of c-number functions and classes of linear operators on Hilbert spaces, as realizations of the quantum-mechanical scheme.

We consider the Hilbert space of the irreducible representation of the algebra A of the creation and annihilation operators for one mode and operators Q that can be written formally as

$$Q = \sum_{n,m} C^{(K)}_{nm}\{a_n a^{\dagger m}\}_{K(a)} \tag{1}$$

where $\{a^n a^{\dagger m}\}_{K(a)}$ denotes, in general, a sum of terms homogeneous in each variable, which is obtained by combining n a's and m $a^\dagger$'s in a well-prescribed order $K(a)$. The c-number function corresponding to Q is

$$f^Q_K(\xi, \xi^*) = \sum_{n,m} C^{(K)}_{nm}\xi^n \xi^{*m} \qquad \xi \in C \tag{2}$$

(1) G. S. Agarwal and E. Wolf: *Phys. Rev. D*, **2**, 2161, 2187, 2206 (1970).
(2) K. E. Cahill and R. J. Glauber: *Phys. Rev.*, **177**, 1857, 1882 (1969).

Among the various rules of association we single out arbitrarily Weyl's rule [3]. Then to a Hilbert-Schmidt operator on $L_2(R)$ there corresponds one and only one function $f^Q_W \in L_2(R\times R)$ and *vice versa* [4]. By the action of suitable operators on f^Q_W it is possible to obtain the images of Q under other rules of association, *i.e.* we will write formally

$$f^Q_K(\xi, \xi^*) = \int K(\xi, \xi^*, \eta, \eta^*) f^Q_W(\eta, \eta^*)\, d^2\eta \tag{3}$$

$K(\xi, \xi^*, \eta, \eta^*)$ is a kernel that takes into account only the structure of A in the passage from one ordering to another, so that it depends only on these. In the following we shall use the simpler notation

$$K(\xi, \xi^*, \eta, \eta^*) \equiv K(\xi, \eta)$$

First, some considerations on the structure of the kernels K of (3)

a) Physics requiring only 1-1 rules of association, it follows that K has an inverse.

b) Q can be written also in terms of other operators b and $b^\dagger$, that constitute another representation of A, unitarily equivalent to the one used in (1), *i.e.*

$$\begin{aligned} b &= a + \gamma\,, \qquad\qquad \gamma \in C\,,\\ b^\dagger &= a^\dagger + \gamma^*\,,\\ Q &= \sum C'^{(K)}_{mn}\{b^n b^{\dagger m}\}_{K(b)} \end{aligned}$$

and

$$'(\xi, \xi^*) = \sum C'_{nm}\xi^n \xi^{*m}$$

We restrict the analysis to those orderings satisfying the following condition

$$\{b^n b^{\dagger m}\}_{K(b)} = \{(a+\gamma)^n(a^\dagger+\gamma^*)^m\}_{K(b)} = \sum \binom{n}{t}\binom{m}{s}\gamma^{n-t}\gamma^{*\,m-s}\{a^t a^{\dagger s}\}_{K(a)}\,. \tag{4}$$

All the orderings so far proposed belong to such class [5].

[3] E. WIGNER: *Phys. Rev.*, **40**, 749 (1932); J. C. T. POOL: *Journ. Math. Phys.*, **7**, 66 (1966).

[4] In agreement with ref. [2], $f(R\times R)$ will be written as $f(c)$ with the condition

$$f(\xi, \xi^\dagger)^2\, d^2\xi < \infty, \text{ where } d^2\xi \equiv dI\xi\, dR\xi.$$

[5] An example of ordering which does not satisfy (4) is the following:

$$\{a^n a^{\dagger m}\}_K = \begin{cases} a^n a^{\dagger m}\,, & \text{if } n+m \text{ even}\,,\\ a^{\dagger m} a^n & \text{if } n+m \text{ odd} \end{cases}$$

