Algebraic Study of a Class of Relativistic Wave Equations*

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First-order relativistic wave equations are considered whose irreducible matrix coefficients satisfy the simplest (except for the Dirac algebra) unique mass condition, \((\beta \cdot p)^2 = p^2(\beta \cdot p)\), which is also sufficient to guarantee causality in a minimally coupled external electromagnetic field. All of the associated representations of \(SL(2,\mathbb{C})\) are classified and studied up to and including those which are the direct sum of four irreducible components, \((n, m)\), with either \(n\) or \(m\) less than two. A large number of families of representations are found which permit the algebraic condition to be satisfied. These are tabulated according to whether a Hermitian choice for \(\gamma^a\) is possible and their spin content is given. If a unique spin is described, then the only possible representations are

1.\( (n, 0) \oplus (n - \frac{1}{2}, \frac{1}{2})\)
2.\( (n, 0) \oplus (n + \frac{1}{2}, -\frac{1}{2})\)
3.\( (n + \frac{1}{2}, \frac{1}{2}) \oplus (n, 0) \oplus (n - \frac{1}{2}, -\frac{1}{2})\)
4.\( (1, 0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0, 1)\)

and their conjugates. If, in addition, the representation is assumed to be self-conjugate, then only the Dirac and Petiau–Duffin–Kemmer equations survive.

1. INTRODUCTION

The easiest way of avoiding the difficulties caused by subsidiary conditions when an interaction is coupled to a free relativistic wave equation is simply not to have any. This reasoning and the fact that any \(n\)th order differential equation is equivalent to a first-order system of equations has led many authors to consider free-particle relativistic wave equations of the form [1]

\[(i\gamma^\mu \partial_\mu - m) \phi(x) = 0,\]  \[(1)\]

where \(\phi(x)\) is an \(N\)-component wave function and \(\gamma^\mu\) and \(m\) are \(N\)-dimensional matrices with constant coefficients. For simplicity, the general form, (1), may be restricted by the requirement that \(m\) be nonsingular and thus, without loss of generality, a nonzero constant. Relativistic covariance further restricts the structure

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of Eq. (1) since under a homogeneous Lorentz transformation, \( x' = A x \), \( \phi(x) \) transforms according to

\[
\phi_{\alpha}(x) \rightarrow \phi_{\alpha}'(x') = S_{\alpha\beta}(A) \, \phi_{\beta}(x); \quad \alpha, \beta = 1, \ldots, N, \tag{2}
\]

where \( S(A) \) is an \( N \)-dimensional matrix representation of \( SL(2, C) \). Thus we must have

\[
S(A)^{-1} \, \beta^\mu \, S(A) = (A \beta)^\mu \tag{3}
\]

if Eq. (1) is to be covariant. Written in terms of the generators, this last relation becomes

\[
\begin{align*}
[J_i, \beta_j] &= 0, \tag{4a} \\
[J_i, \beta_j] &= i\epsilon_{ijk} \beta_k, \tag{4b} \\
[N_i, \beta_j] &= i\beta_i, \tag{4c} \\
[N_i, \beta_j] &= i\delta_{ij} \beta_0 \tag{4d}
\end{align*}
\]

where \( J_i \) and \( N_i \) are the \( N \)-dimensional matrix generators of \( SL(2, C) \) (in the representation \( S(A) \) which correspond to the rotations and pure Lorentz transformations respectively.

A further restriction may be imposed upon Eq. (1) if we demand that its solutions describe a particle with unique mass, i.e., if we assume that Eq. (1) implies

\[
(\Box^2 + m^2) \phi_\alpha(x) = 0; \quad \alpha = 1, \ldots, N. \tag{5}
\]

This requires [2, 3] that \( \beta^\alpha \) satisfy the condition

\[
(\beta \cdot p)^n = p^n (\beta \cdot p)^{n-2}, \tag{6}
\]

where \( p \) is an arbitrary 4-momentum.

