Statistical mechanics of parafermi systems

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The statistical mechanics of a collection of parafermi harmonic oscillators considered as a canonical ensemble is examined. The factorization property of the corresponding partition function selects the representations of the parafermi ring allowed to furnish the energy levels of the oscillators. While the representations corresponding to Green's ansatz are in an obvious way perfect candidates, it is shown however that, contrary to expectations, they do not exhaust all the possibilities. The general conditions on the representations are given and special families of representations are outlined.

I. INTRODUCTION

Following the classic work of Green, paraquantization has received special attention.[1] The representation of parafermi rings for an arbitrary finite number of degrees of freedom has been completely worked out by Ryan and Sudarshan.[2] They have shown that a parafermi ring for ν degrees of freedom is realized by the representations of the orthogonal group B_ν in $2\nu+1$ dimensions. In particular, the Green ansatz of order p is the pth Kronecker power of the fundamental spinor representation ${}^\nu\Delta$ of B_ν. For $p=1$, we get the standard fermi system represented by ${}^\nu\Delta$ of dimension 2^ν; this representation is, of course, irreducible.

For $p>1$, the Green ansatz does not furnish irreducible representations. In a sense the Green ansatz for a parafermi system of order p may be thought of as a fermi ring with $\nu\cdot p$ degrees of freedom with a "hidden" label which can take on distinct values. This may be seen in the original construction of Green.[1]

It is therefore not surprising that if we are to consider the statistical mechanics of a system of parafermi oscillators realized by the Green ansatz no Gibbs paradox[3] arises: The partition function for $\nu=\nu_1+\nu_2$ noninteracting oscillators at any fixed temperature β^{-1} equals the product of the partition functions for ν_1 oscillators at the same temperature β^{-1} and that for ν_2 oscillators. The entropy in this case is thus simply additive.

It is therefore relevant to ask what are the general requirements on the representations of the orthogonal group realizing the parafermi oscillator system for which statistical mechanics can be studied without the Gibbs paradox arising. We propose to investigate this question in this paper. Specifically, we consider a collection of ν parafermi harmonic oscillators, the energy levels of which being given by some representation of the parafermi ring involving ν degrees of freedom. As a canonical ensemble, this system is seen to be in equilibrium with a heat bath at a given temperature. If we consider this system as constituted by two separate subsystems, then the partition function of the whole system can be expressed as the product of the partition function of the two subsystems. This factorization property of the partition function will select the representations of the parafermi ring allowed to describe the ν parafermi harmonic oscillators.

As mentioned above, one can see easily that the representations corresponding to Green's ansatz are perfect candidates. But in this case the system under consideration is essentially of *Fermi type* since it looks like a collection of $\nu\cdot p$ Fermi harmonic oscillators, p being the order of the Green ansatz. If the only solutions were given by Green's ansatz, no system of "essentially parafermi" oscillators should exist. It is worthwhile to see whether or not Green's ansatz furnishes the only representations compatible with the factorization property of the partition function.

In the next section, we show that indeed Green's ansatz leads trivially to possible solutions. Then, in Sec. III we give a general treatment, and it is seen that in fact there exists representations distinct from Green's ansatz for which the factorization property of the partition function holds. As an illustration, some examples are investigated. In the last section, we summarize our results.

II. SYSTEM OF PARAFERMI HARMONIC OSCILLATORS

A collection of ν parafermi harmonic oscillators is defined through the hamiltonian

$$H=\frac{1}{2}\sum_{k=1}^{\nu}\omega_k(a_k^+a_k-a_ka_k^+), \tag{II.1}$$

where a_k^+ and a_k are creation and annihilation operators satisfying the parafermi commutation relations

$$[[a_k,a_l^+],a_j]=2\delta_{lj}a_k, \tag{II.2}$$

$$[[a_k,a_l],a_j]=0 \tag{II.3}$$

for all $j,k,l=1,2,\ldots,\nu$.

From the representation theory[2] of parafermi rings, we know that the representations of the operators are obtained by means of the isomorphism between the Lie algebra of the orthogonal group B_ν in $2\nu+1$ dimensions and the Lie algebra generated by the operators

$$H_k=\tfrac{1}{2}[a_k^+,a_k], \tag{II.4a}$$

$$Q_{+k}=a_k^+, \tag{II.4b}$$

$$Q_{-k}=a_k, \tag{II.4c}$$

together with

$$N_{\pm k}\equiv[Q_{\pm k},Q_{\pm k}],\quad N_{\pm k,\pm k'}\equiv[Q_{\pm k},Q_{\mp k'}],\quad k\neq k'.$$

We use here the standard basis of the Lie algebra of B_ν for which the H_k are diagonal in the adjoint representation. In other words, each unitary representation of B_ν leads through (II.4) to a representation of the parafermi ring (generated by the a_k and their adjoints a_k^+).

