IL NUOVO CIMENTO Vol. 12 A, N. 3 1 Dicembre 1972

Analyticity in Quantum Field Theory (*).

I. - The Triangle Graph Revisited.

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(ricevuto il 19 Maggio 1972)

Summary. — The leading singularity is studied in scalar triangle graphs. We observe that in some examples of practical interest like the « form factors » the anomalous singularity does not arise when we perform the momentum integrations in the Feynman-Stückelberg amplitudes, indicating that quantum field theory will not necessarily yield the so-called analytic scattering amplitudes. However, the analytic triangle form factor which shows the anomalous singularity and the corresponding Feynman-Stückelberg amplitude coincide everywhere above the normal thresholds, especially in the « physical region » of the variables. But we should be warned against the uncritical use of conventional methods and concepts, even in simple perturbation structures.

1. – Introduction: field-theoretical graphs and Landau equations.

In recent years one has dealt in considerable detail with the question of the analytic properties of scattering amplitudes. One began by investigating these quantities in the perturbation approximation of quantum field theory which has been developed into the present form especially by Stückelberg and Feynman [1]. They write the terms of the perturbation series in the form

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

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[1] E. C. G. Stückelberg: *Helv. Phys. Acta*, **15**, 23 (1942); D. Rivier and E. C. G. Stückelberg: *Helv. Phys. Acta*, **22**, 215 (1949); R. P. Feynman: *Phys. Rev.*, **76**, 749 (1949).

of $4l$-fold integrals like

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^4 k_1 \ldots \int_{-\infty}^{+\infty} \mathrm{d}^4 k_l \frac{1}{q_1^2 - m_1^2 + i\varepsilon} \cdots \frac{1}{q_N^2 - m_N^2 + i\varepsilon}, \tag{1}$$

multiplied by suitable factors including the coupling constants or coupling matrices when spin and internal symmetries are to be included. The integration paths go over *real* values of the four-momenta $k_1, \ldots, k_l$ which are called the « loop » momenta. In addition, the internal momenta $q_1, \ldots, q_N$ are real if we are only interested in the physical region of the process; this follows from the fact that the momenta of all the in- and outgoing particles are real. The Stückelberg-Feynman integral (1) describes the contribution to a scattering amplitude due to the exchange of a well-defined system of intermediate states. As an example, let us consider the decay of a particle 1 into two particles 2 and 3 (2). This process is physical if the masses of the particles satisfy

$$M_1 > M_2 + M_3 . \tag{2}$$

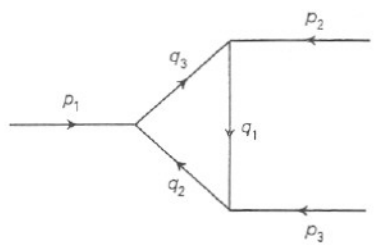

Fig. 1. – Triangle graph: $\pm p_i$ momenta of the in- and outgoing particle with masses M_i; $(\pm) q_i$ momenta of the exchanged (anti)particles with masses m_i.

A possible field-theoretic graph contribution to this decay is drawn in Fig. 1. In this structure an interesting situation occurs when

$$p_1^2 = M_1^2 \geqslant p_{1\mathrm{th}}^2 = (m_2 + m_3)^2 , \tag{3}$$

which means that physically also the intermediate particles with masses m_2 and m_3 can and will be created. One says the field-theoretical amplitude, corresponding to Fig. 1, shows a « normal » threshold in the variable p_1^2 if the equal sign in eq. (3) holds.

(2) We forget about the fact that the mass of the decaying particle 1 should have an imaginary part in the conventional theory.

The normal thresholds are not always the only interesting features of the triangle amplitudes. Several authors have observed another peculiar situation occurring under certain circumstances, the «anomalous» thresholds ([3]). The easiest access to this phenomenon is provided by the general approach which LANDAU and others have initiated in 1959 ([4]). Using the Feynman trick we can rewrite the integral (1) as

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^4 k_1 \ldots \int_{-\infty}^{+\infty} \mathrm{d}^4 k_i \int_0^1 \mathrm{d}\alpha_1 \ldots \int_0^1 \mathrm{d}\alpha_N \delta\Big(\sum_{i=1}^N \alpha_i - 1\Big) \Psi^{-N}$$

with

$$\Psi(p, k, \alpha) = \sum_{i=1}^N \alpha_i (q_i^2 - m_i^2) . \tag{4}$$

The additional integration variables α_i $(0 \leqslant \alpha_i \leqslant 1)$ are called «Feynman» parameters. Now LANDAU claimed that the integral (4) is regular everywhere except when the following equations hold:

$$\begin{cases} q_i^2 = m_i^2 & (i = 1, \ldots, r \leqslant N), \\ \alpha_i = 0 & (i = r+1, \ldots, N), \end{cases} \tag{5}$$

with a permutation of the indices i whenever necessary, and

$$\sum_{j\lambda} \alpha_{j_\lambda} q_{j_\lambda} = 0 \qquad (\lambda = 1, \ldots, l), \tag{6}$$

where j_λ includes all the internal momenta in the loop λ. Applying Landau's analysis to the triangle graph we expect:

i) a «leading» or «triangle» singularity if eqs. (5) and (6) can be satisfied with all $a_i \neq 0$ $(i = 1, 2, 3)$,

ii) lower singularities which are due to one of the α_i equal to zero and will be identified with the «normal» thresholds.

To understand properly what «singularity» means in the present context, let us note that the integral (4) is not singular for physical values of the in- and outgoing particles because of the $i\varepsilon$ prescription. Thus the triangle am-

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([3]) Y. NAMBU: *Nuovo Cimento*, **9**, 610 (1958); R. KARPLUS, C. M. SOMMERFIELD and E. H. WICHMANN: *Phys. Rev.*, **111**, 1187 (1958).

([4]) L. D. LANDAU: *Nucl. Phys.*, **13**, 181 (1959); J. D. BJORKEN: Ph. D. Thesis (Stanford, 1959); J. MATHEWS: *Phys. Rev.*, **113**, 381 (1959); N. NAKANISHI: *Progr. Theor. Phys.*, **21**, 135 (1959).

plitude *e.g.* is, for physical and real p_1^2, perfectly regular; all Landau singularities will occur infinitesimally below the real p_1^2-axis. However, one normally identifies the integral $I(p_1^2, \ldots)$ with the function $\hat{I}(s_1, \ldots)$ of the complex variables $s_1, \ldots$, and defines

$$I(p_1^2, \ldots) \equiv \hat{I}(s_1, \ldots) \qquad \text{with} \qquad s_1 = p_1^2 + i\varepsilon\,. \tag{7}$$

(We shall drop the hat in $\hat{I}(s_1, \ldots)$ later on deliberately!) Thus all singularities of the Feynman-Stückelberg amplitudes which are located infinitesimally below the real (and physical) p_1^2-axis now come to lie on the real s_1-axis, and physical values of s_1 have to be taken slightly above the real axis.

In this paper we shall study the singularities of the triangle graph in quantum field theory, and we shall discuss in particular the existence of the triangle thresholds. First, the triangle graph is evaluated by the covariant method of FEYNMAN; an anomalous threshold will come out if certain conditions are satisfied by the parameters of the triangle *and if some assumptions with respect to the path of four-momentum integration hold.* That these assumptions are not trivial we shall explain in Sect. **3**, where we deal explicitly with the example of the form factors; there the « derived » anomalous threshold lies in the unphysical region of the external variables, and we have to continue the field-theoretical integral into that region. If we stick to the *real* loop-momentum Stückelberg-Feynman path, we shall only find the normal thresholds; the anomalous singularity must be obtained from a new path prescription which we identify. Finally we turn to a triangle amplitude which might contribute to two-particle elastic scattering and should show, according to the Landau equations, a triangle threshold in the physical region; the explicit calculation of the field-theoretical integral with the « causal » prescription in fact reproduces that singularity, which in addition will be found in a specific term of the unitarity sum. In the same Sect. **4** also the application and interpretation of a discontinuity rule, given by CUTKOSKY, is studied.

2. – Covariant « evaluation » of the triangle graph.

The triangle graph is the simplest perturbative structure of quantum field theory in which more elaborate thresholds may show up. Unfortunately one cannot calculate it in closed form like the single loop; there the explicit four-dimensional loop momentum integration yields the same result as the well-known procedures in covariant perturbation theory [5]. In this Section we evaluate the scalar triangle graph by the covariant method. If certain con-

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[5] See, *e.g.*, A. M. GLEESON, R. J. MOORE, H. RECHENBERG and E. C. G. SUDARSAN: *Phys. Rev. D*, **4**, 2242 (1971).

ditions concerning the path of integration are satisfied, then the obtained result can be expressed in terms of nonelementary functions as reported in the literature ([6]).

