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Superpropagator for a Nonpolynomial Field

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By using a method originally due to Okubo we calculate the momentum-space superpropagator for a nonpolynomial field $U(x) = 1/[1+f\phi(x)]$ both for a massless and a massive neutral scalar $\phi(x)$ field. For the massless case we obtain a representation that resembles the weighted superposition of propagators for the exchange of a group of scalar fields $\phi(x)$ as is intuitively expected. The exact equivalence of this representation with the propagator function which has been obtained earlier through the use of the Fourier transform of a generalized function is established. For the massive case we determine the asymptotic form of the superpropagator.

After the partial success of a typical nonpolynomial chiral Lagrangian theory, many fundamental properties of the nonpolynomial interaction have been under study.[1] In particular, some detailed investigations have been made in respect to the interactions of the types

$$\bar{\psi}\exp[g\phi(x)]\psi$$

and

$$\bar{\psi}\{1/[1+f^2\phi^2(x)]\}\psi,$$

where $\phi(x)$ is a scalar field.

As has been pointed out by Salam, the basic problems in these types of interaction are the difficulties associated with renormalizability and the asymptotic behavior of scattering amplitudes. Some light regarding the renormalization aspect has been shed recently in a series of papers by Salam and his collaborators. Okubo[2] investigated, quite in detail, the various aspects of the interaction $\exp[g\phi(x)]$ as early as 1954 (see also Green[3]). In all these investigations the first major problem one encounters is the determination of the propagator for the field $U(x)$, which is either $\exp[g\phi(x)]$ or $1/[1+f\phi(x)]$, or any other nonpolynomial expression involving the field $\phi(x)$; the propagator for the field $\phi(x)$ itself being well known from the study of the usual field theory. All recent investigations in this regard are based on the involved mathematical developments due to Volkov, Efimov, Fradkin, and others.[4] However, the technique adopted by Okubo[2] for the calculation of the superpropagator [i.e., the propagator for the field $U(x)$] is much simpler and entirely different from those used recently in the literature.

We discuss here how a variant of Okubo's method could be used to calculate the superpropagator for the case of $U(x)=1/[1+f\phi(x)]$. It may be pointed out that the significant departure in this method is that one converts the "Borel transform" of the x-space superpropagator into an integral equation in momentum space, instead of representing it by means of the Sommerfeld-Watson integral. For the mass-zero case the integral equation can be solved exactly and one can obtain the standard result as obtained by Salam through the use of the Fourier transform of a generalized function. An interesting feature of our calculation is that the superpropagator can be expressed by means of an integral representation of the type of a weighted sum of propagators corresponding to the group of particles exchanged.

The most important advantage of the present technique is that it can be easily extended to the case of a massive $\phi(x)$ field. Even though the integral equation cannot be solved exactly in this case, one can nevertheless easily obtain the asymptotic form of the superpropagator by the perturbation method. We present here the leading behavior of the result to second order.

To illustrate our method of calculating the superpropagator we start with the "Borel transform" of the x-space superpropagator for the field $U(x)=1/[1+f\phi(x)]$, where f is the minor coupling constant and $\phi(x)$ is a massless scalar field. The x-space Borel sum for the superpropagator $F(\Delta)$, where $\Delta(x)$ is the standard Feynman propagator for the scalar field, is given by[5]

$$F(\Delta)=\int_0^\infty d\xi\, e^{-\xi}\sum_{n=0}^{\infty}(f^2\xi\Delta)^n\,. \tag{1}$$

Instead of using the standard Sommerfeld-Watson integral representation of (1) we consider the sum

$$u(x)=\sum_{n=0}^{\infty}(f^2\xi\Delta)^n\,. \tag{2}$$

It is easy to see that (2) is the iteration solution of the following algebraic equation:

$$u(x)=1+(f^2\xi)\Delta(x)u(x)\tag{3}$$

Following Okubo[2] we transform this equation into an integral equation in momentum space,

$$u(p)=(2\pi)^4\delta(p)-\frac{if^2\xi}{(2\pi)^4}\int\frac{d^4p'}{(p-p')^2}u(p'),\tag{4}$$

where

$$u(p)=\int u(x)e^{ipx}d^4x.$$

In obtaining Eq. (4) we have used the fact that

$$\Delta(x)|_{m=0}=\frac{-i}{(2\pi)^4}\int\frac{e^{ipx}}{p^2-i\epsilon}d^4p.$$

To solve Eq. (4) we proceed as follows: First, we analytically continue the equation into the Euclidean domain by employing the Wick-rotation trick. Then, if we set

$$u(p)=(2\pi)^4\delta(p)+g(p),\tag{5}$$

it is seen that $g(p)$ satisfies the following integral equation:

$$g(p)=\frac{f^2\xi}{p^2}+\frac{f^2\xi}{(2\pi)^4}\int\frac{d^4p'}{(p-p')^2}g(p').\tag{6}$$