The simplest example of the algebraic condition, (6), occurs for \( n = 2 \). We then get

\[
(\beta \cdot p)^2 = p^2, \tag{7}
\]

or, equivalently,

\[
\beta^\alpha \beta^\mu + \beta^\mu \beta^\alpha = 2g^{\alpha\mu}, \tag{8}
\]

i.e., the Dirac algebra. Since this algebra admits only one irreducible representation (up to equivalence), there is a unique choice of \( \beta^\alpha \) which will satisfy (8). This choice yields, through Eqs. (4), the representation \( S(A) \) and thus one is led to the representation, \( \frac{1}{2}, 0 \) \( \bigotimes \) \( (0, \frac{1}{2}) \) of \( SL(2, C) \) according to which \( \phi(x) \) must transform. The solutions, \( \phi(x) \), will describe the positive and negative energy states of a particle with a unique spin, \( \frac{1}{2} \). It is clear from this point of view why the Dirac equation stands out as the relativistic wave equation par excellence!
It is also clear that if we wish to describe particles with spin $\neq \frac{1}{2}$, then we must consider algebras, (6), with $n > 2$. Thus the simplest case after the Dirac algebra is the third-order algebra

$$(\mathbf{\beta} \cdot \mathbf{p})^3 = p^a (\mathbf{\beta} \cdot \mathbf{p}),$$

or, equivalently,

$$\sum_\mathbf{p} (\mathbf{\beta} \cdot \mathbf{p})^3 - g^{\alpha \beta} \mathbf{p} \mathbf{p} = 0,$$

where $\sum_\mathbf{p}$ represents a sum over the six permutations of the vector indices.

This algebra shares a nice feature with the Dirac algebra since if we assume that Eq. (1) in addition to having a unique mass also has a diagonalizable $\mathbf{\beta}^\alpha$, then either the Dirac algebra or the algebra, (9), must be satisfied [2]. The algebra, (9), has the further nice property (also shared with the Dirac algebra) that if a minimally coupled electromagnetic field ($\partial_{\mu} \rightarrow \partial_{\mu} + ieA_{\mu}(x)$) is incorporated into Eq. (1), then the characteristic surfaces of the resulting equation will continue to be the light cone and hence its solutions will propagate causally [4–7]. We wish to note however that in the presence of nonminimal interactions, difficulties may still arise [4, 6] and that these problems may or may not be evidenced by the existence of spacelike characteristic surfaces [8].

Because of the particularly simple structure of the $\mathbf{\beta}$-matrices which satisfy (9), we may further anticipate that there will be no secondary constraints upon the solutions to the wave equation and that therefore their accompanying difficulties may be expected to be absent [3,9].

Thus Eq. (1) with the condition (9) represents the simplest way of describing a particle with spin $\neq \frac{1}{2}$ which is guaranteed to be consistent and causal in a minimally coupled external electromagnetic field. In this report we shall systematically investigate the representations of this algebra and discuss their physical content. We shall find that there exist many families of inequivalent representations but only a small number of families which describe a particle with a unique spin. Specifically, we shall classify and investigate all possible representations of $SL(2, C)$ up through direct sums of four irreducible components of the form $(n, m)$, where either $n$ or $m$ is less than 2. The results are tabulated in Tables I–XIII.\(^1\)

We shall consider these representations independently of any assumptions about their self-conjugacy or the associated existence of an Hermitianizing matrix. Thus we may circumvent restrictions on the order of the $\mathbf{\beta}$-algebra no matter how

\(^1\) In Tables I–XIII we present the results for the classes of representations studied. $H(N)$ signifies that an Hermitian choice for $\mathbf{\beta}_N$ is (is not) possible, $X$ means that the algebraic condition (12) may not be met for these representations, and subscripts denote the number of spin values described by the formalism. None of the 4-representations of the classes (c-1), (c-2), (c-5), (c-7), or (c-8) permit the algebraic requirements to be met and hence their tables are omitted.
high the spin value [10, 11]. In fact, we shall find three families of representations of the algebra (9) which describe a unique, arbitrarily high spin. The simplest of these has received some recent attention [12].

If we demand that the $SL(2, C)$ representation be self-conjugate, then we shall find that only the Dirac ($s = \frac{1}{2}$) and Petiau–Duffin–Kemmer ($s = 0, 1$) equations can describe a unique spin and satisfy the algebra irrediclably. These cases also have the properties that their $\beta$-matrices satisfy a more restrictive relation than (9) and that their $\beta$-algebra is finite.

