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Nuclear Physics B50 (1972) 269–284. North-Holland Publishing Company

COMPLETENESS OF STATES, SHADOW STATES, HEISENBERG CONDITION, AND POLES OF THE S-MATRIX

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Received 7 August 1972

Abstract: A study of the connection between poles of the S-matrix and states of the Hamiltonian of non-relativistic quantum mechanical systems is made with a view to elucidate the concept of shadow states which have been used by one of the authors for the elimination of divergences in quantum field theory with the aid of an indefinite metric. By specific examples we demonstrate that there exist non-relativistic systems for which the S-matrix has poles which correspond to shadow (redundant) states which are not needed in the completeness relation. Systems with such states do not fulfill a condition on S-matrix which was derived by Heisenberg. It is further shown that there exist phase equivalent systems in which these very poles of the S-matrix correspond to genuine bound states which are absolutely necessary to complete the set of states of the system, and these discrete states may have positive or negative norm depending on the choice of the S-matrix.

1. INTRODUCTION

In the description of interactions in particle physics the spectrum of particles themselves should more or less account for their mutual interactions; the "potential", even in the context of low energy elastic processes, is only used as a phenomenological artifice, convenient at times, that may be used. The basic interaction is to be discussed either in terms of a local field theory or in terms of an analytic crossing-symmetric S-matrix. In a field-theoretic framework the fields determine both the spectrum of particles and their dynamics. In the S-matrix approach the singularities of the analytic transition amplitudes are directly related to the existence of physical states; and the inter-relationships of the strengths of the singularities determine the dynamics. The residues at poles and the discontinuities across branch cuts are related to appropriate physical quantities.

In developing these latter notions the non-relativistic non-crossing-symmetric analytic S-matrix of potential scattering has been a valuable guide. In the earlier work

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related to dispersion theory and double dispersion relations and the later work on complex angular momentum, the insight gained in the study of Schrödinger equations with non-relativistic potentials has been very useful.

In the context of quantum field theory we have continually encountered problems to deal with local interactions with relativistic fields. These difficulties are as old as quantum field theory itself; and have their origins in even earlier theories: After all the self-energy difficulty of the Lorents electron is with us still! These infinities can be altogether avoided in a theory which entertains a negative metric for certain species of fields. Such infinite metric theories have been extensively studied in recent times [1]. The basic problem that appears in such theories is the disposition of the "states with negative probability". One of the present authors (E.C.G.S.) has solved this basic problem in terms of the notion of "shadow states". These shadow states, are mathematical states which contribute to the dynamics, but which do not appear in the complete set of physical states, as manifested in the unitarity relation. At the present time it appears to us that the concept of shadow states provides the necessary step to construct a finite quantum field theory of interactions.

We have sought, in this context, for the analogous concept in potential scattering. To our pleasant surprise we find that much of the mathematical work necessary was already done by several authors in the nineteen forties in connection with redundant poles of the S-matrix and the concept of phase equivalent potentials [2].

Redundant poles of the S-matrix in potential theory are poles of S-matrix for energies for which no bound states exist. Normally the poles of the S-matrix in the upper half plane of the complex momentum variable (i.e., physical sheet of the complex energy variable) correspond to genuine bound states of the system; and a set of states must include these bound states before they constitute a complete set. At first sight it appears surprising that such redundant states can exist in view of the following consideration.

Let $u_k(r)$ and $u_n(r)$ be respectively the radial wave functions for the continuous and discrete energy states. They satisfy the completeness relation

$$\sum_n u_n^*(r)u_n(r^1) + \int_0^\infty u_k^*(r)u_k(r^1)\,\mathrm{d}k = \delta(r-r^1) \tag{1.1}$$

Since eq. (1) is valid for all values of r and r^1, on using the asymptotic expressions of these wave functions:

$$u_n(r) \sim c_n(2\pi)^{-\frac{1}{2}} \exp(-|k_n|r)\,,$$

$$u_k(r) \sim (2/\pi)^{\frac{1}{2}} \sin(kr + \delta(k))\,,$$

where c_n is a constant and $\delta(k)$ is the phase shift, one finds the following relation

$$\sum_n |c_n|^2 e^{-|k_n|(r+r^1)} \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{2i\delta(k)+ik(r+r^1)}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ik(r-r^1)}=\delta(r-r^1)$$

which is equivalent to ‡

$$\int_{-\infty}^{+\infty} dk\, S(k)\, e^{ikR} = \sum_n |c_n|^2 e^{-|k_n|R} \qquad .2)$$

This general condition on S-matrix was obtained by Heisenberg [3]. We shall call it the Heisenberg condition. We note that in deriving eq. (1.2) from eq. (1.1) we have made formal interchange of the limit for large r and r^1 and integration with respect to k.