We choose as loop variable k the internal momentum q_1, given in Fig. 1, and write the triangle integral

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^4k[k^2 - m_1^2 + i\varepsilon]^{-1}[(k+p_3)^2 - m_2^2 + i\varepsilon]^{-1}[(k-p_2)^2 - m_3^2 + i\varepsilon]^{-1}. \tag{8}$$

Now we replace the denominators by the parameter integrals ([7])

$$I = 2\int_{-\infty}^{+\infty} \mathrm{d}^4k \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2 \int_0^1 \mathrm{d}\alpha_3\, \delta\Big(\sum_{i=1}^{3} \alpha_i - 1\Big)\, \Phi^{-3}$$

with

$$\Phi = [k + \alpha_2 p_3 - \alpha_3 p_2]^2 + \alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_3 p_1^2 + \alpha_1\alpha_2 p_3^2 - \sum_{i=1}^{3} \alpha_i m_i^2 + i\varepsilon\,. \tag{9}$$

By this procedure we keep in mind the $-i\varepsilon$, added to the mass of the exchanged particles to describe the causal propagation, even if we do not always display it explicitly from now on.

2·1. *Covariant four-momentum integration.* – According to prescriptions of STÜCKELBERG and FEYNMAN the four-dimensional k-integration goes along the real axis of the momentum components k_0, k_1, k_2 and k_3. Now one usually performs a *translation* of the momentum k to k':

$$k' = k + \alpha_2 p_3 - \alpha_3 p_2\,. \tag{10}$$

In addition the time component k_0' is *Wick-rotated*, or

$$k_0'' = -ik_0', \qquad k_i'' = k_i' \qquad (i = 1, 2, 3). \tag{11}$$

([6]) A. C. T. WU: *Kon. Dansk. Vid. Selsk. Mat. Fys. Medd.*, **33**, No. 3 (1961); see also J. S. FREDERIKSEN and W. S. WOOLCOCK: *Nucl. Phys.*, **28** B, 605 (1971); G. KÄLLÉN and A. S. WIGHTMAN: *K. Dansk. Vid. Selsk. Mat. Fys. Skr.*, **1**, No. 6 (1958), succeeded to compute a certain sum of partial derivatives of the scalar triangle graph in closed form; their expression contains all the possible singularities of the analytic triangle amplitude.

([7]) See, *e.g.*, R. P. FEYNMAN: *Phys. Rev.*, **76**, 769 (1949).

With those two transformations, (10) and (11), the denominator Φ finally becomes

$$\Phi = -\sum_{\nu=0}^{3} k_\nu'' + \alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_3 p_2^2 + \alpha_1\alpha_2 p_3^2 - \sum_{i=1}^{3} \alpha_i m_i^2 . \tag{12}$$

The « Euclidean » four-momentum integration over k'' can be done immediately *if also the transformed paths of integration are running along the real axis of every component.* This condition is satisfied provided i) the momenta of the in- and outgoing particles are physical, and ii) in the rotation of the k_0'-path by 90° we do not cross any poles in the energy plane. In the « physical region » the latter assumption is not a problem since the energy poles of the Feynman-Stückelberg integral lie below the positive real $k_0(k_0')$-axis and above the negative $k_0(k_0')$-axis. The situation might change drastically in unphysical regions, and we will return to this situation in the next Section. For the moment we take the conditions as guaranteed and obtain

$$I = i\pi^2 \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2 \int_0^1 \mathrm{d}\alpha_3 \delta\Big(\sum_{i=1}^{3} \alpha_i - 1\Big)\Big[\alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_3 p_2^2 + \alpha_1\alpha_2 p_3^2 - \sum_{i=1}^{3} \alpha_i m_i^2\Big]^{-1}. \tag{13}$$

2·2. *The single-integral representation.* – The trivial integration over the third variable is easily carried out by replacing the integration variables

$$\alpha_1 = 1 - x , \qquad \alpha_2 = x - y , \qquad \alpha_3 = y . \tag{14}$$

The triangle integral becomes

$$I = i\pi^2 \int_0^1 \mathrm{d}x I'(x)$$

with

$$I'(x) = \int_0^x \frac{\mathrm{d}y}{D} . \tag{15}$$

Before proceeding with the y-integration we take a closer look at the denominator D which is a quadratic function of y:

$$D = Ay^2 + By + C$$

with

$$(16)\qquad \begin{cases} A = -s_1\,, \\ B = xs_1 + (1-x)(s_2 - s_3) + m_2^2 - m_3^2\,, \\ C = (1-x)(xs_3 - m_1^2) - xm_2^2\,. \end{cases}$$

(Here $s_i = p_i^2 + i\varepsilon$.)

The denominator D has, in general, two real or complex roots y_+ and y_- (case i)); for special values of the external variables s_i and the internal parameters m_i the two roots may coincide in a real double root y_0 (case ii)); finally, one of the external momentum squared, say s_1, may be zero, and D becomes only a linear function of y yielding a single root (case iii)).

The y-integration in the simplest case iii) gives

$$(17)\qquad I_3'(x) = [(1-x)(s_2 - s_3) + m_2^2 - m_3^2]^{-1} \log\left\{\frac{x(1-x)s_2 - (1-x)m_1^2 - xm_3^2}{x(1-x)s_3 - (1-x)m_1^2 - xm_2^2}\right\}.$$

We shall not carry out the x-integration explicitly; the triangle amplitude, even in case iii), looks rather messy containing dilogarithms due to the terms

$$(18)\qquad \int_0^1 \mathrm{d}x \frac{1}{x+a} \log(x+b) = \int_a^{1+a} \frac{\mathrm{d}x'}{x'} \log[x' + (b-a)] = \\ = \left[\log(b-a)\log x' - \mathscr{L}_2\left(-\frac{x'}{b-a}\right)\right]_{x'=a}^{1+a}.$$

In the most general case i), *i.e.* with two roots, we obtain after the y-integration [8]

$$I_1'(x) = \frac{1}{\sqrt{\Delta}} \log\left\{\frac{Bx + 2C + x\sqrt{\Delta}}{Bx + 2C - x\sqrt{\Delta}}\right\}$$

with

$$(19)\qquad \Delta = B^2 - 4AC\,.$$

The special case of two coinciding roots for the denominator D may be derived from the solution eq. (19) by taking the limit $\Delta \to 0$, and we shall discuss it explicitly below.

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(8) In the situation in which Δ becomes smaller than zero, eq. (19) still holds for complex arguments of the logarithm and leads to the same singularities as described in Subsect. 2·4 below.

(9) Without loss of generality we may assume that the factor, multiplying the logarithm in eq. (17), is finite.

2·3. *Normal thresholds.* – The Feynman-Stückelberg integral (8) does not become singular for real values of the variables p_i^2 and m_i^2; singularities may appear, however, on the real axes of the complex variables s_1, s_2 and s_3, if we read the integral as $\hat{I}$, according to eq. (7). These singularities are branch points which we shall call thresholds, with branch cuts attached to them.

Let us discuss first the example in which one of the external particles, say particle 1, has mass zero. Then the triangle amplitude becomes a single integral with the integrand given by eq. (17), and its singularities are created by the roots of the denominator and the numerator under the logarithm ([9]). The zeros in the numerator provide the normal threshold in the variable s_2, whereas the denominator yields the normal threshold in the variable s_3:

$$s_2^{\text{th}} = (m_1 + m_3)^2 \qquad \text{and} \qquad s_3^{\text{th}} = (m_1 + m_2)^2 . \tag{20}$$

The cuts attached to these branch points go in the correct direction in the complex s_2 and s_3 planes.