The relation between f and f' is such that under the transformation from a and $a^\dagger$ to b and $b^\dagger$, $f(\xi)$ goes into $f'(\xi+\gamma)$. Since the kernel $K(\xi, \eta)$ depends only on the orderings, it connects the images of Q under the Weyl and K ordering, irrespective of the basic operators a, $a^\dagger$ and b, $b^\dagger$ in terms of which Q can be written. As a consequence of this, K has to be translationally invariant,

$$K(\xi, \eta) = K(\xi - \eta, \xi^* - \eta^*) \tag{5}$$

c) Another transformation on a and $a^\dagger$ that ought to leave the structure constants of A unchanged is

$$a \to \gamma a\,, \qquad a^\dagger \to \gamma^* a^\dagger\,, \qquad |\gamma| = 1\,,$$

and following *b*) we conclude that

$$K = K[(\xi - \eta)(\xi^* - \eta^*)] = K(|\xi - \eta|^2)$$

d) Under any ordering, to the operator a^n must correspond the function ξ^n. This implies that

$$\eta^n = \int K(|\xi - \eta|^2)\, \xi^n\, d^2\xi$$

and it is immediately seen that a necessary and sufficient condition is

$$\int K(|\xi|^2)\, \xi^n\, d^2\xi = 0\,, \qquad n > 0$$
$$\int K(|\xi|^2)\, d^2\xi = 1$$

We next start studying the structure of another kernel $L_K(\alpha, \beta, \gamma)$, $\alpha, \beta, \gamma \in C$, which is required to originate, in the function space for the K-ordering, the operation corresponding to the usual multiplication of two linear operators on the Hilbert space. For a given Q_1 and Q_2, we have

$$f_K^{Q_1 Q_2}(\gamma) = \int f_K^{Q_1}(\alpha)\, f_K^{Q_2}(\beta)\, L_K(\alpha, \beta, \gamma)\, d^2\alpha\, d^2\beta \tag{6}$$

The kernel L_W for the Weyl ordering is known to be

$$L_W(\alpha, \beta, \gamma) = \exp\left[2[(\alpha - \gamma)(\beta - \gamma)^* - (\alpha - \gamma)^*(\beta - \gamma)]\right] \tag{7}$$

Rewriting (6) in the Weyl-ordering language and then returning to the K-ordering, it is easy to obtain the relation

$$L_K(\alpha, \beta, \gamma) = \int K^{-1}(|\alpha - \alpha'|^2)\, K^{-1}(|\beta - \beta'|^2)\, K(|\gamma - \gamma'|^2)\, L_W(\alpha', \beta', \gamma')\, d^2\alpha'\, d^2\beta'\, d^2\gamma'\,, \tag{8}$$

and due to (5) and (7

$$L_K(\alpha, \beta, \gamma) \quad L_K(\alpha \quad \gamma, \beta-\gamma) \tag{9}$$

We note that the expression (8) for L_K ensures that it satisfies the associativ property since L_W does.

2. – By this way we are led to consider the inverse problem, *i.e.* to find out which are the kernels that are of the form (9) and define an associative product. An analogous problem has been stated and solved in ref. [6]. The associative property determines a functional equation for $L(\alpha-\gamma, \beta-\gamma)$. Given three arbitrary functions f_1, f_2, f_3, we have

$$(f_1 \times f_2)\times f_3(\alpha) - f_1 \times (f_2 \times f_3)(\alpha) \quad \int f_1(\gamma) f_2(\varepsilon) f_3(\eta)$$
$$\cdot[L(\delta-\gamma, \delta-\varepsilon) L(\alpha-\delta, \alpha-\eta) - L(\delta-\varepsilon. \delta- \quad \eta) L(\alpha \quad \gamma, \alpha \quad \delta)]\, d^2\gamma\, d^2\delta\, d^2\varepsilon\, d^2\eta$$

Due to the arbitrariness of the functions involved, we have

$$\int d^2\delta[L(\alpha-\gamma, \alpha-\delta) L(\delta \quad \varepsilon. \delta \quad \eta) \quad L(\alpha-\delta, \alpha-\eta) L(\delta \quad \gamma, \delta \quad \varepsilon)] \quad 0 \tag{10}$$

Introducing the Fourier transform of L,

$$W(x, y) = \pi^{-2}\int L(\alpha, \beta) \exp[x\alpha^* \quad x^*\alpha \quad y\beta^* - y^*\beta]\, d^2\alpha\, d^2\beta$$

we can transform (10) into an equation for W which reads as follows

$$W(x+y, z) W(x, y) = W(x, y+z) W(y, z \tag{11}$$

In the following we will search for the solutions of (11). We propose to use elementary techniques, and, hence, restrict our attention to a class of functions. We find a general solution, which may be valid more generally than within the restricted class.