It is interesting to note that the r.h.s. of the expression (1.2) receives contributions from the genuine bound state poles only. However, if one considers non-relativistic Schrödinger equation with an attractive potential, this general condition (1.2) is not satisfied. To see this let us consider the non-relativistic S-wave scattering by an attractive exponential potential, $V(r) = -V_0\, e^{-r/a}$. The Schrödinger wave function for this potential which vanishes at the origin is

$$u_k(r) = i(2\pi)^{-\frac{1}{2}}\, |\Gamma(1+\rho)/J_{i\rho}(\alpha)\,|\{J_{-i\rho}(\alpha e^{-r/2a})$$
$$-J_{i\rho}(\alpha)\, J_{i\rho}(\alpha e^{-r/2a})\} \qquad (1.3)$$

and the corresponding S-matrix is

$$S(k) = \frac{J_{i\rho}(\alpha)\,\Gamma(1+\rho)}{J_{i\rho}(\alpha)\,\Gamma(1-i\rho)}(\tfrac{1}{2}\alpha)^{-2i\rho} \qquad (1.4)$$

where J's are Bessel functions, $\rho = 2ak$ and $\alpha = 2a(2mV_0)^{\frac{1}{2}}h$.

The states corresponding to the poles of the S-matrix on the imaginary k-axis of the upper-half plane are obtained from (i) the vanishing of $J_{|\rho|}(\alpha) = 0$ and (ii) from the poles of $\Gamma(1+i\rho)$ at $1 + i\rho = -n$; n is positive integer. It appears at first sight that they should all correspond to the genuine bound states and should contribute to the completeness conditions. However, at $1 + i\rho = -n$, though we can construct

‡ Condition (1.2) as it stands is incomplete, we should include a term on the r.h.s. from the discontinuity due to the presence of a cut on the upper imaginary k-axis.

a non-trivial solution to the Schrödinger equation, the wave function (1.3) vanishes identically; as a result, the states corresponding to these poles of S-matrix do not contribute to the completeness conditions. It is easy to show that the set of states given by eq. (1.3) do satisfy the completeness condition with the inclusion of the bound state poles alone; the redundant states do not contribute. Thus we find from this simple exponential potential theory that the redundant states which occur dynamically (they are dynamical in the sense that they disappear when the potential is cut off at large distances) in the S-matrix as shown in eq. (1.3), do not contribute to the completeness condition. On the other hand these states make their presence felt through the Heisenberg condition. The right hand side of eq. (1.2) contains additional terms arising from the redundant poles at $k = i\,|k_r|$ of the S-matrix. The Heisenberg condition for this problem is found to be [4]

$$\int_{-\infty}^{+\infty} \mathrm{d}k S(k) \mathrm{e}^{ikR} = \sum_n |C_n|^2 \,\mathrm{e}^{-|k_n|R} - \sum_r \left(\frac{\mathrm{d}S}{\mathrm{d}k}\right)^{-1}\Bigg|_{k=-i|k_r|} \mathrm{e}^{-|k_r|R} \tag{1.5}$$

We wish to point out the close connection that exists between the shadow states in quantum field theory and the redundant poles of S-matrix in potential scattering theory. More specifically, we wish to point out that we can have two phase equivalent systems i.e., two different potentials which give rise to the same scattering phase shifts for all real momenta, but the complete sets of states of one potential includes only the scattering states, while for the other potential there is a bound state that must be included. In other words, for the same S-matrix with one theory we may have a redundant pole while in the other theory there is a state which is responsible for the pole and hence the pole ceases to be redundant.

The circumstance is not restricted to local potential problems alone: as an example of greater generality we study the question of phase equivalence between a separable potential model and a Lee model. What is a redundant state in one theory turns out to be a bound state in the other theory; it is a matter of choice whether the residue at this point is positive or negative, as far as the Lee model is concerned.