Turning to the general case i), we should at least expect normal thresholds in all external variables s_1, s_2 and s_3, for arbitrary values of the other triangle variables. Excluding the possibility of $\Delta = 0$ which we shall discuss below, the only singularities of the triangle amplitude emerge again from the expression under the logarithm, eq. (19). The roots of the numerator and denominator under the logarithm can be due to either a point *within* the domain or at the *end* of the x-integration, and both situations occur. For the upper limit $x = 1$ one finds

$$I_I'(1) = \lambda^{-1} \log \left\{ \frac{s_1 - m_2^2 - m_3^2 + \lambda}{s_1 - m_2^2 - m_3^2 - \lambda} \right\}$$

with

$$\lambda(s_1, m_2^2, m_3^2) = [s_1 - (m_2 - m_3)^2]^{\frac{1}{2}} [s_1 - (m_2 + m_3)^2]^{\frac{1}{2}} . \tag{21}$$

This expression gives rise to the normal branch cut in the variable s_1, starting at the normal threshold $s_1^{\text{th}} = (m_2 + m_3)^2$. The other singularities of the integrand (19) occur for internal points of the x-integration and are due to the four roots of the equation

$$(Bx + 2C)^2 - x^2 \Delta = 4C[Ax^2 + Bx + C] = 0 . \tag{22}$$

The zeros of the first factor,

$$C \equiv (1 - x)\, x s_3 - (1 - x)\, m_1^2 - x m_2^2 , \tag{23}$$

yield the normal-threshold singularity plus the branch cut in the variable s_3,

while those of the second factor,

$$Ax^2 + Bx + C \equiv (1-x)\, xs_2 - (1-x)\, m_1^2 - xm_3^2\,, \tag{24}$$

provide the normal-threshold singularity in the variable s_2.

2·4. *The triangle threshold.* – The normal thresholds are the only singularities in the triangle amplitude, provided the expression

$$\Delta = B^2 - 4AC \equiv [xs_1 + (1-x)(s_2 - s_3) + m_2^2 - m_3^2]^2 + \\ + 4s_1[(1-x)(xs_3 - m_1^2) - xm_2^2] \tag{25}$$

will never become zero. What happens in the contrary case ii) may be guessed from taking the limit $\Delta \to 0$, so that $x \to x_\pm$, where x_+ and x_- refer to the two roots of the quadratic expression (25). Then the first factor in the integrand (19) diverges, but at the same time also the logarithm tends to zero with the same power. This peculiar situation creates a new singularity, different from the normal thresholds, in the triangle integral.

To simplify the discussion let us study the special case of equal and fixed external parameters s_2 and s_3 $(=s)$ and look at possible singularities in the variable $s_1 = z$ (10). Two conditions must be satisfied if the roots of Δ are to cause a singularity in the triangle amplitude. The two roots x_+ and x_- have to be real, and they must assume values within the interval $0 < x < 1$ of the x-integration. For *positive* external variables the first condition holds as long as

$$m_1^2 z^2 + z[(s - m_1^2)^2 + m_2^2 m_3^2 - (s + m_1^2)(m_2^2 + m_3^2)] + s(m_2^2 - m_3^2)^2 \geqslant 0\,. \tag{26}$$

The *anomalous threshold* z_a is obtained as the largest root of the expression (26), provided the corresponding double root x_0 of Δ lies between 0 and 1. The latter requirement leads to the condition (11)

$$s > m_1^2 + m_2 m_3 \qquad \big(0 < s < \min\{(m_1 + m_2)^2,\ (m_1 + m_3)^2\}\big)\,. \tag{27}$$

Considering a simple triangle amplitude in which all internal masses are equal $(m_1 = m_2 = m_3 = m)$, the anomalous threshold becomes

$$z_a = \frac{s(4m^2 - s)}{m^2} \qquad \text{for } 2m^2 < s < 4m^2. \tag{28}$$

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(10) This example has been considered first by NAMBU when he discussed the deuteron form factor in ref. (3).

(11) In the general case when $s_2 \neq s_3$, the condition (27) has to be replaced by $(s_2 m_2 + s_3 m_3)/(m_2 + m_3) > m_1^2 + m_2 m_3$.

Starting from this branch point a cut develops on the real z-axis, extending as far as the corresponding values of x,

$$x_{\pm} = \frac{1}{2(s - z/4)}\left[s \pm \sqrt{s^2 - 4m^2\left(s - \frac{z}{4}\right)}\right], \tag{29}$$

stay below 1 for $z > z_a$. Now x_+ increases for growing z from the value x_0 (at z_a) to 1 (at $z = 4m^2$), and we conclude that the anomalous branch cut at least extends from z_a to the normal threshold z_{th}. We cannot derive more information from eq. (29), since x_- stays below 1 for all $z > z_a$. However, we are able to close this gap by studying the anomalous threshold as a function of the parameters in eq. (28). One finds that z_a increases from the value zero, for $s = 4m^2$, to the value $4m^2$, for $s = 2m^2$; the anomalous threshold will never exceed the normal threshold $z_{\text{th}} = 4m^2$ unless the parameter s becomes imaginary and therefore unphysical. This reflects the fact that a triangle threshold can occur above the normal threshold only if $s_2 > 4m^2$ and $s_3 < 0$, or *vice versa*, which cannot be satisfied in our example where $s_2 = s_3 = s > 0$. As a consequence from this consideration the anomalous cut will exist only between the anomalous threshold z_a and the normal threshold z_{th} ([12]). We shall deal with other triangle singularities occurring above the normal threshold in Sect. **4**.

3. – Anomalous thresholds in quantum field theory.

In the covariant evaluation of the quantum-theoretical integral we had to assume that the two shifts of the path, expressed by eqs. (10) and (11), can be made without ever crossing the poles in the complex energy-plane. A sufficient condition for this requirement to hold is that we evaluate the amplitude in the physical region of its (external) variables, *i.e.* for real physical momenta of the in- and outgoing particles. But do the anomalous thresholds obtained in the last Section always fall in the physical region? The answer is no, in some typical examples of the triangle amplitudes. In those cases we have to reconsider the amplitudes, given by eq. (8), since an *ambiguity* arises as to how to continue the quantum-theoretical integrals to unphysical values of the external momenta. One possibility seems to stick to the original Stückelberg-Feynman prescription, *i.e.* leave the loop momentum paths real even if some of the internal momenta now become complex and the poles of the integrand in the triangle amplitude (8) wander into the complex energy-plane ([13]). We shall

([12]) This holds equally in the general case. We refer to the detailed treatment of R. J. EDEN, P. V. LANDSHOFF, D. I. OLIVE and J. C. POLKINGHORNE: *The Analytic S-Matrix*, Chap. 2.3 (Cambridge, 1966).

([13]) Since the loop momenta are defined only modulo a suitable linear sum of external momenta, this choice implies a fixed line momentum being real. Then the others may contain complex momenta, but the imaginary parts of these momenta remain fixed throughout the loop integration.

demonstrate explicitly that with this continued Stückelberg-Feynman prescription the anomalous threshold in the unphysical region does not arise. Another possible prescription is to distort the paths of four-momentum integration into the complex planes in such a manner that we obtain the anomalous singularity in the unphysical region. This prescription is equivalent to the analytic continuation of the physical-region amplitude, *keeping the set of intermediate states always complete by adding some unphysical states with complex momenta.* It is interesting to note that the normal thresholds, even when lying below the physical boundary, will be found from both prescriptions.

3·1. « *Anomalous thresholds* » *in the unphysical region.* – The anomalous thresholds have been introduced first in connection with electromagnetic

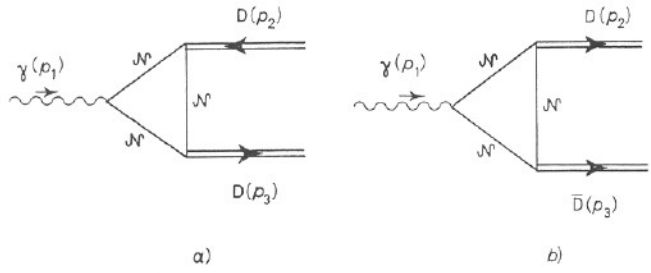

Fig. 2. – Deuteron « form factor »: *a*) deuteron form factor and *b*) deuteron production.

form factors, especially with that of the deuteron (Fig. 2). The anomalous threshold, as discussed in the form factors, does not occur in the physical region of the amplitude (Fig. 2 *a*)), but in its analytic continuation to timelike photons with positive momentum transfer squared $t = p_1^2$ (or $z = p_1^2 + i\varepsilon$). One argues, however, that the anomalous threshold z_a, calculated from eq. (28), is only slightly away from the physical region $t < 0$, therefore it should be visible in the physical deuteron form factor. Without discussing the meaning of this argument at the moment let us enter a « naive » evaluation of the triangle « form factor », represented by Fig. 2 *b*), as if it were corresponding to a physical process $\gamma \to \mathrm{D} + \overline{\mathrm{D}}$ with all momenta being timelike.