Using (II.4a), the Hamiltonian (II.1) reads

$$H=\sum_{k=1}^{\nu}\omega_k H_k, \tag{II.5}$$

the operators H_k being a complete set of commuting operators which generate a maximal Abelian subalgebra of B_ν. Thus from the definition of the partition function of a canonical ensemble, we see that the partition function of the ν parafermi harmonic oscillators $Z(\beta,\nu)$ takes the form

$$Z(\beta,\nu)=\mathrm{Tr}\left(\prod_{k=1}^{\nu}e^{-\beta\omega_k H_k}\right)\quad\text{with } \beta=\frac{1}{\kappa T} \tag{II.6}$$

in some representation of B_ν.

Let us denote by (ν) the system of ν parafermi harmonic oscillators and consider two systems (ν_1) and (ν_2) both in equilibrium with a heat bath at a given temperature T. Then, from the additivity of the entropy,[3,4] one deduces that the partition function of the system $(\nu_1+\nu_2)$ is the product of the partition functions $Z(\beta,\nu_1)$ and $Z(\beta,\nu_2)$ of (ν_1) and (ν_2), repectively, that is

$$Z(\beta,\nu_1+\nu_2)=Z(\beta,\nu_1)\times Z(\beta,\nu_2). \tag{II.7}$$

This factorization property will select the representations of the parafermi ring capable of describing the system of ν parafermi oscillators.

As a particular case, let us consider the ensemble of ν *Fermi* harmonic oscillators. The corresponding representation of B_ν is the fundamental spinor representation ${}^{\nu}\Delta$ in which the eigenvectors of the Hamiltonian (I.1) reads $|\pm\frac{1}{2},\pm\frac{1}{2},\ldots,\pm\frac{1}{2}>$, where the plus and minus signs are *arbitrarily* distributed, the labels $\pm\frac{1}{2}$ referring to the eigenvalues of the H_k defined in (II.4a). Hence using the definition (II.6) of the partition function, its expression reads

$$Z(\beta,\nu)=2^{\nu}\prod_{k=1}^{\nu}\cosh\beta\omega_k,$$

which shows that, in fact, the partition function factorizes entirely, each parafermi harmonic oscillator being in the fundamental spinor representation ${}^{1}\Delta\equiv{}^{1}D_{1/2}$ of $O(3)$. Since the Green ansatz of order p is given by the Kronecker product $[{}^{\nu}\Delta]^{p}$, it immediately follows that Green's ansatz leads to the factorization of the corresponding partition function.

So, the representations corresponding to Green's ansatz are in an obvious way solutions for our problem. We shall show in the next section that in fact the complete factorization of the partition function into the product of the "particle" partition functions is a necessary condition.

III. GENERAL TREATMENT

Let us investigate the consequences of the factorization (II.7) of the partition function. A representation ${}^{\nu_1+\nu_2}D$ of the system $(\nu_1+\nu_2)$ being given, we denote by $|L>$ the eigenstates of the corresponding Hamiltonian (II.1) with $L=(l_1,l_2,\ldots,l_{\nu_1+\nu_2})$, l_k being an eigenvalue of H_k. The energy of the state $|L>$ then reads

$$E_L=\sum_{k=1}^{\nu_1+\nu_2}\omega_k l_k.$$

Likewise, let $|L_1>$ and $|L_2>$ be the eigenstates of the Hamiltonians of the systems (ν_1) and (ν_2), respectively. Since the Hamiltonian of the whole system is simply the sum of the Hamiltonians for each subsystem, to the state $|L>$ there corresponds two states $|L_1>$ and $|L_2>$ such that $L_1=(l_1,l_2,\ldots,l_{\nu_1})$ and $L_2=(l_{\nu_1+1},\ldots,l_{\nu_1+\nu_2})$. But the factorization property (II.7) written in the form

$$Z(\beta,\nu_1+\nu_2)=\sum_{L_1}\sum_{L_2}\exp[-\beta(E_{L_1}+E_{L_2})], \tag{III.1}$$

where the summations on L_1 and L_2 are *independent*, shows that, conversely, to the states $|L_1>$ and $|L_2>$ there corresponds a state $|L>$ such that $L=(L_1,L_2)$. In other words, the weight space of the representation ${}^{\nu_1+\nu_2}D$ must be the direct product of the weight spaces of the representations ${}^{\nu_1}D$ and ${}^{\nu_2}D$ of the system (ν_1) and (ν_2). This property can be recasted in terms of characters. Indeed, let ${}^{\nu_1+\nu_2}\chi$ be the character of the representation ${}^{\nu_1+\nu_2}D$. It is defined[5] by