Next, noting that for the Euclidean case[6]

$$\Box_p\left(\frac{1}{p-p'}\right)^2=-4\pi^2\delta^{(4)}(p-p'),$$

we convert Eq. (6) into a differential equation (for nonvanishing p^2),

$$\left(\Box_p+\frac{f^2\xi}{4\pi^2}\right)g(p)=0.\tag{7}$$

Since $g\equiv g(p^2)$, the last equation may be rewritten as

$$\left(4t\frac{d^2}{dt^2}+8\frac{d}{dt}+\frac{f^2\xi}{4\pi^2}\right)g(t)=0,\tag{8}$$

where $t=p^2$. Finally, the introduction of a new function,

$$v=wg(w),$$

where

$$w=\left[-\left(\frac{f^2\xi}{4\pi^2}\right)t\right]^{1/2},$$

enables us to obtain

$$w^2\frac{d^2v}{dw^2}+w\frac{dv}{dw}-(1+w^2)v=0.\tag{9}$$

Equation (9) is recognized as the equation for the modified Bessel functions; we can thus obtain the relevant solution for $g(p)$ (which incorporates the boundary condition) as

$$g(p)=\frac{(-1/8\pi^2)(f^2\xi)^2}{(-p^2f^2\xi/4\pi^2)^{1/2}}K_1\left(\left(\frac{-p^2f^2\xi}{4\pi^2}\right)^{1/2}\right).\tag{10}$$

The momentum-space superpropagator, $\tilde F(p^2)$, is given by the Fourier transform of Eq. (1); from Eqs. (1) and (2) it follows that

$$\tilde F(p^2)=\int_0^\infty d\xi e^{-\xi}u(p,\xi).$$

Using Eqs. (5) and (10), we obtain

$$\tilde F(p^2)=(2\pi)^4\delta(p)+(f^2)^2(p^2f^2)^{-1}\exp\left(\frac{-p^2f^2}{32\pi^2}\right)W_{-2,1/2}\left(\frac{-p^2f^2}{16\pi^2}\right).\tag{11}$$

It is not difficult to see that Eq. (11) is nothing but the standard result which one obtains by using Gelfand-Shilov formula and the Fourier transform of the generalized function $\Delta^z(m=0)=r^{-2z}$ in the range $0<\mathrm{Re}z<2$. To demonstrate this explicitly let us start with the expression for $\tilde F(p^2)$ obtained by Salam[5]:

$$\tilde F(p^2)=(2\pi)^4\delta(p)+\tfrac12 i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\,\frac{(-f^2)^z(-p^2)^{z-2}\Gamma(z+1)}{\sin\pi z\,\Gamma(z)\Gamma(z-1)}.\tag{12}$$

Noting that

$$(-f^2)^z(-p^2)^{z-2}=(-f^2)^2\frac{1}{\Gamma(2-z)}\int_0^\infty dx\,x^{1-z}e^{-p^2f^2x},$$

we may rewrite Eq. (12) as

$$\tilde F(p^2)=(\tfrac12 i)(-f^2)^2(2\pi i)\left(-\frac{1}{\pi^2}\right)\int_0^\infty dx\,xe^{-p^2f^2x}\left\{\frac{1}{2\pi i}\int_{\alpha-i\alpha}^{\alpha+i\infty}dz\,x^{-z}[\Gamma(z+1)\Gamma(1-z)]\right\}+(2\pi)^4\delta(p).\tag{13}$$

The expression in the curly brackets is the inverse Mellin transform[7] of $\Gamma(z+1)\Gamma(1-z)$, so that Eq. (13) becomes

$$\tilde F(p^2)=\frac{1}{\pi}(-f^2)^2\int_0^\infty dx\,e^{-p^2f^2x}\frac{x^2}{(1+x)^2}+(2\pi)^4\delta(p).$$

Evaluating the last integral[8] we finally obtain

$$\tilde{F}(p^2) = \frac{2}{\pi}(-f^2)^2(p^2f^2)^{-1}\exp(\tfrac{1}{2}p^2f^2)W_{-2,1/2}(p^2f^2) + (2\pi)^4\delta(p). \qquad (14)$$

The asymptotic behavior of $\tilde{F}(p^2)$ for $p^2 \to +\infty$ can be easily inferred from the known asymptotic behavior of the Whittaker function $W_{K,m}(z)$ and is given by

$$\tilde{F}(p^2) \underset{p^2\to\infty}{\sim} (-f^2)^2(p^2f^2)^{-3}. \qquad (15)$$