For the deuteron case the intermediate masses are in very good approximation equal to the average mass of the nucleon ($m_i = m_{\mathcal{N}}$). From the covariant evaluation in the above Subsect. **2·4** we obtained an anomalous threshold

$$z_a = \frac{s(4m^2 - s)}{m^2} = \frac{m_{\mathrm{D}}^2(4m_{\mathcal{N}}^2 - m_{\mathrm{D}}^2)}{m_{\mathcal{N}}^2} \ll 4m_{\mathrm{D}}^2 , \tag{30}$$

which is in fact very small. Now the interesting situation with the anomalous threshold, due to eq. (30), is that it does not occur in the physical region of the

process $\gamma \to D + \overline{D}$ either. This becomes immediately clear if we pass to the rest frame of our « photon », *i.e.*

$$p_1 = (2E, 0)\,, \qquad p_2 = (E, \boldsymbol{p})\,, \qquad p_3 = (E, -\boldsymbol{p})\,. \tag{31}$$

If the process were physical, we had to require real three-momentum $\boldsymbol{p}$, hence

$$z_{\text{phys}} = 4E^2 = 4(m_D^2 + \boldsymbol{p}^2) \geqslant 4m_D^2\,. \tag{32}$$

Comparing the physical boundary in eq. (32) with the anomalous threshold (30) we recognize that no *physical process* can happen at z_a, since to the anomalous threshold there corresponds an unphysical $\boldsymbol{p}$ with

$$p^2 = -\frac{1}{4}\frac{m_D^4}{m_N^2}\,. \tag{33}$$

Since we have not evaluated the triangle amplitude in Sect. **2** for such unphysical momenta, we do not know whether the anomalous threshold, given by eq. (30), actually exists.

We can even go further and consider « anomalous thresholds », as formally « calculated » in Sect. **2**, in more general triangle amplitudes. Actually not all the intermediate masses in the deuteron form factor are equal, but we have in Fig. 2 a neutron connecting the two deuterons and two protons coupled to the photon. Generalizing the « form factor » triangle to arbitrary internal masses m_1, m_2 and m_3, we obtain from the condition (27) the inequality

$$(m_2 + m_3)^2 > z_a = 4E^2 \geqslant 4s > 4(m_1^2 + m_2 m_3)\,, \tag{34}$$

which leads to the condition, if we take the extreme left- and right-hand sides,

$$(m_2 - m_3)^2 > 4m_1^2 \tag{35}$$

between the internal masses. At the same time we require the parameters s below their normal thresholds. Using that information, we derive two inequalities from the relation (34), one of which cannot be satisfied. Hence there is no anomalous threshold in arbitrary « form factors » connected with physical values of the four-momenta of the external particles.

With respect to most general triangle amplitudes represented by the graph of Fig. 1 we have not succeeded in demonstrating that the anomalous thresholds « derived » in Sect. **2** are *always* in the unphysical region of the process in question. However, one has to impose a rather strong condition on the triangle parameters if the anomalous threshold, say s_1^a, should occur for physical momenta, namely

$$(s_2 - s_3 + m_2^2 - m_3^2)^2 - 4m_1^2 s_1^a \geqslant 0\,. \tag{36}$$

The requirement (36) is derived from the observation that the Feynman-parameter value x_0, which occurs as a double root of the expression (25), must be larger than zero, and that in the physical region always

$$s_1 \geqslant (\sqrt{s_2} + \sqrt{s_3})^2 . \tag{37}$$

The condition (36) is not easy to satisfy; *e.g.* in the special case of equal internal masses $m_i = m$ $(i = 1, 2, 3)$ it never will hold. Therefore it seems to us very difficult, if not impossible, to find an anomalous threshold in the physical region of a triangle amplitude. This remark does not refer to other triangle singularities as we shall discuss in Sect. **4**.

3·2. *Continuation of the Feynman-Stückelberg amplitude.* – Having recognized that the formally « derived » anomalous threshold often (or always?) falls in the unphysical region of the triangle amplitude, we are now confronted with the task of finding a continuation of the field-theoretical integral (8) to unphysical values of its variables. Let us study as a typical and simple example the « form factor » with equal internal masses (Fig. 2 *b*)). This amplitude describes a physical process for values of the variable z above $4s$, since otherwise we have to include complex three-momenta of the outgoing states. This means that in the unphysical region the shift of the original real four-momentum path (of STÜCKELBERG and FEYNMAN) includes in general a complex translation $\alpha_2 p_3 - \alpha_3 p_2$, according to eq. (10). The details of this situation become clear if we specify the internal momenta for convenience:

$$\begin{cases} q_1 = (k_0, \boldsymbol{k} + \boldsymbol{p}) , \\ q_2 = (k_0 - E, \boldsymbol{k}) , \\ q_3 = (k_0 + E, \boldsymbol{k}) . \end{cases} \tag{38}$$

In the physical region $E^2 \geqslant s$, the reality of the loop momentum k does not impose any restriction on the calculation from eq. (9) to eq. (13). Then the three-momentum $\boldsymbol{P}$ is real because of

$$|\boldsymbol{P}| = \sqrt{E^2 - s} . \tag{39}$$

In the unphysical region $\boldsymbol{P}$ becomes a complex three-vector, and we can assume without any loss of generality that it has only a third component,

$$\boldsymbol{P} = (0, 0, P) \qquad \text{with} \qquad P = ip . \tag{40}$$

So, starting with the Feynman parametrization, eq. (9), and choosing cylindrical co-ordinates (l, φ, k_z), for the three-momentum $\boldsymbol{k}$, we obtain after per-

forming the trivial angle integration

$$I = \int_{-\infty}^{+\infty} \mathrm{d}k_z \int_0^{\infty} \pi \mathrm{d}(l^2) \int_{-\infty}^{+\infty} \mathrm{d}k_0 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, 2\Phi^{-3}$$

with

$$\Phi = (1-x)[k_0^2 - l^2 - (k_z + P)^2 - m^2] + \\ + (x-y)[(k_0 - E)^2 - l^2 - k_z^2 - m^2] + y[(k_0 + E)^2 - l^2 - k_z^2 - m^2] + i\varepsilon\,. \tag{41}$$

The l^2-integration gives further

$$I = \pi \int_{-\infty}^{+\infty} \mathrm{d}k_z \int_{-\infty}^{+\infty} \mathrm{d}k_0 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \Psi^{-2}$$

with

$$\Psi = (1-x)[k_0^2 - (k_z + P)^2 - m^2] + \\ + (x-y)[(k_0 - E)^2 - k_z^2 - m^2] + y[(k_0 + E)^2 - k_z^2 - m^2] + i\varepsilon\,. \tag{42}$$

Finally the y-integration yields

$$I = \pi \int_{-\infty}^{+\infty} \mathrm{d}k_z \int_{-\infty}^{+\infty} \mathrm{d}k_0 \int_0^1 \mathrm{d}x \frac{1}{4k_0 E} \{X_1^{-1} - X_2^{-1}\}$$

with

$$\left\{ \begin{array}{l} X_1 = (1-x)[k_0^2 - (k_z + P)^2 - m^2] + x[(k_0 - E)^2 - k_z^2 - m^2] \\ \text{and} \\ X_2 = (1-x)[k_0^2 - (k_z + P)^2 - m^2] + x[(k_0 + E)^2 - k_z^2 - m^2]\,. \end{array} \right. \tag{43}$$

The integrand of the k_0-integral contains a pole at $k_0 = 0$ and two pairs of poles from the quadratic denominators X_1 and X_2. The latter singularities do not coincide with the $k_0 = 0$ pole unless the condition

$$(1-x)[(k_z + P)^2 + m^2] + x[k_z^2 + m^2 - E^2] = \\ = [k_z + (1-x)P]^2 + m^2 - x(1-x)s - x^2 E^2 = 0 \tag{44}$$

is satisfied.

In the case of *real and physical* three-momentum P, eq. (44) would lead to a lowest threshold, given by

$$m^2 - x(1-x)s - x^2 E^2 = 0\,. \tag{45}$$

That is the « normal » threshold

$$E_{\text{th}} = m\,, \qquad \text{or} \qquad z_{\text{th}} = 4E^2 = 4m^2\,. \tag{46}$$

From the considerations in the above Section, eq. (32), we know that also this threshold does not lie in the physical region; hence it cannot, in principle, be obtained with real three-momentum P.

For *imaginary* P, on the other hand, the pinch condition, eq. (44), is satisfied only if $x = 1$ since we stick to real k_z. Again we obtain as singularity of the integral (43) the « normal » threshold, given by eq. (46). Thus the « normal » threshold will be obtained from the Feynman-Stückelberg amplitude (8), if we continue to complex values of P or to unphysical values of the covariant variable z.