It would be idle to multiply such examples; rather one should attend to the construction of finite and meaningful quantum field theories of particle phenomena. These theories will have poles with residues of the "wrong" sign; and branch cuts with the "wrong" sign of discontinuities. We should however associate them with "shadow" states so that there will be no "physical" states with negative norm. One of us has long pursued this aim; and has periodically reported about it elsewhere. But these efforts may now be understood better in view of their similarity with potential theories where the S-matrix has redundant poles. The same S-matrix may be associated with two (or more) distinct complete sets of states. It is up to us to decide which set is physical and which is not. It has been pointed out that the choice of physical states in an indefinite metric theory is part of the dynamical problem.

The plan of the paper is as follows: In sect. 2 we display two distinct potentials which are phase equivalent and hence have the same analytic structure; but the complete set of states are quite distinct in the two cases. We display the complete set of states; and show that only in one of the two cases is the Heisenberg condition satisfied. In sect. 3 we study the separable potential and show that the S-matrix poles associated with the vanishing of the D-function correspond to genuine states but those corresponding to the poles of the N-function are redundant poles. We explicitly display the case of two-redundant poles and show how to construct a larger theory in which these poles are associated with bound states and thus cease to be redundant. In sect. 4 we pose and solve the inverse problem: given a modified solvable model with 2 poles to construct a reduced theory with these poles demoted to the status of redundant poles. We see that such a construction is very similar to the reduction of a convergent indefinite metric theory to a non-local theory with only positive norm states. In the concluding section we comment on the results of this paper in relation to a purely S-matrix theory of strong interactions; and in relation to general "theorems" of quantum field theory.

2. PHASE EQUIVALENT POTENTIALS

In this section we examine the role of redundant states in potential theory. In particular we discuss here two different phase-equivalent potentials which give rise to the same S-matrix having the same analytic property but with different completeness conditions.

We consider *S-wave* scattering by the following potentials:

$$\text{(i)}\ V_1(r) = -2\beta\lambda^2 \frac{e^{-\lambda r}}{(\beta e^{-\lambda r}+1)^2}\,, \qquad 0>\beta>-1, \lambda>0\,,$$

$$\text{(ii)}\ V_2(r) = \frac{6(r-\alpha)\,[(r-\alpha)^3 - 2\gamma^3]}{[(r-\alpha)^3+\gamma^3]^2}$$

The normalized Schrödinger wave function which vanishes at the origin is given by

$$\phi(k,r) = \frac{1}{2i|f(k)|}\{f(k,0)f(-k,r) - f(-k,0)f(k,r)\}\,,$$

where $f(k,r)$ is the Jost function with asymptotic behaviour $e^{ikr}f(k,r) \underset{r=\infty}{\rightarrow}$ The S-matrix can be written as

$$S(k) = \frac{f(k, 0)}{f(-k, 0)} = \frac{f(k)}{f(-k)}$$

For the two potentials listed above, the Jost functions are [2]

$$f_1(k, r) = e^{-ikr} \frac{2k + i\lambda\left(\dfrac{\beta e^{-\lambda r} - 1}{\beta e^{-\lambda r} + 1}\right)}{2k - i\lambda} \tag{2.5}$$

$$f_2(k, r) = e^{-ikr} \frac{\dfrac{4k^2 - 12ik(r-\alpha)^2}{(r-\alpha)^3 + \gamma^3} - \dfrac{12(r-\alpha)}{(r-\alpha)^3 + \gamma^3}}{}$$

With proper choice of α and γ these two Jost functions lead to the same S-matrix

$$S(k) = S_1(k) = S_2(k) = \frac{(2k + i\nu)\,(2k + i\lambda)}{(2k - i\lambda)\,(2k - i\nu)}$$

where

$$\nu = \lambda \frac{\beta - 1}{\beta + 1} < 0$$

It will be seen that this S-matrix has two poles on the imaginary axis in the complex momentum plane – one of them being in the lower half plane at $k = \frac{1}{2}i\nu$ (note that $\nu < 0$) and the other in the upper half plane at $k = \frac{1}{2}i\lambda$. The latter should correspond to a bound state but this is not always the case, as will be clear from the following discussion. One finds from eq. (2.4) that the poles of the S-matrix in the upper half plane correspond either to the zero of $f(k)$ in the lower half plane or to the pole of $f(k)$ in the upper half plane. For the potential V_1 (See eq. (2.1) we have from eq. (2.5):