$$\begin{aligned}{}^{\nu_1+\nu_2}\chi(\phi_1,\phi_2,\ldots,\phi_{\nu_1+\nu_2})\\ =\sum_{(l_1,l_2,\ldots,l_{\nu_1+\nu_2})}\exp[i(l_1\phi_1+l_2\phi_2+\cdots+l_{\nu_1+\nu_2}\phi_{\nu_1+\nu_2})]\end{aligned} \tag{III.2}$$

with $\phi_i\in[0,2\pi]$. It then follows that ${}^{\nu_1+\nu_2}\chi$ can be written as

$$\begin{aligned}{}^{\nu_1+\nu_2}\chi(\phi_1,\ldots,\phi_{\nu_1+\nu_2})\\ =\sum_{(l_1,\ldots,l_{\nu_1})}\exp[i(l_1\phi_1+\cdots+l_{\nu_1}\phi_{\nu_1})\\ \times\sum_{(l_{\nu_1+1},\ldots,l_{\nu_1+\nu_2})}\exp[i(l_{\nu_1+1}\phi_{\nu_1+1}+\cdots+l_{\nu_1+\nu_2}\phi_{\nu_1+\nu_2})].\end{aligned}$$

The right-hand side is seen to be the product of the characters ${}^{\nu_1}\chi$ and ${}^{\nu_2}\chi$ of the representations ${}^{\nu_1}D$ and ${}^{\nu_2}D$. We thus have

$$\begin{aligned}{}^{\nu_1+\nu_2}\chi(\phi_1,\ldots,\phi_{\nu_1+\nu_2})={}^{\nu_1}\chi(\phi_1,\ldots,\phi_{\nu_1})\\ \times{}^{\nu_2}\chi(\phi_{\nu_1+1},\ldots,\phi_{\nu_1+\nu_2}).\end{aligned} \tag{III.3}$$

Therefore, the factorization property (II.7) is equivalent to the factorization (III.3) of the character of the representation ${}^{\nu_1+\nu_2}D$ describing the whole system. From (III.3) we deduce easily that the character ${}^{\nu_1+\nu_2}\chi$ factorizes entirely into the product $\Pi_{k=1}^{\nu=\nu_1+\nu_2}{}^{1}\chi(\phi_k)$ in which ${}^{1}\chi$ is the character of the representation ${}^{1}D$ describing each parafermi harmonic oscillator. But then the partition function $Z(\beta,\nu)$ factorizes entirely, namely,

$$Z(\beta,\nu)=\prod_{k=1}^{\nu}Z_k(\beta); \tag{III.4}$$

$Z_k(\beta)$ being the partition function of the kth oscillator. This result is not surprising since it shows that the factorization property (II.7) is actually independent of the choice of the two subsystems (ν_1) and (ν_2) as it was expected.

Finally, the system of ν parafermi harmonic oscillators will be conveniently described by the representation ${}^{\nu}D$ such that their corresponding character ${}^{\nu}\chi$ can be written as

$${}^{\nu}\chi(\phi_1,\ldots,\phi_\nu)=\prod_{k=1}^{\nu}{}^{1}\chi(\phi_k); \tag{III.5}$$

${}^{1}\chi$ being the character of the representation ${}^{1}D$ of $O(3)$ which gives the energy levels of each oscillator. In fact, Eq. (III. 5) gives a way to construct the representations ${}^\nu D$ if we know the representation ${}^1D$ for each oscillator. Therefore, a representation ${}^1D$ and its character ${}^1\chi$ being given, we have to see if the product $\Pi_{k=1}^{\nu}\,{}^1\chi(\phi_k)$ is either a character for some representation ${}^\nu D$ of B_ν or not.

First of all, let us note that if we consider a representation ${}^1D$ which contains *both* integer and half integer spins, then the product $\Pi_{k=1}^{\nu}\,{}^1\chi(\phi_k)$ will *not* be a character for B_ν. Indeed, we know that a character ${}^\nu\chi$ of B_ν reads

$$ {}^\nu\chi(\phi_1,\dots,\phi_\nu)=\sum \exp[i(j_1\phi_1+\cdots+j_\nu\phi_\nu)], \qquad \text{(III. 6)} $$

where the components $j_1, j_2, \dots, j_\nu$ of the weights are either all integers or all half integers. But Eq. (III. 5) tells us that $j_1, j_2, \dots, j_\nu$ are, in fact, the weights contained in ${}^1\chi$ so that ${}^1\chi$ must contain either integer or half integer spins.