Perhaps it provides a better understanding if we reach the same conclusions from another integral representation of the triangle amplitude. The denominator Ψ in eq. (42) has the form

$$\Psi = A + B$$

with

$$A = [k_0 - (x - 2y)E]^2 \tag{47}$$

and

$$B = -[k_z + (1-x)P]^2 - (x-2y)^2 E^2 + xE^2 - x(1-x)P^2 - m^2\,.$$

In the case of imaginary P the condition for a singularity reads ($x = 1$)

$$[k_0 - (1-2y)E]^2 - k_z^2 - m^2 + E^2 y(1-y) = 0\,. \tag{48}$$

This requirement does not yield any singularity below the normal threshold. The quantity B is always negative for $E < m$, and we may integrate the expression (42) in this region with respect to k_0, using the identity

$$\int_{-\infty}^{+\infty} \mathrm{d}k_0 \{[k_0 - (x-2y)E]^2 - b^2 + i\varepsilon\}^{-2} = -\frac{1}{4b^3} \log \frac{\zeta - b}{\zeta + b}\bigg|_{-\infty}^{+\infty} = \pm \frac{\pi i}{4b^3}$$

with

$$b = \sqrt{-B}\,. \tag{49}$$

The final result is then

$$I = \pi i \int_{-\infty}^{+\infty} \mathrm{d}k_z \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \{m^2 - x(1-x)s - 4y(x-y)E^2 + [k_z + (1-x)P]^2\}^{-\frac{3}{2}}. \tag{50}$$

3·3. *Analyticity properties of the causal triangle « form factor ».* – It has been demonstrated above that the continued Feynman-Stückelberg triangle « form factor » contains the normal-threshold singularity but not the anomalous one. The crucial point turned to be the following fact: when we continue the external momenta to unphysical values but keep the reality of the original three-momentum path k_z, then the shifted momentum

$$k'_z = k_z + (1-x)\,P \tag{51}$$

became complex for unphysical (imaginary) P and arbitrary x. The reason why the « normal » threshold, though also in the unphysical region, came out is that this singularity is due to the corresponding Feynman parameter being zero, or $x=1$, $k'_z = k_z$. But for the anomalous (triangle) singularity of the « form factor » the complex momentum q_1 plays the decisive role if we stick to the Stückelberg-Feynman prescription of real k_z. If we, on the other hand, had required a *real* k'_z regardless of what P and x enter in eq. (51), then we had found the anomalous threshold in the triangle integral, as given by eq. (28). Because the singularity condition in this case would read, instead of eq. (48),

$$m^2 - x(1-x)\,s - zy(x-y) = 0\,, \qquad \text{where} \qquad z = 4E^2\,. \tag{52}$$

Equation (52) is identical with the Landau condition for the triangle singularity provided x and y stay in the interior of their integration intervals, and it yields the « anomalous » threshold in the unphysical region.

Thus the « anomalous » singularity in the form factor would follow only from *choosing a complex initial three-momentum* k_z, which we must regard as a *new prescription* of integration different from that initiated by STÜCKELBERG and FEYNMAN. This new prescription creates the *analytic continuation* of the causal field-theoretic « form factor » amplitude in unphysical regions (of the variable z) below the normal threshold $z_{\text{th}} = 4m^2$. It follows that the so-called « analytic » triangle amplitude is, in examples of practical interest, at variance with the original causal prescription ([14]). Both amplitudes coincide, however, above the normal threshold which is in our example still below the physical boundary. For values of the variable z which are smaller than $4m^2$ the Feynman-Stückelberg « form factor » shows no singularity, neither for real z nor for complex z in the left-hand plane. The « analytic » amplitude, on the other hand, demonstrates a cut on the real z-axis from z_a to $4m^2$.

([14]) This statement does not contradict the work of KÄLLÉN and WIGHTMAN, ref. ([6]), since these authors looked for the *maximal* analytic extension rather than for an extension of the Stückelberg-Feynman prescription.

In fact, it is straightforward to find that the Feyman-Stückelberg scattering amplitude evaluated below the normal threshold is not the same analytic function as that computed above. If we study the poles of the « form factor » integral (42) in the complex k_0-plane, we discover the poles $\pm E_1$ on the wrong side of the causal path. (We choose positive k_z and positive Im P in Fig. 3!)

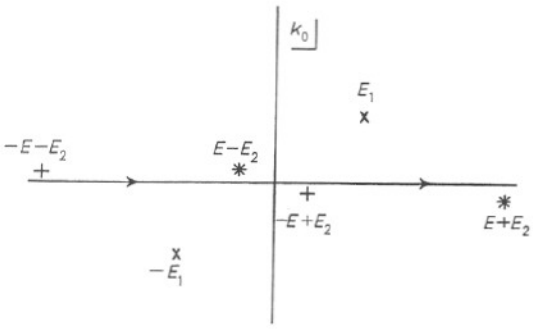

Fig. 3. – Energy poles in the unphysical plane: $E_1 = \sqrt{(k_z + p)^2 + m^2 - i\varepsilon}$ and $E_2 = \sqrt{k_z^2 + m^2 - i\varepsilon}$.

Hence we cannot go through the steps eqs. (12) and (13) when evaluating the triangle integral. On the other hand, we are able to proceed as far into the unphysical region as the normal threshold because the latter comes from pinching the poles at $E - E_2$ and $-E + E_2$. Therefore, the « analyticity » domain of the Feynman-Stückelberg amplitude can be extended to the normal threshold. Below it the « wrong » position of the above-mentioned poles E_1 and $-E_1$ becomes crucial, changing the amplitude function to another which is analytically different from the analytic continuation of the physical amplitude. In common terminology we should call the continued Feynman-Stückelberg triangle « form factor » a *piecewise analytic* function.

Our efforts to distinguish clearly between the continued Feynman-Stückelberg amplitude and the analytic continuation of the field-theoretical integral might seem rather academic since the alleged difference does only concern the unphysical region ([15]). Moreover, one will point out that there exist two empirical facts in favor of the analytic continuation, the dispersion relations and the deuteron form factor consideration which gave birth to the anomalous threshold. We shall deal with the question of dispersion relations in a separate

([15]) Of course, we see one important reason to undergo all this labor, namely to complement and improve the statements which have been made in standard literature. We refer, *e.g.*, to the lectures given by J. HAMILTON: *The Dynamics of Elementary Particles and Pion-Nucleon Interaction* (Copenhagen, 1968).

([16]) H. RECHENBERG and E. C. G. SUDARSHAN: in preparation.

([17]) See Y. NAMBU: *Nuovo Cimento*, **9**, 610 (1958), p. 621.

paper ([16]), and confine ourselves here to answer the deuteron form factor argument which suggests a close relation between the low-lying anomalous threshold in the deuteron «form factor» and the loose deuteron binding. As representing the inverse of a binding force one considered the «radius» ([17])

$$r = \frac{1}{\sqrt{z_0}} = \frac{1}{4\sqrt{m_N(2m_N - m_D)}} \qquad (z_0 = \text{lowest threshold!})\,. \tag{53}$$

In the pion form factor the lowest threshold is the normal threshold $4m_N^2$ and the binding, therefore, very strong. The deuteron radius (53) also appears in the deuteron wave function; hence the anomalous threshold in the form factor seems justified. But this «physical» reasoning is to be viewed with caution. On the one hand, when talking about the anomalous threshold we are concerned with the unphysical region of the form factor. The singularity structure of the analytic amplitude is rather different when the momentum-transfer variable t changes to positive values. Secondly, it is well known from other examples in physics that one can deal with some *unstable* phenomena by using the analytic continuation of a function which is generally valid, that is, gives the *stable* solution, only in a different physical region. For instance, the equilibrium transition region between a liquid and a nonideal gas in the $(p$-$v)$-plane is certainly not described by the analytic Van der Waals isotherm though its continuation into that region accounts nicely for the unstable supercooling or superheating phenomena. In the same sense we can still use the Nambu picture of the weak deuteron binding, even if we refuse to accept the analytically continued amplitude as *the* expression which follows from quantum field theory in the unphysical domain.

4. – Physical-region triangle singularity and interpretation.

To complete the analysis of the triangle amplitudes let us turn to a well known structure, shown in Fig. 4, which might contribute to the two-particle scattering. If we so choose, we may also associate with it the three-particle scattering, provided we split both the external lines 2 and 3 into particle pairs. Here the covariant variable t is taken to be negative and fixed. From the Landau

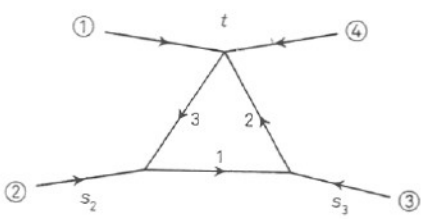

Fig. 4. – Two-particle scattering amplitude.

equations (5) and (6) one obtains, under certain conditions for the triangle parameters, a triangle singularity in the variables s_2 and s_3 which lies in both cases above the normal threshold. Further on, we shall choose a situation in which all singularities appear in the physical region. Then the triangle threshold will be calculated in the field-theoretical amplitude, and it will correspond to a term in the unitarity sum. The usual physical interpretation of the higher singularity can be given.