2. Methods of Classification and Construction of Representations

The two best known $\beta$-algebras are those of Dirac, (8), and Petiau–Duffin–Kemmer (P-D-K) [13–15]:

$$\beta^a \beta^b + \beta^b \beta^a = g^{ab} \beta^a + g^{ab} \beta^b. \quad (11)$$

$\beta^a$ which satisfy either of these algebraic conditions will also satisfy the condition of interest here:

$$\sum_p (\beta^p \beta^a - g^{ap} \beta^p) = 0. \quad (12)$$

The converse is, of course, not true. In fact, the algebra defined by (12) differs from the algebras (8) and (11) in a very important way [2, 14], viz., its order is infinite. The conditions (8) and (11) allow one to relate an arbitrarily large product of $\beta$-matrices to a product of lower order and thus the number of linearly independent elements of the $\beta$-algebra when (8) or (11) holds is finite. The Dirac algebra has 16 linearly independent elements and the P-D-K algebra has 126 such elements. On the other hand, the algebraic relation (12) has each $\beta^a$ occurring on the left or right of a product in more than one term. Thus (12) cannot be used to reduce the order of an arbitrary product of $\beta$-matrices and so we may expect that the general algebra defined by (12) in the absence of any further conditions will be infinite.

Since the algebras defined by (8) and (11) are finite and semisimple, their inequivalent irreducible representations are numbered by the number of linearly independent elements in their centers and the squares of the dimensions of these irreducible representations must sum to the square of the order of the algebra [13, 15]. Thus, for the Dirac algebra, we have one irreducible representation of dimension 4 (spin-$\frac{1}{2}$), $4^2 = 16$, and for the P-D-K algebra we have three inequivalent irreducible representations of dimension 1 (trivial representation), 5 (spin-0) and 10 (spin-1); $1^5 + 5^1 + 10^2 = 126$. 
The algebra (12), however, is of infinite order and so we cannot expect its representations to be enumerated by the above scheme. In fact, we expect that there are an infinite number of inequivalent irreducible representations and that these can be of arbitrarily high dimension. In view of these facts and expectations, we shall in the present work assume a "head-on" approach to the problem. We shall systematically investigate the representations of (12) based upon the accompanying representations of \( SL(2, C) \). Thus we shall seek all possible representations of (12) according to whether the representation \( S(A) \) under which the wave function transforms is a direct sum of one, two, three, etc., irreducible representations of \( SL(2, C) \). We shall consider increasingly complicated \( S(A) \) and determine directly whether or not there exist \( \beta^u \) which satisfy (12).

In order to find matrices \( \beta^u \) which satisfy Eqs. (4) and (12) for a given representation \( S(A) \), we focus our attention on \( \beta_0 \) and consider the equivalent problem of finding a \( \beta_0 \) such that

\[
[J_1, \beta_0] = 0, \quad (13)
\]

and

\[
[[N_3, \beta_0], N_3] = \beta_0, \quad (14)
\]


\[
\beta_0^3 = \beta_0. \quad (15)
\]

The first two relations with the definition, \( \beta_1 = i[\beta_0, N_3] \), are equivalent to (4) and the last relation is equivalent to (12) once the covariance properties are guaranteed by (13) and (14) [2].

In general, \( S(A) \) is of the form

\[
S = (n_1, m_1) \oplus \cdots \oplus (n_N, m_N),
\]

where we have represented the irreducible components according to the integer or half-integer values labeling its two commuting \( SU(2) \) components. Each irreducible representation of \( SL(2, C) \) may be further reduced if one restricts oneself to its \( SU(2) \) subgroup, i.e., the subgroup corresponding to spatial rotations. This reduction will exhibit the spins which the various wave function components describe. The representation \( (n, m) \) of \( SL(2, C) \) when restricted to its \( SU(2) \) subgroup may be written in the completely reduced form:

\[
D^{[n+m]}(R) \oplus \cdots \oplus D^{[n-m]}(R).
\]