So we define ${}^1D$ as

$$ {}^1D\equiv\sum_{j=\epsilon}^{l} a_j\,{}^1D_j, \qquad \text{(III. 7)} $$

where ${}^1D$ denotes the irreducible representation of spin j of $O(3)$, a_j being its multiplicity and ϵ is 0 or $\frac{1}{2}$ according to whether we are considering integer or half-integer spins, respectively, l being the highest spin in ${}^1D$ assuming $a_l\neq 0$. The character ${}^1\chi$ of ${}^1D$ reads

$$ {}^1\chi(\phi)=\sum_{k=\epsilon}^{l} c_k\cos k\phi \qquad \text{(III. 8)} $$

with

$$ c_k=2\sum_{j=k}^{l} a_j \quad \text{if } k\geqslant 1 \text{ or } k=\epsilon=\tfrac{1}{2}, $$

$$ c_0=\sum_{j=0}^{l} a_j \quad \text{when } \epsilon=0. $$

In order to identify a character of B_ν with the product (III. 5), let us consider its general expression recalling that a finite-dimensional unitary representation of B_ν is fully reducible, that is, of the form

$$ {}^\nu D=\sum_{\{k\}} a_{\{k\}}\,{}^\nu D_{\{k\}}, \qquad \text{(III. 9)} $$

where ${}^\nu D_{\{k\}}$ denotes the irreducible representation of B_ν with highest weight

$$ \{k\}=\{k_1,\dots,k_n\}k_1\geqslant k_2\cdots\geqslant k_\nu\geqslant 0; \qquad \text{(III. 10)} $$

$k_1, k_2, \dots, k_\nu$ being either all integers or all half integers. From (III. 9), we see that the character ${}^\nu\chi$ of ${}^\nu D$ reads

$$ {}^\nu\chi(\phi_1,\dots,\phi_\nu)=\sum_{\{k\}} a_{\{k\}}\frac{A_{\{k\}}(\phi_1,\dots,\phi_\nu)}{A_0(\phi_1,\dots,\phi_\nu)}, \qquad \text{(III. 11)} $$

where

$$ {}^\nu\chi_{\{k\}}(\phi_1,\dots,\phi_\nu)=\frac{A_{\{k\}}(\phi_1,\dots,\phi_\nu)}{A_0(\phi_1,\dots,\phi_\nu)} \qquad \text{(III. 12)} $$

is the simple character of B_ν corresponding to the representation ${}^\nu D_{\{k\}}$, $A_{\{k\}}$ and A_0 being alternating elementary sums[6] (A. E. S.) corresponding to the highest weight $\{k\}$ and to the weight $\{0, 0, \dots, 0\}$ of the identify representation, respectively. Let us recall that an A. E. S. of B_ν is given by

$$ A_{\{k\}}(\phi_1,\dots,\phi_\nu)=\sum_{\sigma\in S_\nu} P(\sigma)\sin l_{\sigma1}\phi_1\sin l_{\sigma2}\phi_2\cdots\sin l_{\sigma\nu}\phi_\nu, $$

where S_ν is the permutation group of ν objects, $P(\sigma)$ the partity of the permutation σ, and $l_i=k_i+\nu+\frac{1}{2}-i$.

In order to satisfy Eq. (III. 5), we have to prove that the quantity

$$ A(\phi_1,\dots,\phi_\nu)\equiv A_0(\phi_1,\dots,\phi_\nu)\prod_{k=1}^{\nu}{}^1\chi(\phi_k), \qquad \text{(III. 13)} $$

where ${}^1\chi$ is given in (III. 8), is a sum of A. E. S. *with positive coefficients*. A simple but lengthy computation shows that the quantity (III. 13) is indeed a sum of A. E. S. Let us indicate the proof: From the expression of $A_0(\phi_1,\dots,\phi_\nu)$ and Eq. (III. 8), we get

$$ \begin{aligned} A(\phi_1,\dots,\phi_\nu)=\frac{1}{2^\nu}&\sum_{j_1,j_2,\dots,j_\nu=\epsilon}^{l} c_{j_1}c_{j_2}\cdots c_{j_\nu}\sum_{\sigma\in S_\nu}P(\sigma)\\ &\times[\sin(r_{\sigma_1}+j_1)\phi_1+\sin(r_{\sigma_1}-j_1)\phi_1]\cdots\\ &\times[\sin(r_{\sigma_\nu}+j_\nu)\phi_\nu+\sin(r_{\sigma_\nu}-j_\nu)\phi_\nu] \end{aligned} $$

with $r_i=\nu+\frac{1}{2}-i$.