In what follows we ignore the numerical factors occurring in the exponent and the argument of the Whittaker function in Eq. (11); if we now use the following representation,[8]

$$W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma(\mu-\lambda+\frac{1}{2})}\int_0^\infty t^{\mu-\lambda-1/2}e^{-t}(1+t/z)^{\mu+\lambda-1/2}dt$$

$$[\mathrm{Re}(\mu-\lambda) > -\tfrac{1}{2}, \quad |\arg z| < \pi],$$

we immediately find that $\tilde{F}(p^2)$ can be represented as

$$\tilde{F}(p^2) = (f^2)^2\int_0^\infty \frac{\sigma(t)dt}{(t+p^2f^2)}, \qquad (16)$$

where

$$\sigma(t) = \tfrac{1}{2}[\delta(t) - e^{-t}(2-t)].$$

It is thus seen that the superpropagator resembles the weighted superposition of the propagators for the exchange of a group of scalar fields $\phi(x)$ as is intuitively expected.

To find the superpropagator for the massive field $U(x) = 1/[1+f\phi(x)]$, where $\phi(x)$ corresponds to a scalar field with mass m, we consider the analog of Eq. (4) for the massive case, which reads

$$u(p) = (2\pi)^4\delta(p) - \frac{if^2\xi}{(2\pi)^4}\int\frac{d^4p'}{(p-p')^2+m^2}u(p'). \qquad (17)$$

Following Okubo[2] we rewrite $u(p)$ as

$$u(p) = u_0(p) + \frac{im^2f^2\xi}{(2\pi)^8}\int\frac{d^4p'd^4p''u_0(p-p')u(p'')}{[(p'-p'')^2+m^2](p'-p'')^2}, \qquad (18)$$

where

$$u_0(p) \simeq (f^2\xi)^2\frac{1}{(-p^2f^2\xi)^{1/2}}K_1[(-p^2f^2\xi)^{1/2}] + (2\pi)^4\delta(p).$$

To solve Eq. (18) we use the following representation for the K function[8]:

$$K_{\nu-\mu}(ab) = \frac{2^\mu\Gamma(\mu+1)}{a^{\nu-\mu}b^\mu}\int_0^\infty\frac{J_\nu(bx)x^{\nu+1}}{(x^2+a^2)^{\mu+1}}dx$$

$$[-1 < \mathrm{Re}\,\nu < \mathrm{Re}(2\mu+\tfrac{3}{2}), \quad a>0, \quad b>0].$$

Note that for the space-preferred metric employed by us the quantity $(-p^2f^2\xi)$ is positive; to avoid having to write the minus signature under the radical sign let $-\xi = \xi'$. Then

$$u_0(p) = (2\pi)^4\delta(p) + \frac{(f^2\xi')^2}{p^2f^2\xi'}\int_0^\infty\frac{x^2J_1(x)}{x^2+p^2f^2\xi'}dx. \qquad (19)$$

Substituting Eq. (19) into (18) we can find the perturbation solution for $u(p,\xi)$ to second order. For this purpose the integrations over four-momenta are carried out by means of the Feynman integration technique,[9] followed by integrations over x and y variables. Recalling that the momentum-space superpropagator is given by

$$\tilde{F}(p^2) = \int d\xi e^{-\xi}u(p,\xi),$$

we next integrate over ξ. This yields Whittaker functions as before. If we now take the limit $p^2\to\infty$, the p^2 dependence of $\tilde{F}(p^2)$ separates out from the integration over the auxiliary Feynman variables, and the leading behavior is found to be

$$\tilde{F}(p^2)|_{m\neq 0} \underset{p^2\to\infty}{\sim} (m^2f^2)(p^2)^{-2}. \qquad (20)$$

We wish to conclude by pointing out that the expressions obtained for the superpropagator in this paper find an interesting application in the problem of scattering of two particles via the exchange of the nonpolynomial field $U(x)$ in the ladder approximation. The study of the asymptotic behavior of such amplitudes has been the subject of some of the recent investigations.[10] We will report on this elsewhere.

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[1] For important references on the subject, see Abdus Salam, in *High Energy Physics* (Proceedings of the Schladming Conference, 1970), edited by P. Urban (Springer, New York, 1970); Abdus Salam and J. Strathdee, Phys. Rev. D 1, 3296 (1970). G. V. Efimov, CERN Report No. 1087, 1969 (unpublished).

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[4] See references in Ref. 1 above for details.

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