$$f_1(k) = \frac{2k + i\nu}{2k - i\lambda}$$

Since $\nu < 0, f_1(k)$ can never have any zero in the lower half imaginary axis. However it has a pole in the upper half plane, i.e., at

$$k = \tfrac{1}{2}\lambda \qquad \lambda > 0\,,$$

which corresponds to a pole of the S-matrix in the upper half-plane. This would ordinarily imply the existence of a proper bound state. It may be noted, however, that at this point the wave function, ϕ_1, computed from (2.5) and (2.3) vanishes identically. The pole of the S-matrix appearing through the infinity of $f_1(k)$ in the upper half-plane leads to a redundant state; and the real bound state corresponds to a vanishing of $f_1(k)$ in the upper half-plane. Furthermore, if we now calculate the completeness integral

$$\int_0^\infty \phi_1^*(k,r)\,\phi_1(k,r^1)\,\mathrm{d}k$$

using eqs. (2.3), and (2.5) and employing Ma's techniques we obtain

$$\int_0^\infty \phi_1^*(kr)\phi_1(kr^1)\,\mathrm{d}k = \int_0^\infty g_1(k,r,r^1)\mathrm{d}k = \delta(r \quad r^1)$$

$$+\, 2\pi i \,\mathrm{Res}\, g_1(k,r,r^1)|k = \tfrac{1}{2}i\lambda \tag{2.10}$$

The residues of $g_1(k,r,r^1)$ at the pole $k = \frac{1}{2}i\lambda$ is found to vanish. The completeness condition, therefore, becomes

$$\int_0^\infty \phi_1^*(kr)\phi_1(kr^1)\mathrm{d}k = \delta(r - r^1)\,, \tag{2.11}$$

showing thereby that the redundant state does not contribute to the completeness relation. The Heisenberg condition (discussed earlier) consistent with this completeness relation should be

$$\int_{-\infty}^{\infty} S(k)\,\mathrm{e}^{ikr}\mathrm{d}k = 0 \tag{2.12}$$

However, on actual computation, using the expression for $S(k)$ given in eq. (2.7) we find

$$\int_{-\infty}^{\infty} S(k)\,\mathrm{e}^{ikr}\mathrm{d}k$$

$$= 2\pi i \;\mathrm{Res}\;\; S(k)\,\mathrm{e}^{ikr}|_{k=\frac{1}{2}i\lambda}$$

$$= -\,4\pi\lambda\beta\mathrm{e}^{-\frac{1}{2}\lambda r} \qquad \beta < 0 \tag{2.13}$$

This violation of the Heisenberg's condition is brought about by the presence of the redundant state.

For the potential V_2 we have from eq. (2.6)

$$f_2(k) = \frac{(2k + i\lambda)\,(2k + i\nu)}{4k^2} \tag{2.14}$$

which has a zero at $k = \frac{1}{2}i\lambda$ but no pole in the upper half plane. That is, we have only one bound state and no redundant state. The completeness relation becomes in this case

$$\int_0^\infty \phi_2^*(k_1 r)\phi_2(k, r^1)\,\mathrm{d}k = \delta(r - r^1) - \phi_2^*(\tfrac{1}{2}i\lambda, r)\,\phi_2(\tfrac{1}{2}i\lambda, r^1)$$

The *Heisenberg condition* for this case *is same* as that of the previous case *and is compatible with the completeness condition.*

The above example shows a close connection between the shadow state of the quantum field theory and the redundant state in potential theory in that, the redundant state makes its appearance through dynamics, nevertheless it does not manifest itself in the completeness integral. Furthermore the S-matrix which has a well defined analytic property, fails to distinguish between different types of interactions.