After some algebraic manipulations, we get the following results:

$$ A(\phi_1,\dots,\phi_\nu)=\sum_{\{k\}=\{\epsilon,\epsilon,\dots,\epsilon\}}^{\{l,l,\dots,l\}} a_{\{k\}}A_{\{k\}}(\phi_1,\dots,\phi_\nu), \qquad \text{(III. 14)} $$

where the coefficients $a_{\{k\}}$ expressed in terms of the a_j occurring in the representation (III. 7) of each oscillator are given by

$$ a_{\{k\}}=\sum_{\sigma\in S_\nu}P(\sigma)\prod_{i=1}^{\nu}\sum_{j_i=|k_i+\sigma i-i|}^{2\nu+k_i-i-\sigma i} a_{j_i}. \qquad \text{(III. 15)} $$

Therefore from Eqs. (III. 14) and (III. 12), the representation of B_ν we are looking for is seen to be

$$ {}^\nu D=\sum_{\{k\}=\{\epsilon,\epsilon,\dots,\epsilon\}}^{\{l,l,\dots,l\}} a_{\{k\}}\,{}^\nu D_{\{k\}} \qquad \text{(III. 16a)} $$

provided of course that the coefficients $a_{\{k\}}$ given in (III. 15) are all nonnegative. This is the *only* condition the a_j introduced in (III. 7) must fulfill.

In the following subsections, we shall investigate successively the case for which the representations of the parafermi ring involving ν degrees of freedom are irreducible, the case for which the representations are given by a "sum" of Green's ansatz, and finally the most general representations.

A. Irreducible representations

In this case, following our notation, the representations of B_ν are denoted by ${}^\nu D_{\{k\}}$ where $\{k\}$ is defined in (III. 10). From Eq. (III. 16a) we see that if this representation satisfies the factorization condition (III. 5), then it must be characterized by the weight $\{k\}=\{l, l, \dots, l\}$. Indeed from (III. 15), we deduce that

$$ a_{\{l,l,\dots,l\}}=a_l^\nu, \qquad \text{(III. 16b)} $$

and since we assumed that $a_l\neq 0$, this coefficient does not vanish so that in (III. 16a) the representation ${}^\nu D_{\{l,l,\dots,l\}}$ appears in any case. But since we assume ${}^\nu D$ irreducible, a_l must be equal to 1 and all the coefficients $a_{\{k\}}$ with $\{k\}\neq\{l,l,\ldots,l\}$ must vanish. Let us compute the coefficients $a_{\{l,l,\ldots,l,k\}}$ with $\epsilon\leq k<l$. From (III.15), we get

$$a_{\{l,l,\ldots,l,k\}}=a_l^{\nu-1}a_k=a_k. \qquad \text{(III.17)}$$

These coefficients vanish if $a_k=0$ for $\epsilon\leq k<l$, which means that the representation ${}^1D$(III.7) is irreducible and contains the spin l. So, at this stage the representations ${}^\nu D$ and ${}^1D$ are seen to be

$${}^\nu D\equiv{}^\nu D_{\{l,l,\ldots,l\}},\quad {}^1D\equiv{}^1D_l.$$

If we now look directly at the factorization condition (III.5), we see that the restriction of ${}^\nu D$ to $O(3)$ must be a multiple of the representation ${}^1D$. Using the branching theorems which give the reduction of an irreducible representation of B_ν in terms of irreducible representations of $B_{\nu-1}$, one can be easily convinced that the restriction of ${}^\nu D$ to $O(3)$ will be a multiple of the representation ${}^1D$ *only* in the case $l=\frac{1}{2}$ for which we have

$${}^\nu D\equiv{}^\nu\Delta\to 2^\nu\,{}^1D_{1/2};$$

in other words, only for the fundamental spinor representation.

Thus the result is the following: If we are considering irreducible representations for a parafermi ring, then the only representation satisfying the factorization condition corresponds in fact to a Fermi ring.

B. Green's ansatz

In this section, we investigate the case for which the representation ${}^\nu D$ is given by a "sum" of Green ansatz, that is to say,

$${}^\nu D=\sum_{i=1}^{n}\alpha_i[{}^\nu\Delta]^{p_i}, \qquad \text{(III.18)}$$

where α_i is the multiplicity of the Green ansatz of order p_i, the order being either all even or all odd and such that

$$p_1<p_2<\cdots<p_n. \qquad \text{(III.19)}$$

The character of the representation (III.18) reads

$${}^\nu\chi(\phi_1,\ldots,\phi_\nu)=\sum_{i=1}^{n}\alpha_i[{}^\nu\chi_{\{1/2,1/2,\ldots,1/2\}}(\phi_1,\ldots,\phi_\nu)]^{p_i}. \qquad \text{(III.20)}$$

Using the factorization property of the fundamental spinor representation we get

$${}^\nu\chi(\phi_1,\ldots,\phi_\nu)=\sum_{i=1}^{n}\alpha_i\prod_{j=1}^{\nu}[{}^1\chi_{1/2}(\phi_j)]^{p_i} \qquad \text{(III.21)}$$

from which we deduce

$${}^\nu\chi(\phi,0,\ldots,0)=\sum_{i=1}^{n}\alpha_i 2^{p_i-1}[{}^1\chi_{1/2}(\phi)]^{p_i}. \qquad \text{(III.22)}$$