4·1. *Calculation of the anomalous threshold.* – To deal with the graph in Fig. 4 explicitly we go into the reference frame (Breit frame) in which the momentum-transfer vector has only a z-component, or

$$(p_1 + p_4) = (0, 0, 0, 2q)\,,$$
$$p_2 = (E, 0, p, -q)\,,$$
$$p_3 = (-E, 0, -p, -q)\,,$$

hence

$$s_2 = s_3 = s = E^2 - p^2 - q^2 \qquad \text{and} \qquad t = -4q^2\,. \tag{54}$$

We have specialized for simplicity to equal masses for the external particles 2 and 4 (we consider, however, s as a variable!), and we have also rotated the x and y co-ordinates suitably.

Choosing as loop a four-momentum k such that

$$\begin{cases} q_1 = (k_0 + E,\, k_x,\, k_y + p,\, k_z)\,, \\ q_2 = (k_0,\, k_x,\, k_y,\, k_z - q)\,, \\ q_3 = (k_0,\, k_x,\, k_y,\, k_z + q)\,, \end{cases} \tag{55}$$

we find for the triangle integral, with all exchanged masses being equal to m, the expression (we omit the $+i\varepsilon$ in each denominator)

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^3 k \int_{-\infty}^{+\infty} \mathrm{d}k_0 \,[(k_0 + E)^2 - k_x^2 - (k_y + p)^2 - k_z^2 - m^2]^{-1} \cdot \tag{56}$$
$$\cdot [k_0^2 - k_x^2 - k_y^2 - (k_z - q)^2 - m^2]^{-1} [k_0^2 - k_x^2 - k_y^2 - (k_z + q)^2 - m^2]^{-1}\,.$$

Using the Feyman trick we obtain

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^3 k \int_{-\infty}^{+\infty} \mathrm{d}k_0 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \Psi^{-3}$$

with

$$\Psi = [k_0 + (1-x)E]^2 - k_x^2 - [k_y + (1-x)p]^2 - \\ - [k_z - (x-2y)q]^2 + x(1-x)s + y(x-y)t - m^2 . \tag{57}$$

The denominator Ψ has a root for a minimal s_a given by

$$s_a = 2m^2\left(1 + \sqrt{1 - \frac{t}{2m^2}}\right) > s_{\text{th}} = 4m^2 \qquad (t < 0) . \tag{58}$$

This triangle threshold s_a follows from an evaluation of the integral as suggested in Sect. 2. In that case the noncovariant terms in Ψ drop out for the special loop momentum

$$k = \big(-(1-x)E,\ 0,\ -(1-x)p,\ (x-2y)q\big) , \tag{59}$$

which is in the Stückelberg-Feynman path of integration since E, p and q are chosen to be real. Again the situation would be different and no triangle singularity would show up in the causal amplitude, if s_a were in the unphysical region of the process. In contrast to the earlier examples of «form factor» triangles, however, we have to deal here in general with the physical region of two-particle scattering amplitudes.

4·2. *The interpretation of the unitary sum.* – In nonrelativistic quantum theory the singularities of a scattering amplitude are created by intermediate states. Can we generalize this explanation? There are two ways along which one might proceed. The first is to prove the assertion in the framework of the so-called old-fashioned perturbation theory which in turn can be shown to be equivalent to the covariant formalism. We start again with the Feynman-Stückelberg integral, choosing for convenience the external energy variables

$$p_{10} = -S , \qquad p_{20} = -Q , \qquad p_{30} = -R \qquad (Q + R + S = 0), \tag{60}$$

thus we refer to all particles as outgoing. Then the integrand of

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^3 k \int_{-\infty}^{+\infty} \mathrm{d}k_0 \, [k_0^2 - E_1^2]^{-1} [(k_0 - R)^2 - E_2^2]^{-1} [(k_0 + Q)^2 - E_3^2]^{-1}$$

with

$$\begin{cases} E_1 = \sqrt{\boldsymbol{k}^2 + m_1^2 - i\varepsilon} , \qquad E_2 = \sqrt{(\boldsymbol{k} - \boldsymbol{p}_3)^2 + m_2^2 - i\varepsilon} , \\ E_3 = \sqrt{(\boldsymbol{k} + \boldsymbol{p}_2)^2 + m_3^2 - i\varepsilon} , \end{cases} \tag{61}$$

has six poles in the complex k_0-plane, as indicated in Fig. 5. In performing the k_0-integration we replace the path on the real k_0-axis by the half-circle

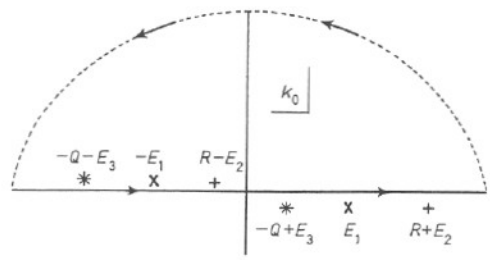

Fig. 5. – Poles of the triangle graph in the complex energy-plane.

at infinity in the upper plane, which does not contribute, and the residue of the poles at

$$k_0 = -E_1\,,\ R-E_2\,,\ -Q-E_3\,. \tag{62}$$

The result is

$$I = \int_{-\infty}^{+\infty} \mathrm{d}^3k\, 2\pi i\, \frac{\Sigma}{8E_1E_2E_3} \tag{63}$$

with

$$\begin{aligned}\Sigma = &\left(\frac{1}{R+E_1-E_2} - \frac{1}{R+E_1+E_3}\right)\left(\frac{1}{Q-E_1+E_3} - \frac{1}{Q-E_1-E_3}\right) + \\ &+ \left(\frac{1}{R+E_1-E_2} - \frac{1}{R-E_1-E_2}\right)\left(\frac{1}{S+E_2-E_3} - \frac{1}{S+E_2+E_3}\right) + \\ &+ \left(\frac{1}{Q-E_1+E_3} - \frac{1}{Q+E_1+E_3}\right)\left(\frac{1}{S+E_2-E_3} - \frac{1}{S-E_2-E_3}\right).\end{aligned}$$

The sum Σ contains 12 terms each of which is a product of two denominators. We can rearrange and reduce the sum to six terms using the identity

$$\frac{1}{AB} + \frac{1}{AC} + \frac{1}{BC} = 0\,, \qquad \text{if } A+B+C=0\,, \tag{64}$$

which is satisfied in our case due to the energy conservation, because we always choose the denominators A, B and C such that

$$A+B+C = Q+R+S\,,$$

e.g.

$$A = R+E_1-E_2\,, \quad B = Q-E_1+E_3\,, \quad C = S+E_2-E_3\,, \text{ etc.} \tag{65}$$

The *unitarity sum* finally becomes

$$\Sigma = \frac{1}{Q+E_1+E_3}\frac{1}{R-E_1-E_2} + \frac{1}{Q-E_1-E_3}\frac{1}{R+E_1+E_2} + \\ + \frac{1}{Q+E_1+E_3}\frac{1}{S-E_2-E_3} + \frac{1}{Q-E_1-E_3}\frac{1}{S+E_2+E_3} + \\ + \frac{1}{R+E_1+E_2}\frac{1}{S-E_2-E_3} + \frac{1}{R-E_1-E_2}\frac{1}{S+E_2+E_3}\,. \tag{66}$$

Every term in the unitarity sum corresponds, in this sequence, to an old-fashioned perturbation graph where a time order is introduced from left to right (Fig. 6). The intermediate states are found on the time cuts (dashed

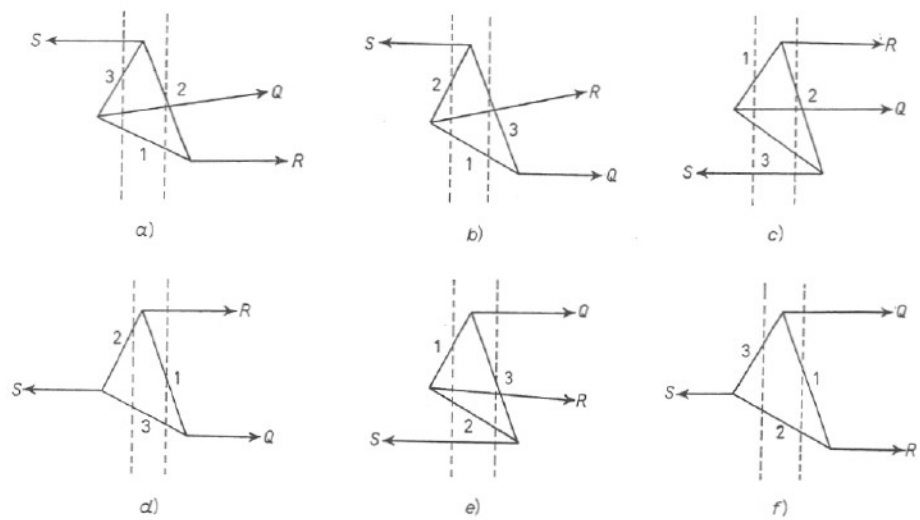

Fig. 6. – Terms of the unitarity sum.

lines), where 1 corresponds to E_1, etc. Thus we verify the equivalence of the unitarity sum method (in old-fashioned perturbation theory) with the field-theoretical integral ([18]).