3. SEPARABLE POTENTIAL AND THE LEE MODEL

We shall now construct a separable potential model in which there occur two redundant poles of the S-matrix and show that a modified Lee Model can be constructed which has identical S-matrix but in which these poles correspond to genuine bound states. In the Hilbert space of functions $\psi(k)$ with

$$\int_1^\infty \mathrm{d}k\,\rho(k)\,\psi^*(k)\,\psi(k) < \infty$$

the Hamiltonian matrix of the separable potential theory is

$$\omega(k)\,\delta(k - k^1) + \eta g(k^1)\,, \tag{3.1}$$

where η and $g(k)$ are yet to be specified. The eigenvalue equation is solved by

$$\psi_q(k) = \delta(k - q) + \frac{\eta P(k)\,g(q)\,g(k)}{(\omega(q) - \omega(k) + i\epsilon)\,\beta(\omega(q) + i\epsilon)}\,, \tag{3.2}$$

in

$$\beta(z) = 1 + \eta \int_{1}^{\infty} \mathrm{d}k \frac{g^2(k)\,\rho(k)}{\omega(k) - z} \tag{3.3}$$

The scattering matrix is

$$S(\omega_q) = \frac{\beta(\omega_q - i\epsilon)}{\beta(\omega_q + i\epsilon)} = 1 + \frac{2\pi\rho(q)g^2(q)}{\beta(\omega_q + i\epsilon)} = \quad + \frac{2\pi i N(q)}{D(q)} \tag{3.4}$$

$$N(q) = \rho(q)\,g^2(q)\,, \qquad D(q) = \beta(\omega_q) \tag{3.5}$$

Poles of the S-matrix can arise as a result of the vanishing of the D-function or due to a pole of the N-function. We now choose η to be positive so that D-function given by eq. (3.3) does not vanish so that there are no genuine bound states. With the choice

$$g^2(k) = \frac{f^2(k)\,\{m_2 + \xi m_1 - \omega(k)\}}{(\omega(k) - M_1)(\omega(k) - M_2)}\,, \tag{3.6}$$

the N-function has two poles at $z = M_1$ and $z = M_2$ which are also poles of the S-matrix. However the scattering states $\psi_q(k)$ by themselves can be easily shown to be complete [4]. Hence these two poles are redundant.

We now construct a phase equivalent system with the scattering states labelled by the same continuous momentum label but for which there are two genuine bound states. This system is a modified Lee model with discrete states V_1, V_2 and continuum $N\theta$ states with a Hilbert space spanned by 3-component wave functions

$$\{U_\lambda^{(1)}, U_\lambda^{(2)}, \phi_\lambda(k)\}\,.$$

with the inner product

$$(U_\lambda^{(1)*}, U_\lambda^{(1)}) + \xi(U_\lambda^{(2)*}, U_\lambda^{(2)}) + \int^{\infty} \rho(k)\phi^*_{\lambda'}(k)\phi_\lambda(k)\,\mathrm{d}k\,,$$

and for which the Hamiltonian matrix is

$$\begin{vmatrix} m_1 + \Delta_{11} & \Delta_{12} & f(k^1) \\ \Delta_{21} & m_2 + \Delta_{22} & \\ f^*(k) & \xi f^*(k) & \omega(k)\,\delta(k-k^1) \end{vmatrix} , \tag{3.7}$$

where Δ is the mass renormalization matrix and $\omega(k)$ is the energy function. Temporarily ignoring Δ, the eigenvalue equations are

$$(\omega(q) - m_1^0)\, U_1(q) = \int \mathrm{d}k^1 f(k^1)\, \rho(k^1)\, \phi_q(k^1)\, r_1(q)\,,$$

$$(\omega(q) - m_2^0) U_2(q) = \int \mathrm{d}k^1 f(k^1)\, \rho(k^1)\, \phi_q(k^1)\, r_2(q)$$

$$(\omega(q) - \omega(k)\; \phi_q(k) = f^*(k)\; U_1(q) + \xi f^*(k)\; U_2(q)\,, \tag{3.8}$$

and their solutions are

$$\phi_q(k) = \frac{U_1(q) f^*(k) + \xi\, U_2(q) f^*(k)}{\omega(q) - \omega(k) + i\epsilon} + \delta(k-q)\,,$$

$$U_1(q) = \frac{(\omega(q) - m_2) f(q)\, \rho(q)}{(\omega(q) - m_1 - \gamma)(\omega(q) - m_2 - \xi\gamma) - \xi\gamma^2} \tag{3.9}$$

$$U_2(q) = \frac{(\omega(q) - m_1) f(q)\, \rho(q)}{(\omega(q) - m_1 - \gamma)(\omega - m_2 - \xi\gamma) - \xi\gamma^2}$$

where

$$\gamma(z) = \int_1^\infty \mathrm{d}k \frac{|f(k)|^2\, \rho(k)}{z - \omega(k)}\,. \tag{3.10}$$