From the factorization condition (III.5), it is easily derived that the character ${}^1\chi$ for each oscillator must be of the form

$${}^1\chi(\phi)=\sum_{i=1}^{n}\beta_i[{}^1\chi_{1/2}(\phi)]^{p_i},\quad \beta_n\neq 0, \qquad \text{(III.23)}$$

so that the factorization condition reads

$$\sum_{i=1}^{n}\alpha_i\prod_{j=1}^{\nu}[{}^1\chi_{1/2}(\phi_j)]^{p_i}=\prod_{j=1}^{\nu}\sum_{i=1}^{n}\beta_i[{}^1\chi_{1/2}(\phi_j)]^{p_i}. \qquad \text{(III.24)}$$

Via (III.19), we see that the simple character ${}^1\chi p_{n/2}(\phi_1)$ will appear on the right-hand side of (III.24) with the coefficient

$$\alpha_n\prod_{j=2}^{n}[{}^1\chi_{1/2}(\phi_j)]^{p_n} \qquad \text{(III.25)}$$

and on the left-hand side with the coefficient

$$\beta_\nu\prod_{j=2}^{\nu}\sum_{i=1}^{n}\beta_i[{}^1\chi_{1/2}(\phi_j)]^{p_i}. \qquad \text{(III.26)}$$

Identifying these coefficients, we get

$$\alpha_n=\beta_n^\nu, \qquad \text{(III.27)}$$

$$\beta_i=0\ \text{ for } i<n;$$

by using again the fact that in $[{}^1\chi_{1/2}(\phi_j)]^{p_n}$ the simple character ${}^1\chi p_{n/2}(\phi_j)$ appears only once.

Hence, it follows from (III.23) and (III.27) that each oscillator is described by a representation which contains only the Green ansatz of order p_n and so is the representation for the whole system.

C. General case

Let us go back to the general treatment we gave at the beginning of this section. We shall show that there exists representations other than Green's ansatz which are compatible with the factorization condition.

Let us define the polynomials $P_{\{k\}}(x_\epsilon,\ldots,x_{l-1})$ of l variables by

$$P_{\{k\}}(x_\epsilon,\ldots,x_{l-1})\equiv\sum_{\sigma\in S_\nu}P(\sigma)\prod_{i=1}^{\nu}\sum_{j_i=|k_i+\sigma i-i|}^{2\nu+k_i-i-\sigma i}x_{j_i} \qquad \text{(III.28)}$$

with $x_l=1$.

Then the coefficients (III.15) occurring in the representation ${}^\nu D$ (III.16a) are given by

$$a_{\{k\}}=a_l^\nu P_{\{k\}}(x_\epsilon,\ldots,x_{l-1}) \qquad \text{(III.29)}$$

with

$$x_i=a_i/a_l,\quad i=\epsilon,\ldots,l-1.$$

Our problem is to find the sets of nonnegative rational numbers $(x_\epsilon,\ldots,x_{l-1})$ for which all the polynomials $P_{\{k\}}$ are nonnegative. Then if we choose a_l to be the least common multiple of the denominators of the rational numbers $(x_\epsilon,\ldots,x_{l-1})$, it follows that we obtain a set of nonnegative integers $(a_\epsilon,\ldots,a_l)$ for which $a_{\{k\}}$ are all nonnegative integers.

First of all, let us note that there exists a set of integers $y=(y_\epsilon,\ldots,y_{l-1})$ for which all the polynomials are strictly positive, namely, the set corresponding to Green ansatz[7] of order $2l$. Thus, for each polynomial there exists a neighborhood $\mathscr{O}_{\{k\}}(y)$ of y such that

$$P_{\{k\}}(x)>0\ \text{ for all } x\in\mathscr{O}_{\{k\}}(y).$$

Then defining

$$\mathscr{O}(y)=\bigcap_{\{k\}=\{\epsilon,\epsilon,\ldots,\epsilon\}}^{\{l,l,\ldots,l\}}\mathscr{O}_{\{k\}}(y),$$

which is obviously not empty, it is clear that all the polynomials $P_{\{k\}}$ are strictly positive in $\mathscr{O}(y)$. Hence, to every set of rational numbers in $\mathscr{O}(y)$ there corresponds a representation ${}^{\nu}D$ for which the partition function factorizes.