Writing $x = x_1 + ix_2$ (and analogously for y and z) and $W(x, y) \equiv W(x_1 x_2 y_1 y_2)$, we differentiate (11) with respect to z_1 and z_2, setting $x_1 = x_2 = y_1 = y_2 = 0$, and obtain respectively

$$\frac{\partial}{\partial z_1}[w(0, 0, 0, 0) w(0, 0, z_1, z_2) \quad w(0, 0, z_1, z_2)^2] \equiv 0, \tag{12}$$

$$\frac{\partial}{\partial z_2}[w(0, 0, 0, 0) w(0, 0, z_1, z_2) \quad w(0, 0, z_1, z_2)^2] \equiv 0 \tag{12}$$

[6] M. S. Shriram and T. S. Shankara: Bangalore preprint

From where we conclude that

$$w(0, 0, z_1, z_2) \quad w(0, 0, 0, 0) \quad C$$

Let us now consider two alternatives depending on whether C is nonzero:

a) $C \neq 0$. We assume that all the first-order partial derivatives of w exist. We will use a notation where each index j attached to a function denotes the first-order partial derivative of the latter with respect to the j-th variable and $w_{ij} = (w_i)_j$. Differentiating (11) with respect to x_1 and setting $x_1 = x_2 = 0$, we have

$$\frac{w_1(y_1 y_2 z_1 z_2)}{w(y_1 y_2 z_1 z_2)} = \frac{f(y_1 + z_1, y_2 + z_2) \quad f(y_1 y_2)}{C} \tag{13}$$

with $f(y_1 y_2) = w(0, 0, y_1, y_2)$ and, after integration,

$$w(y_1 y_2 z_1 z_2) \quad \exp\left[\varphi(y_2, z_1, z_2) + \int_0^{y_1} \frac{f(t + z_1, y_2 + z_2) \quad f(t, y_2)}{C}\, dt\right]$$

with $\varphi(y_2, z_1, z_2)$ an arbitrary function.

Through the same procedure, differentiating (11) with respect to x_2, setting $x_1 = x_2 = y_1 = 0$ and $g = w_2(00z_1z_2)$, and comparing the new solution with (14) we have

$$w(y_1 y_2 z_1 z_2) = \tag{15}$$

$$= C \exp\left[\frac{1}{C}\left(\int_0^{y_1} [f(t + z_1, y_2 + z_2 \quad f(t, y_2)]\, dt + \int_0^{y_2} [g(z_1, t + z_2) \quad g(0, t)]\, dt\right)\right],$$

which can be written as

$$w(y_1 y_2 z_1 z_2) = K\Omega^0(y_1 y_2)\, \Omega^0(z_1 z_2)\, \Omega^{0^{-1}}(y_1 + z_1, y_2 \quad z_2)$$

$$\exp\left[\chi(z_1, y_2 + z_2) - \chi(0, y_2) \quad \chi(z_1 z_2)\right]$$

with

$$\frac{\partial}{\partial x} F(x, y) = f(x, y), \qquad \frac{\partial}{\partial y} G(x, y) = g(x, y), \qquad \chi(x, y) \equiv G(x, y) \quad F(x, y)$$

$$\Omega^0(x, y) = \exp[-F(x, y)], \qquad K = \exp[G(0, 0)]$$

We remark now that products and quotients of solutions of (11) are again solutions of the same equation and that

$$\Omega^0(y_1 y_2)\, \Omega^0(z_1 z_2)\, \Omega^{0^{-1}}(y_1 + z_1, y_2 \quad z_2)$$

is a solution, and conclude that

$$\exp[\chi(z_1, y_2 + z_2) \quad \chi(0_1 y_2) \quad \chi(z_1 z_2)] \quad w^0(y_2, z_1, z_2$$

is a solution of (11).