The T-matrix can be evaluated by taking the coefficient of the $(\omega_q \quad \omega_L + i\epsilon)^{-1}$ term taken on the energy shell to yield

$$T(\omega_q + i\epsilon) = f(q)\, (U_1(q) + \xi U_2(q)) \Big|_{\omega_q = \omega_k}$$

$$= \frac{\rho(q) f^2(q) \{(\omega(q) - m_2) + \xi(\omega - m_1)\}}{(\omega(q) - m_1 - \gamma)(\omega - m_2 - \xi\gamma) - \xi\gamma^2} \tag{3.11}$$

from which we get

$$S = 1 + 2\pi i T = \frac{2\pi i p f^2 \{(\omega - m_2) + \xi(\omega - m_1)\}}{(\omega - m_1 - \gamma^*)(\omega - m_2 - \xi\gamma) - \xi\gamma^2}$$

$$\frac{(\omega - m_1 - \gamma^*)(\omega - m_2 - \xi\gamma^*) - \xi\gamma^{*2}}{(\omega - m_1 - \gamma)(\omega - m_2 - \xi\gamma) - \xi\gamma^2}$$

Define the denominator function

$$D(z) = (z \quad m_1)(z - m_2) - \gamma(z)(z - m_2) + \xi(z - m_1\}$$

In the limit of sufficiently small coupling, there are obviously two zeroes in the vicinity of m_1 and m_2. Without bothering to investigate the detailed conditions under which these two zeroes continue to exist, let us assume that there are in fact two zeroes at two points $z = M_1$, and $z = M_2$. Then by a trivial series of manipulations we can write:

$$D(z) = (z - M_1)(z \quad M_2) F(z) ,$$

$$F(z) = 1 + \int \frac{|f(k)|^2 (m_2 + \xi m_1 - \omega(k_1)\, \rho(k)\, \mathrm{d}k}{(z - \omega(k))(M_1 - \omega(k))(M_2 - \omega(k))} ,$$

and the bare mass parameters m_1 and m_2 can be determined from the simultaneoı equations

$$D(M \quad = (m_1 - M_1)(m_2 - M_1) + \gamma(M_1)(\xi m_1 + m_2 - (1 + \xi)M_1) = 0$$

$$D(M_2) = (m_1 - M_2)(m_2 - M_2) + \gamma(M_2)(\xi m_1 + m_2 - (1 + \xi)M_2) = 0 , \qquad (3.14)$$

which, when considered as equations in m_1, m_2 are purely algebraic, whose solutions are essentially unique.

The S-matrix for this model for real physical energies can be written in the form

$$S(\omega_q + i\epsilon) = \frac{D(\omega + i\epsilon)}{D(\omega + i\epsilon)} = \frac{F(\omega - i\epsilon)}{F(\omega + i\epsilon)} , \qquad (3.15)$$

which is identical to the S-matrix for the separable potential case given by eq. (3.4) with the choice (3.6) of $g^2(k)$. Off the real axis this function has the analytic continuation

$$S(z) = \frac{F(z) + 2\pi i \{|f|^2 (m_2 + \xi m_1 - z)\, \rho \frac{\mathrm{d}k}{\mathrm{d}\omega} z/(z - M_1)(z - M_2)\}}{F(z)} \qquad (3.16)$$

This analytic function has, in addition to the unitarity branch cut along the real axis for +1 to +∞, and any geometric singularities introduced by the form factor $f(k)$, two poles at $z = M_1$ and $z = M_2$. (The singularities of the phase space factor are only along the unitarity cut). Without loss of generality we may ignore the form factor in these considerations. We may then assert that all the singularities of the S-matrix are associated with physical states – the unitarity cut with N scattering states; the two poles with the two genuine bound states. These states have the wave functions

$$U_1^{\mathrm{I}} = N^{\mathrm{I}}(M_1 - m_2)\,,$$

$$U_2^{\mathrm{I}} = N^{\mathrm{I}}(M_1 - m_1)\,,$$

$$\phi^{\mathrm{I}}(k) = \frac{N^{\mathrm{I}} f(k)(M_1 - m_2) + \xi(M_1 - m_1)}{M_1 - \omega(k)} \tag{3.17}$$

$$U_1^{\mathrm{II}} = N^{\mathrm{II}}(M_2 - m_2)$$

$$U_2^{\mathrm{II}} = N^{\mathrm{II}}(M_2 - m_1)$$

$$\phi^{\mathrm{II}}(k) = N^{\mathrm{II}} \frac{f(k)(M_2 - m_2) + \xi(M_2 - m_1)}{M_2 - \omega(k)} \tag{3.18}$$