IV. EXAMPLES

Before considering some examples, it is useful to derive a recurrence formula expressing the coefficients $a_{\{k\}}$ occurring in the representation ${}^{\nu}D$ of ν oscillators in terms of the coefficients occurring in the representation ${}^{\nu-1}D$ of $\nu-1$ oscillators. From Eq. (III.15), using the decomposition of a permutation $\sigma\in S_\nu$ in terms of a product of transpositions, we get the following recurrence formula:

$$a_{\{k_1,\ldots,k_\nu\}}=\sum_{i=1}^{\nu}(-1)^{i+1}a_{\{k_1+1,\ldots,k_{i-1}+1,k_{i+1}+1,\ldots,k_\nu\}}\times\sum_{j_i=|k_i+1-i|}^{k_i+2\nu-1-i}a_{j_i},\tag{IV.1}$$

where the coefficients $a_{\{k'_1,k'_2,\ldots,k'_{\nu-1}\}}$ appearing on the right-hand side belong to the representation ${}^{\nu-1}D$ of $\nu-1$ oscillators. This relation is useful to construct step by step the representation of ν oscillators.

For example, let us consider the case for which the representation of the parafermi ring corresponding to one oscillator is given by

$${}^1D=a_0\,{}^1D_0+a_1\,{}^1D_1+a_2\,{}^1D_2,\quad a_2\neq0.\tag{IV.2}$$

According to Eq. (III.16a), the representation of ν oscillators will read

$${}^{\nu}D=\sum_{\{k\}=\{0,0,\ldots,0\}}^{\{2,2,\ldots,2\}}a_{\{k\}}\,{}^{\nu}D_{\{k\}}.\tag{IV.3}$$

For $\nu=2$ oscillators, the coefficients appearing in (IV.3) are given by (III.15) as well as (IV.1). We get

$$\begin{aligned}&a_{22}=a_2^2,\quad a_{21}=a_2a_1,\quad a_{20}=a_2a_0,\\&a_{11}=a_1^2-a_0a_2-a_2^2,\\&a_{10}=a_2a_0-a_2a_1+a_1a_0,\\&a_{00}=a_0^2+a_0a_1+a_0a_2-a_1^2.\end{aligned}\tag{IV.4}$$

From the decomposition

$$({}^1\Delta)^4=2\,{}^1D_0+3\,{}^1D_1+{}^1D_2,\tag{IV.5}$$

we see that Green's ansatz of order 4 corresponds to the values

$$a_0=2,\quad a_1=3,\quad a_2=1,\tag{IV.6}$$

from which the decomposition of $({}^2\Delta)^4$ can be obtained through (IV.3) and (IV.4). According to the discussion of Sec. III.C, the positiveness of the coefficients (IV.4) reads

$$x_1^2-x_0-1\geq0,\tag{IV.7a}$$

$$x_0-x_1+x_1x_0\geq0,\tag{IV.7b}$$

$$x_0^2+x_0x_1+x_0-x_1^2\geq0\ \text{ with } x_0=a_0/a_2,\quad x_1=a_1/a_2.\tag{IV.7c}$$

From this set of conditions, we see that x_1 and x_0 cannot vanish which means that in the representation (IV.2) all the spins up to 2 must appear. In fact this is quite general as it can be seen directly in (IV.1) with $\nu=2$. Moreover from (IV.7a) we must have $x_1>1$, that is, a_1 is always greater than a_2. From these simple considerations, we thus see that many representations do not lead to the factorization property. However from the next example, we shall see that there exists an arbitrarily large number of solutions even for a large number of oscillators.

Let us consider the case for which the representation ${}^1D$ contains only the spins 0 and 1, namely, we put

$${}^1D\equiv a_0\,{}^1D_0+a_1\,{}^1D_0.\tag{IV.8}$$

Then the representation ${}^{\nu}D$ for ν oscillators is given by

$${}^{\nu}D=\sum_{k=0}^{\nu}a_{\{1^k0^{\nu-k}\}}\,{}^{\nu}D_{\{1^k0^{\nu-k}\}},\tag{IV.9}$$

where $\{1^k0^{\nu-k}\}$ denotes the weight in which the first k components are equal to 1 and the last $\nu-k$ components are zero. Using the recurrence formula (IV.1), we get

$$a_{\{1^k0^{\nu-k}\}}=a_1^ka_{\{0^{\nu-k}\}},\quad k\geq1,\tag{IV.10a}$$

$$a_{\{0^\nu\}}=(a_0+a_1)a_{\{0^{\nu-1}\}}-a_1^2a_{\{0^{\nu-2}\}}\tag{IV.10b}$$

in which the coefficients $a_{\{0^{\nu-k}\}}$ refer to the representation ${}^{\nu-k}D$ for $\nu-k$ oscillators. Therefore, the conditions for the existence of the representations (IV.9) are simply

$$P_{\{0^k\}}(x)\geq0\ \text{ for } 2\leq k\leq\nu\ \text{ with } x=a_0/a_1,\tag{IV.11}$$