It remains to specify the structure of χ, for which we derive from (11) the following functional equation:

$$\chi(z_1, x_2 + y_2 + z_2) \quad \chi(0, x_2 \quad y_2) + \chi(y_1, x_2 + y_2) \quad \chi(y_1, y_2) =$$
$$\chi(y_1 + z_1, x_2 + y_2 + z_2) \quad \chi(y_1 \quad z_1, y_2 + z_2) + \chi(z_1, y_2 \quad z_2) \quad \chi(0, y_2)$$

and with the same technique as before one obtains

$$\chi_{11}(y_1, x_2 + y_2 + z_2) \quad \chi_{11}(y_1, y_2 \quad z_2),$$

the integral of which is

$$\chi(x, y) = m(x) \quad h(y)x + n(y)$$

with $m(x)$, $h(x)$, $n(x)$ arbitrary functions.

Using in (17) this expression for χ and taking the derivative of (17) with respect to z_2 we find

$$h_1(y_2) \quad h_1(x_2 \quad y_2)$$

so that

$$h(y) = By \quad D$$

with B, D constants. Hence χ becomes

$$\chi(x, y) \quad m(x) \quad Bxy + n(y) + Dx$$

and from (16)

$$w^0(y_2, z_1, z_2) = \exp[-m(0)] \exp[-n(y_2)] \exp[-n(z_2)] \exp[n(y_2 + z_2)] \cdot$$
$$\cdot \exp\left[\frac{B}{2}(z_1 y_2 - z_2 y_1)\right] \exp\left[-\frac{B}{2} y_1 y_2\right] \exp\left[\quad \frac{B}{2} z_1 z_2\right] \exp\left[\frac{B}{2}(y_1 + z_1)(y_2 \quad z_2)\right]$$

Therefore

$$w(y_1 y_2 z_1 z_2) \quad \Omega(y_1 y_2)\Omega(z_1 z_2)\Omega^{-1}(y_1 + z_1, y_2 + z_2) \exp\left[\frac{B}{2}(z_1 y_2 \quad z_2 y_1)\right],$$

as in ref. [1], and where

$$\Omega(y_1, y_2) = K\Omega^0(y_1 y_2) \exp\left[\quad \frac{B}{2} y_1 y_2\right] \exp[m(0)]$$

b) $C = 0$. Let us consider the eq. (11) for $x_1 = -y_1$, $x_2 = -y_2$ and any z_1, z_2. In such points the l.h.s. vanishes. It is now straightforward to show that requiring W to be analytic in all R^4 implies that it is identically zero. Let us suppose, indeed, that there is a point $P \equiv (\bar{y}_1 \bar{y}_2 \bar{z}_1 \bar{z}_2)$ where W is different from zero. For continuity $W \neq 0$ in some neighbourhood of P. Due to the above considerations W vanishes identically in some neighbourhood of $P^1 = (-\bar{y}_1, -\bar{y}_2, \bar{x}_1 + \bar{z}_1, \bar{x}_2 + \bar{z}_2)$ and for analyticity W is identically zero on all R^4.

We conclude with the statement that the only nontrivial solutions of (11) analytic in all R^4 have the form (19).

3. – After the results of the last Section, the kernel L depends on an arbitrary continuous function Ω and a complex constant $K = i(B/4)$, so that in general we write

$$(20)\qquad (f \times g)_{\Omega,K}(z) = \int \hat{f}(\alpha)\,\hat{g}(\beta)\,\Omega(\alpha)\,\Omega(\beta)\,\Omega^{-1}(\alpha \quad \beta) \exp[K(\alpha^*\beta - \alpha\beta^*) \quad z(\alpha^* + \beta^*) \quad z^*(\alpha + \beta)]\,\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,,$$

where

$$\hat{f}(\alpha) \quad \frac{1}{\pi}\int f(\alpha')\exp[\alpha\alpha'^* \quad \alpha^*\alpha']\,\mathrm{d}^2\alpha'$$

$L(\alpha, \beta)$ can be seen as the product of two factors $\Omega(\alpha)\,\Omega(\beta)\,\Omega^{-1}(\alpha + \beta)$ and an exponential depending on K; the noncommutativity has its origin in this second factor.