The question now arises as to how to interpret the terms of the unitarity sum eqs. (63) and (66). We answer it in the special case treated above. With the external momenta given by eq. (55) we obtain two types of terms, those containing denominators with the energy S which vanishes and those containing only denominators with the energies Q and R, that is $\pm E$. The first type of

([18]) See, *e.g.*, A. MACKE: *Zeits. Naturf.*, **8** a, 599 (1953).

products is of the form

$$(67)\qquad [\pm E \pm \sqrt{m^2+k_x^2+(k_y+p)^2+k_z^2}+\sqrt{m^2+k_x^2+k_y^2+(k_z\pm q)^2}]^{-1}\cdot \\ \cdot[\sqrt{m^2+k_x^2+k_y^2+(k_z-q)^2}+\sqrt{m^2+k_x^2+k_y^2+(k_z+q)}]^{-1},$$

and gives rise to the normal thresholds in the variables s_2 and s_3, since we have assumed t to be fixed. Generally speaking, *each denominator in the unitarity sum* (66) *corresponds to a normal threshold in its energy variable.* But where do the triangle thresholds hide? The terms in the unitarity sum which are depicted by Fig. 6 *a*) and *b*) read in our special example

$$(68)\qquad [E \pm (\sqrt{m^2+k_x^2+(k_y+p)^2+k_z^2}+\sqrt{m^2+k_x^2+k_y^2+(k_z-q)^2})]^{-1}\cdot \\ \cdot[E \mp (\sqrt{m^2+k_x^2+(k_y+p)^2+k_z^2}+\sqrt{m^2+k_x^2+k_y^2+(k_z+q)^2})]^{-1}.$$

In the case that $k_z \neq 0$, the two denominators in the expression (68) are different and, in agreement with the earlier statement, exhibit the normal thresholds $s_2 = s_3 = s = 4m^2$. But in the Stückelberg-Feynman path also the points with $k_z = 0$ are included. Then the two denominator factors in (68) have coinciding roots:

$$(69)\qquad E^2 = (\sqrt{m^2+k_x^2+(k_y+p)^2}+\sqrt{m^2+k_x^2+k_y^2+q^2})^2.$$

Rationalization of this equation leads to the condition

$$(70)\qquad s^2-4sm^2+m^2t-4mp^2-4sp(k_y+p)-4(q^2+s)(k_y+p)^2=0\,,$$

for a new singularity in the triangle unitarity sum. As it stands the condition (70) looks terribly noncovariant, though we know that the result which we shall compute by the three-momentum integration in eq. (63) is covariant. Hence we have to ask under which condition the noncovariant part in eq. (70) becomes zero ([19]). This happens for special values of k_y, given by

$$(71)\qquad k_y^{\pm} = -p+\frac{p}{2(q^2+s)}\left(s\pm\sqrt{s^2-4m^2s+m^2t}\right)\,,$$

which are real and lie on the causal path provided

$$(72)\qquad s^2-4m^2s+m^2t \geqslant 0\,.$$

([19]) In the special case of $p=0$ we need not carry out the following considerations because then the minimum threshold following from eq. (70) is obtained for $k_y=0$, and immediately given by the zeros of the covariant term in eq. (70).

Now the covariant part in eq. (70) becomes zero exactly when the equal sign in (72) holds. We have found a new covariant threshold in the unitarity sum, and comparing it with the triangle threshold, eq. (58), we find that both are identical. Thus the triangle threshold *in the physical region* corresponds to a very special intermediate state; the mathematical and physical condition, by which it is created in the unitarity sum, is that *two denominators have coinciding roots.*

4·3. *Causality and discontinuities in the physical region.* – From a quantum-field-theoretical point of view our previous derivation and interpretation of the thresholds in triangle graphs might seem rather clumsy and inelegant, since another manifestly covariant method has been proposed. In a study of the singularities of Feynman amplitudes CUTKOSKY presented a formula which permits us in principle to calculate the discontinuities associated with Landau singularities [20]. Starting from a general Feynman diagram (1) and assuming Landau's conditions (5) and (6) to hold, he derived an equation

$$\text{(73)} \qquad \operatorname{disc}_r I = (-2\pi i)^r \int \mathrm{d}^4k_1 \ldots \mathrm{d}^4k_l \frac{\delta^{(+)}(q_1^2 - m_1^2) \ldots \delta^{(+)}(q_r^2 - m_r^2)}{(q_{r+1}^2 - m_{r+1}^2 + i\varepsilon) \ldots (q_N^2 - m_N^2 + i\varepsilon)}$$

for the difference between the amplitude and its adjoint, due to a singularity of the order r. Cutkosky's formula has two disadvantages. First, it does not tell us about the sheet structure of the scattering amplitude, but only states the value of the discontinuity once a singularity is given [21]. Secondly, one has to say what does the $\delta^{(+)}$-function in eq. (73) mean.

A first guess was, and this is suggested in the original paper, that the superscript $(+)$ refers to taking the positive energy root, $q_{i^0} > 0$, in the mass shell condition $q_i^2 = m_i^2$ [22]. Such a definition is consistent with an interpretation of the singularities in the physical region which has been expressed most clearly by COLEMAN and NORTON [23]. They define the quantity

$$\text{(74)} \qquad \Delta_i^\mu = \alpha_i q_i^\mu \qquad (i = 1, \ldots, r \text{ and } \mu = 0, 1, 2, 3)$$

as a measure of the space-time interval traversed by the i-th particle between the vertices which are connected by it. Since the Feynman parameters are

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(20) R. CUTKOSKY: *Journ. Math. Phys.*, **1**, 429 (1960).

(21) See, *e.g.*, R. CUTKOSKY: *Rev. Mod. Phys.*, **33**, 448 (1961).

(22) R. CUTKOSKY: *Journ. Math. Phys.*, **1**, 429 (1960), p. 432.

(23) S. COLEMAN and R. E. NORTON: *Nuovo Cimento*, **38**, 438 (1965). Earlier statements can be found in articles by I. J. R. AITCHISON and C. KACSER: *Phys. Rev.*, **133**, B 1239 (1964); C. KACSER: *Phys. Lett.*, **12**, 269 (1964); R. E. NORTON: *Phys. Rev.*, **135**, B 1381 (1964).

always positive, and the *intermediate momenta are explicitly taken to be real*,

$$q_i = q_i^* \qquad (i = 1, \ldots, r), \tag{75}$$

the space-time separation (74) between two vertex points becomes always real when Landau singularities are present in the physical region ([24]). The Feynman parameter α_i divided by the particle mass m_i can, therefore, be interpreted as the proper time during which the (intermediate) particle exists between collisions that occur at the vertices. The positivity of α_i means that the *particles considered as classical objects* move forward in time. In fact, one can dilate the distances between the vertices to macroscopic dimensions, once the particles appear on mass shell and do behave like classical objects transporting energy-momentum from one space point to the other. The second Landau condition, eq. (6), states explicitly that «rescattering» does not happen unless there is a space-time coincidence of the relevant particles.