where $P_{\{0^k\}}(x)$ is defined according to (III.28) and (III.29) by

$$P_{\{0^k\}}(a_0/a_1)=a_1^{-k}a_{\{0^k\}}\tag{IV.12}$$

and obey the recurrence formula

$$P_{\{0^\nu\}}(x)=(x+1)P_{\{0^{\nu-1}\}}(x)-P_{\{0^{\nu-2}\}}(x).\tag{IV.13}$$

For $\nu=2$ oscillators, a direct computation or the use of Eq. (III.15) lead to the following representation

$${}^2D=a_1^2\,{}^2D_{\{1,1\}}+a_1a_0\,{}^2D_{\{1,0\}}+(a_0^2+a_1a_0-a_1^2)\,{}^2D_{\{0,0\}}\tag{IV.14}$$

provided that

$$P_{\{0,0\}}(x)=x^2+x-1\geq0.$$

We shall show that the condition

$$x\geq1\tag{IV.15}$$

is strong enough in order to satisfy (IV.11) *for any number of oscillators*. Indeed, let us assume that for a given k, $P_{\{0^{k-1}\}}>P_{\{0^{k-2}\}}>0$. Then from (IV.14) with the condition (IV.15), we get

$$P_{\{0^k\}}>xP_{\{0^{k-1}\}}>P_{\{0^{k-1}\}}>0.\tag{IV.16}$$

Since this is true for $k=3$, then this is true for $k=4$ and finally for any k.

So, provided that $a_0\geq a_1$, the representation

$${}^1D=a_0\,{}^1D_0+a_1\,{}^1D_1$$

leads to a representation ${}^{\nu}D$ for any ν which fulfills the factorization property. Let us note that Green's ansatz of order 2 is given by $a_0=a_1=1$.

As a direct consequence, the Kronecker product

$$\prod_{i=1}^{p}(a_0^{(i)}\,{}^1D_0+a_1^{(i)}\,{}^1D_1)\ \text{ with } a_0^{(i)}\geq a_1^{(i)}\tag{IV.17}$$

will lead to new representations of the system of parafermi harmonic oscillators in the same way the fundamental spinor representation leads to Green's ansatz. Indeed, since the character of a Kronecker product of representations is the product of the characters of the representations, it follows that the character of the representation (IV.17) reads

$$^1\chi(\phi)=\prod_{i=1}^{p}\left[a_0^{(i)}+a_1^{(i)}\,{}^1\chi_1(\phi)\right], \tag{IV.18}$$

so that the product $\Pi_{k=1}^{\nu}\,{}^1\chi(\phi_k)$ will be

$$\prod_{k=1}^{\nu}{}^1\chi(\phi_k)=\prod_{i=1}^{p}\prod_{k=1}^{\nu}\left[a_0^{(i)}+a_1^{(i)}\,{}^1\chi_1(\phi_i)\right]. \tag{IV.19}$$

But we have seen that the product $\Pi_{k=1}^{\nu}[a_0^{(i)}+a_1^{(i)}\,{}^1\chi_1(\phi_k)]$ is a character for a representation, say ${}^{\nu}D^{(i)}$, of the form (IV.9) so that the product $\Pi_{k=1}^{\nu}\,{}^1\chi(\phi_k)$ is the character of the Kronecker product $\Pi_{i=1}^{p}\,{}^{\nu}D^{(i)}$. In particular, for $p=2$ we obtain solutions of the first example we were considering at the beginning of this section, solutions which are valid for any number of oscillators.

V. CONCLUSION

Let us summarize our results following the type of representations of the parafermi ring involving ν degrees of freedom we considered.

(1) Under the assumption of an irreducible representation (which implies a unique vacuum), it turns out that only system leading to the factorization of the partition function is in fact the standard Fermi system.

(2) It has been shown that a sum of Green ansatz of different orders does *not* satisfy the factorization condition so that among all the representations constructed in terms of Green ansatz, the order plays the role of a *selection rule*.

(3) We have shown that there exists infinitely many reducible representations distinct from Green's ansatz for which the factorization condition is fulfilled. Starting with particular representations, we have seen that we can generate other representations by considering Kronecker products of the "basic" representations as one does starting with the fundamental spinor representation. One can be convinced then that the number of possible representations is in fact arbitrarily large.

For these general representations the statistical weights of the various energy levels are very large in contrast with the standard Fermi representation; and to that extent they seem to have no immediate physical application.

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[5]All the properties of the characters of B_ν which will be used are contained for instance in H. Boerner's *Representations of Groups* (North-Holland, Amsterdam, 1963), especially Chaps. VII and VIII.
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[7]This property can be shown by using the reduction of the Kronecker product ${}^{\nu}\Delta\times{}^{\nu}D_{\{k\}}$ in terms of irreducible components. See, for instance, F. D. Murnaghan, *The Theory of Group Representations* (John Hopkins, Baltimore, 1958), Chap. 10, Sec. 4, p. 313.