In what follows we shall consider only the case of K nonzero and real though formally K could be complex. We can then look into the structure of L in a clearer way, and note as before that $(f \times g)_{1,K}(z)$ can be written also as follows:

$$(f \times g)_{1,K}(z) \quad \frac{1}{K^2}\int f(\alpha)g(\beta)\exp\left[\frac{1}{K}[(z-\alpha)(z \quad \beta)^* - (z \quad \alpha)^*(z \quad \beta)]\right]\mathrm{d}^2\alpha\,\mathrm{d}^2\beta$$

We make now a linear transformation on the functions $f(\alpha)$, in order that the transformed functions $f'(\alpha)$ satisfy the following relation:

$$(21)\qquad (f \times g)'_{\Omega,K}(z) \quad (f' \times g')_{1,K/|K|}(z)$$

Such a transformation has to be 1-to-1 and invertible and we will see that starting with any Ω and real K it exists in general, so that only the products of the r.h.s. of (21) need to be considered. We operate on f in two successive steps. We define at first

$$\omega : f(\alpha) \to \widetilde{(\hat{f}\Omega)}(\alpha)$$

where

$$\tilde{f}(\alpha) = \frac{1}{\pi}\int f(y) \exp[K(\alpha y^* - \alpha^* y)]\, d^2y .$$

The following relation holds

$$\omega(f \times g)_{\Omega,K} = [(\omega f \times \omega g)_{1,1/K}](z)$$

Next we define

$$\sigma_{a,b} : f(x) \to a f(bx)$$

with a and b nonzero real numbers depending on K, which can be adjusted in order that

$$\sigma(f \times g)_{1,1/K}(z) = [\sigma f \times \sigma g)]_{1,K/|K|}(z)$$

and it is found that

$$a \quad K , \qquad b = \sqrt{|K|} .$$

For $K/|K| = 1$ the kernel generates quantum mechanics with the usual sign of the commutators. (For a proper choice of a, b the kernel (7) is obtained.) The choice $K/|K| = -1$, on the contrary, generates quantum mechanics with an indefinite metric.

For the Lie algebraic approach to dynamics, we can also conclude that the most general Lie product that derives from an associative kernel is

$$(f \times g)(z) - (g \times f)(z) \equiv (f \times g)_{\text{Lie}}(z) =$$
$$= 2\int f(\alpha) g(\beta)\, \Omega(\alpha)\, \Omega(\beta)\, \Omega^{-1}(\alpha + \beta) \sinh[K(\alpha^*\beta - \alpha\beta^*)] \cdot \exp[z(\alpha + \beta)^* - z^*(\alpha + \beta)]\, d^2\alpha\, d^2\beta$$

with the same conditions on Ω and K as stated at the beginning of this Section. For K real and $\Omega = 1$, we obtain the Moyal kernel, which was shown in ref. (7) by direct calculations to generate a Lie product. We remark that not all the Lie products can be derived from an associative product, as, for instance, the familiar Poisson bracket of the classical dynamics shows. (In this case we can, however, go to a limiting form of $(f \times g)_{\text{Lie}}(z)$ as $K \to 0$ to obtain the classical Poisson bracket.)

(7) T. F. Jordan and E. C. G. Sudarshan: *Rev. Mod. Phys.*, **33**, 515 (1961).

We mention that a similar result is contained in ref. ([8]) where a comprehensive study of the algebras closed under twisted convolutions is made.

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([8]) C. M. EDWARDS and J. T. LEWIS: *Comm. Math. Phys.*, **13**, 131 (1969). We are grateful to Prof. G. F. DELL'ANTONIO for having drawn our attention to this paper.

● RIASSUNTO

Si compie un'analisi dei noccioli integrali che realizzano prodotti fra funzioni dello spazio delle fasi, le quali siano in corrispondenza con variabili quanto-meccaniche di un sistema non relativistico. I noccioli che soddisfano la proprietà associativa vengono trovati come soluzioni di un'equazione funzionale.

О ядрах, реализующих ассоциативные умножения между функциями фазового пространства.

Резюме (*). — Проводится анализ интегральных ядер, которые реализуют умножения между функциями фазового пространства, которые соответствуют квантовомеханическим переменным нерелятивистской системы. Ядра, удовлетворяющие ассоциативному свойству, получаются, как решение функционального уравнения.

(*) *Переведено редакцией.*