As an illustration of this interpretation we discuss the triangle amplitudes, depicted in Fig. 1 and 4, in which we might expect two types of singularities corresponding to normal and triangle thresholds. The interpretation of a normal threshold, which occurs, say, in a two-particle decay of a resonance with mo-

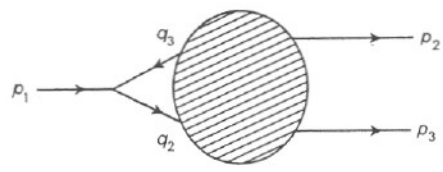

Fig. 7. – Normal-threshold interpretation.

mentum p_1 is shown in Fig. 7. Now, according to the above positive q_{i_0}-root prescription we had to assume a physical process with the energy transfer

$$p_{10} \to q_{20} - q_{30} \qquad (q_{20}, q_{30} > 0). \tag{76}$$

This conclusion is not consistent with the intuitive picture which one develops from the normal singularity, suggesting rather the process

$$p_1 \to q_2 + \bar{q}_3 \qquad (\bar{q}_3 = -q_3, \text{ and } \bar{q}_{30} > 0) \tag{77}$$

to happen, *i.e.* the «decay» into a particle-antiparticle pair $(q_2, \bar{q}_3)$. The same process (77) is obtained if we read Fig. 7 as a term in the unitarity sum and

([24]) This additional condition is often forgotten when one tries to interpret also all Landau singularities in the unphysical region, *i.e.* anomalous thresholds in «form factors», by the same ideas. This extension is not justified.

indicate a time direction from left to right as in old-fashioned perturbation theory ([25]). Therefore we have to develop proper care in handling Cutkosky's $\delta^{(+)}$-function.

The situation becomes even worse if we consider the triangle singularity in the physical region of the two-particle scattering. Two unitarity diagrams, Fig. 6 *a*) and *b*), contribute to this singularity. We redraw them explicitly in Fig. 8. The physical process at the vertex 2 of Fig. 8 *a*) corresponds to the

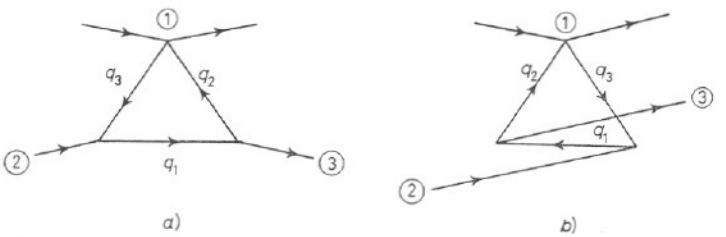

Fig. 8. – Triangle threshold interpretation.

« decay » of the incoming particle 2 into a particle with momentum q_1 and an antiparticle with momentum $-q_3$. At the vertex 1 the antiparticle $(-q_3)$ « creates » a state with momentum p_1 (p_1^2 is timetable) and an antiparticle $(-q_2)$. Finally, at the vertex 3, a particle (q_1) and an antiparticle $(-q_2)$ « annihilate » and the particle 3 (p_3) is « created ». Though it seems from the old-fashioned perturbation graph that such a process can happen only for $(m_1 + m_2)^2 > M_3^2$, *i.e.* s_3 below the normal threshold, this turned out to be wrong since we have derived clearly the opposite result in eq. (58) and from the discussion of the unitarity sum. Of course, one might blame the old-fashioned perturbation theory picture for the inconsistency, but as a consequence we are again back in the difficulty of determining the « proper » root in the Cutkosky rule (73). We should mention that also the discussion of the graph in Fig. 8 *b*) does not lead to a more consistent result.

Several ways out of this dilemma exist. First, one calculates the discontinuities in the physical region from the covariant integration which we developed in Sect. **2**. Second, one uses the not explicitly covariant evaluation leading to the unitarity sum eqs. (63) and (66). A third possibility is to return to the discredited Cutkosky rule and to seek to avoid the « proper » root problem ([26]). We shall demonstrate now that this last procedure (of using only the q_i^2 as integration variables) can be carried out in the special case of the single-loop

([25]) Of course, also the process $p_1 \to \bar{q}_2 + q_3$ is possible which is obtained by inverting the arrows in q_2 and q_3. This term can then be found in old-fashioned perturbation theory, but we should always bear in mind that in some graphs of the latter scheme energy is not conserved.

([26]) This has been suggested already by CUTKOSKY himself, in ref. ([21]), p. 452.

integral and leads to the correct answer. The discontinuity is given, according to Cutkosky's rule, by

$$\Delta(p^2 = s) = -4\pi^2 \int_{-\infty}^{+\infty} \mathrm{d}k_0 \int_{-\infty}^{+\infty} \mathrm{d}^3k\, \delta(k^2 - m_1^2)\, \delta[(p-k)^2 - m_2^2] = \tag{78}$$

$$= -16\pi^3 \int_0^{\infty} \varkappa^2\, \mathrm{d}\varkappa\, \delta(p_0^2 - 2p_0 k_0 + m_1^2 - m_2^2)\,,$$

where we compute for simplicity in the rest frame of the incoming particle with momentum p. Unfortunately the integral on the right-hand side of eq. (78) still shows some trace of the proper-root problem, because we have from the first δ-function the ambiguity

$$k_0 = \pm\sqrt{\varkappa^2 + m_1^2} \qquad (\varkappa^2 = \boldsymbol{k}^2)\,. \tag{79}$$

Changing the variable $\varkappa$ to v,

$$\sqrt{v} = \sqrt{\varkappa^2 + m_1^2}\,, \tag{80}$$

we can perform the integration and obtain finally

$$\Delta(s) = -4\pi^4 s^{-\frac{1}{2}} [s - (m_1 - m_2)^2]^{\frac{1}{2}} [s - (m_1 + m_2)^2]^{\frac{1}{2}}\,, \tag{81}$$

which is identical with the result from the direct calculation ([27]). Note that we have resolved the «proper» root ambiguity of eq. (79) by *choosing the positive root*. The reason for this definition comes from the fact that the *two-body phase-space should come out positive*. We conclude from our simple exercise that one can always apply the practical Cutkosky rule if one respects the positivity of the generalized n-body phase-space entering the discontinuity formula (73).

Finally we should add a brief remark concerning the relation between the analyticity of the scattering amplitudes and macrocausality. Roughly speaking the following result has been derived on the basis of very general assumptions: only scattering amplitudes which are analytic except for the discontinuities related to Landau surfaces with positive Feynman parameters are compatible with macrocausality or the cause-event structure in macroscopic domains ([28]). This theorem, however, does not rule out the piecewise analytic «form factor» obtained from the Feynman-Stückelberg prescription, because in the proof the fact is used that one can put all the intermediate propagators contributing to a singularity on the mass shell and, therefore, obtain a macroscopic prop-

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([27]) See, A. M. GLEESON *et al.*: ref. ([5]), Appendix.

([28]) D. IAGOLNITZER and H. P. STAPP: *Comm. Math. Phys.*, **14**, 15 (1969). For an earlier statement see also C. CHANDLER and H. P. STAPP: *Journ. Math. Phys.*, **10**, 826 (1969).

agation of signals. The latter process implies *physical values* of the four-momenta of *all* quanta which are involved. Correctly, the Iagolnitzer-Stapp theorem emphasizes that only analyticity in the physical domain is required. A further extension of the analyticity region might not be justified from physical reasons. We shall expand on this deep problem in a second paper ([15]).

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The authors would like to thank their friends and colleagues H. P. DÜRR, A. M. GLEESON, A. PAGNAMENTA, J. L. RICHARD, L. M. SIMMONS and H. P. STAPP for discussions.

● RIASSUNTO (*)

Si studia la singolarità principale nei grafici triangolari scalari. Si osserva che in alcuni esempi di interesse pratico come i « fattori di forma » la singolarità anomala non sorge quando si effettuano le integrazioni dell'impulso nelle ampiezze di Feynman-Stückelberg, indicando che la teoria quantistica dei campi non da necessariamente le cosiddette ampiezze di scattering analitiche. Tuttavia il fattore di forma triangolare analitico che presenta la singolarità anomala e la corrispondente ampiezza di Feynman-Stückelberg coincidono dovunque sopra le soglie normali, specialmente nella « regione fisica » delle variabili. Ma bisogna state in guardia contro l'uso indiscriminato dei metodi e concetti convenzionali, anche in semplici strutture perturbative.

(*) *Traduzione a cura della Redazione.*

Аналитичность в квантовой теории поля.
I: Повторное исследование треугольных графиков.

Резюме (*). — Исследуется главная сингулярность в скалярных треугольных графиках. Мы обнаружили, что в некоторых случаях, имеющих практический интерес, подобно « форм-факторам », аномальная сингулярность не возникает, когда мы проводим импульсные интергирования в амплитудах Фейнмана-Стюкельберга, что указывает на тот факт, что квантовая теория поля не обязательно дает так называемые аналитические амплитуды рассеяния. Однако, аналитический треугольный форм-фактор, который обнаруживает аномальную сингулярность, и соответствующая амплитуда Фейнмана-Стюкельберга совпадают всюду выше нормальных порогов, особенно в « физической области » переменных. Но мы хотели бы предостеречь против некритического использования обычных методов и представлений, даже в простых структурах возмущений.

(*) *Переведено редакцией.*