The normalization constant N^{I} is determined by the criterion

$$|1/N^{\mathrm{I}}|^2 = (M_1 - m_2)^2 + \xi(M_1 \quad m_1)^2 + (M_1 - m_2) + \xi(M_1 - m_1)^2$$

$$\times \int \frac{f(k)^2\, \rho(k)\, \mathrm{d}k}{(M_1 - \omega(k))^2} \tag{3.19}$$

with a similar relation for N^{II}. These normalized wave functions together with the continuum of scattering states labeled by q with wave functions

$$U_1^q = f(q)\,\rho(q)\,(\omega(q) - m_2)\,/\,D(\omega(q) + i\epsilon)$$

$$U_2^q = f(q)\,\rho(q)\,(-\,m_1)\,/\,D(m(q) + i\epsilon)$$

$$\phi^q(k) = \delta(k - q) + \frac{\rho(q)\, f(q)\, f(k)\,\{\omega(q) - m_2 + \xi(\omega(q) - m_1)\}}{D(\omega(q) + i\epsilon)\,(\omega(q) - \omega(k) + i\epsilon)} \tag{3.20}$$

can be shown to be complete by a straight-forward but tedious calculation. Hence for this theory, the poles of the S-matrix correspond to physical states. There are no redundant singularities of the S-matrix. We note that if $\xi < 0$, then one of the bound states has negative norm. Thus we see that the redundant poles of the S-matrix for separable potential theory become genuine bound state poles in a modified Lee model.

4. BOUND STATES REDUCED TO THE STATUS OF REDUNDANT STATES

In the previous section we discussed a solvable model with a continuum of scattering states with the S-matrix

$$S(\omega + i\epsilon) = F(\omega \mp i\epsilon) \,/\, F(\omega + i\epsilon)\,,$$

$$F(z) = \quad + \int \mathrm{d}k\, \rho(k)\, |f(k)|^2 \frac{m_2 + m_1 \xi - \omega(k)}{(z - \omega(k))\,(M_1 - \omega(k))\,(M_2 - \omega(k))}\,.$$

The solvable model had, in addition to the scattering states, two bound states at $z = M_1$ and $z = M_2$, the first one with positive norm and the second one with positive or negative norm according as $\xi = +1$ or -1. We wish to avoid having such bound states and have only the continuum states but we wish to retain the same S-matrix. Such a construction is not only of interest in itself; it has been found some time ago that the simplest resolution to the ills of local field theory was to introduce an indefinite metric into the theory and avoid the difficulties of physical interpretation by redefining the set of physical states. The identification of physical states in such a theory is part of the dynamical problem.

We pose ourselves the following problem: Given the S-matrix derived from a theory with two bound states in addition to the continuum of scattering states, construct another theory which yields the same S-matrix but which has only a continuum of scattering states as the complete set of states.

Consider a Hilbert space of (two-particle) states by a continuous momentum label (we restrict attention to S-waves) with the scalar product

$$(\psi_1, \psi_2) = \int \mathrm{d}k\, \rho(k)\, \psi_1^*(k)\, \psi_2(k)\,, \tag{4.1}$$

$$(\psi, \psi) < \infty$$

For the Hamiltonian we choose the linear operator

$$H\psi(k) = \omega(k)\, \psi(k) + \eta g^*(k) \int \mathrm{d}k^1 \rho(k^1)\, g(k^1)\, \psi(k^1)\,.$$

The eigenstates of the Hamiltonian can be easily written down in the forms

$$\psi_\lambda(k) = \delta(q - \omega(k)) = \frac{\eta g(q)\, g^*(k)}{(\lambda - \omega(k) + i\epsilon)\, \beta(\lambda + i\epsilon)},$$

$$H\psi_\lambda = \lambda\psi_\lambda = \omega(q)\psi_\lambda\,,$$

$$\beta(z) = \quad + \eta \int \frac{\mathrm{d}k\, |g(k)|^2\, \phi(k)}{(\omega(k) - z)}$$

These states can be verified to be complete in the sense

$$\psi_\lambda(k)\psi^*_\lambda(k^1)\mathrm{d}\lambda = \delta(k - k^1)$$

provided $\beta(z)$ has no zeroes. From the structure of $\beta(z)$ it can have at most one zero and this happens only if η is negative (attractive potential) and the interaction is sufficiently strong. (This will not happen in our case – see below.) Hence the scattering states, with the bound state if it exists, form a complete set of physical states.

The S-matrix for this system can be calculated in a straightforward manner. We get

$$S(\omega + i\epsilon) = \frac{\beta(\omega - i\epsilon)}{\beta(\omega + i\epsilon)}$$

Comparing eq. (4.5) with the expression (4.1) we recognize that the model solves the problem posed above, provided

$$\eta\, |g(k)|^2 = \frac{m_2 + \xi m_1 - \omega(k)\, |f(k)|^2}{[M_1 - \omega(k)]\, [M_2 - \omega(k)]}\,.$$

The poles of the S-matrix at $z = M_1$ and $z = M_2$ are now seen to be redundant and stemming from the "geometry of the interaction" i.e., the structure of the form factors, in very close similarity with the case of the local potentials like the exponential and the Bargman potentials.

We note in passing that the *unwanted states* of one theory *become embedded in the geometry of the non-local structure functions* of the reduced theory; and this is the situation that we encounter in the *quantum theory of action at-a-distance* [5].

5. CONCLUDING REMARKS

In most cases of quantum scattering theory, the singularities of the scattering amplitude are closely related to the physical spectrum of states. In particular two-body scattering amplitudes have poles corresponding to discrete bound states and branch cuts corresponding to the continuum of scattering states. The onset of a new threshold, say, one due to particle production, is reflected in the onset of new singularities in the scattering amplitude. This association seems so natural that in recent years this correspondence has been elevated to the status of a central axiom in the S-matrix formulation of relativistic particle theories. In fact this physical identification of singularities together with the principle of crossing symmetry constitute the foundation of S-matrix theory of strong interactions.

The study undertaken in this paper shows how cautious one has to be in such a pursuit. Not all singularities are to be interpretable in terms of physical states. We have restricted attention in this paper to poles and their relationship to discrete physical states and shown such a correspondence is not always valid. But, then, the residues of the poles do not always have to have a unique sign since they may not be associated with a physical bound state (of positive norm). It follows that any "theorem" stating a suitable inequality on suitable functionals of the scattering amplitude, including upper limits on renormalized coupling constants, need not be valid in general.

The situation is clearly more general. What if we had branch cuts which were not associated with unitarity either in the direct or in the crossed channels? Could we then not have branch cuts with the "wrong" sign of the discontinuity? Indeed we can and they do occur in indefinite metric field theories with suitable physical interpretations. We had long advocated the need for enlarging the linear space of mathematical states to include some which are associated with a negative metric; and insisted that the physical interpretation includes as an integral step the proper identification of physical states which would be a subset of the mathematical states. We have then a larger space of mathematical states and a smaller subspace of physical states. The unphysical states, called "shadow states" in previous expositions, are by no means irrelevant since they serve to determine the dynamical theory, but they do not enter into the complete set of states needed in the context of unitarity. The redundant poles are the simples manifestations of shadow states.

In the above study we have seen that the redundant states of a quantum system represent the geometry of a non-local interaction. Elsewhere we have developed the generalization of this relationship to relativistic field theories and shown that the shadow quanta represents the geometry of action-at-a-distance.

This insight into the structure of quantum field theory suggests that the negative theorems of quantum field theory are not so decisive! It is after all, not very rational to impose "physical" requirements on the shadow states. There is reason to be optimistic about local relativistic quantum field theory yet.

Two of us (S.N.B. and T.P.) would like to thank Professor E.C.G. Sudarshan for the hospitality at the Center for Particle Theory, University of Texas at Austin during the summer of 1970.

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