We may therefore, in this basis, write the \( \beta_0 \) matrix in terms of blocks, larger ones corresponding to the \( SL(2, C) \) representation \( (SL(2, C)\text{-blocks}) \) and smaller "sub-blocks" corresponding to the \( SU(2) \) decomposition \( (SU(2)\text{-blocks}) \). Thus e.g., for the representation \( (1, 0) + (\frac{1}{2}, \frac{1}{2}) + (0, 1) \) (P-D-K spin-1), we have
RELATIVISTIC WAVE EQUATIONS

\[(1, 0) \quad (\frac{1}{2}, \frac{1}{2}) \quad (0, 1)\]

\[
s = 1 \quad s = 1 \quad s = 0 \quad s = 1
\]

\[
\beta_0 =
\begin{array}{cccc}
\beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\
\beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\
\beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\
\beta_{41} & \beta_{42} & \beta_{43} & \beta_{44}
\end{array}
\]

\[
s = 1 \quad (1, 0) \quad (\frac{1}{2}, \frac{1}{2}) \quad (0, 1)
\]

where, e.g., \( [\beta_{12} \mid \beta_{13}] \) is a \( 3 \times 4 \) SL(2, C)-block connecting the representation \((\frac{1}{2}, \frac{1}{2})\) and \((1, 0)\) \( \beta_{12} \) is a \( 3 \times 3 \) SU(2)-block connecting the \( s = 1 \) part of the \((\frac{1}{2}, \frac{1}{2})\) representation with the pure \( s = 1 \) representation \((1, 0)\), and \( \beta_{13} \) is a \( 3 \times 1 \) rectangular SU(2)-block connecting the \( s = 0 \) part of the \((\frac{1}{2}, \frac{1}{2})\) representation to the \((1, 0)\) representation \((s = 1)\).

Consider Eq. (1) in the form

\[
i \beta \psi \phi = m \phi.
\]

From a group theoretical point of view, the \( \beta \)-matrices in this equation project \( \phi \) out of the direct product \( \partial_n \otimes \phi \):

\[
(\frac{1}{2}, \frac{1}{2}) \otimes S \rightarrow S.
\]

If we call the irreducible components of \( S (n, m) \), we see [16, 15] that two such components may be connected by \( \beta \) only if they are related by

\[
(n, m) \leftrightarrow (n \pm \frac{1}{2}, m \pm \frac{1}{2}).
\]

Traditionally, two such representations are said to be “interlocking.” Thus, the matrix \( \beta \) may have nonvanishing entries only in the SL(2, C)-blocks which connect representations related as in (19). Thus in the example (16), we must have

\[
\beta_0 =
\begin{array}{cccc}
0 & \beta_{12} & \beta_{13} & 0 \\
\beta_{21} & 0 & 0 & \beta_{24} \\
\beta_{31} & 0 & 0 & \beta_{34} \\
0 & \beta_{42} & \beta_{43} & 0
\end{array}
\]

\[
1 \quad 1 \quad 0 \quad 1
\]
We may also see from the above discussion that it is impossible to write a first-order wave equation (with constant coefficients) if the index transformation of the wave function is an irreducible representation of \( SL(2, \mathbb{C}) \). Thus the simplest such equation must have a wave function whose indices transform as the direct sum of two irreducible representations of \( SL(2, \mathbb{C}) \) related as in (19). We shall study such equations in the next section.

To complete our example of the spin-1 P-D-K theory, consider the restrictions on (20) imposed by its rotation properties, (13). As is shown in Appendix B, all nonsquare \( SU(2) \)-blocks must vanish and all square \( SU(2) \)-blocks must be multiples of the identity. Thus we have

\[
\begin{array}{cccc}
1 & 1 & 0 & 1 \\
\hline
\beta_{12} & & & 1 \\
\beta_{21} & & \beta_{44} & 1 \\
\beta_{42} & & & 0 \\
\beta_{41} & & & 1 \\
\end{array}
\]

where all entries are now complex numbers.

The algebraic restriction (15) further requires

\[
x(\beta_{12}\beta_{21} + \beta_{42}\beta_{41}) = x,
\]

where

\[
x = \beta_{11}, \beta_{21}, \beta_{41} \text{ or } \beta_{41},
\]

and these conditions admit nontrivial solutions (e.g., \( x = 1/\sqrt{2} \)) which differ only in the (arbitrary) scale associated with each component representation.

Thus we may conclude that the representation \((1, 0) \oplus (1, \frac{1}{2}) \oplus (0, 1)\) does permit the existence of covariant \( \beta \)-matrices which satisfy the condition (12). By considering the eigenvalues of (21), we may further conclude that the associated wave equation will describe the positive and negative energy states of a spin-1 particle. This is the familiar P-D-K spin-1 formalism.

In the following sections we shall apply these same straightforward techniques to systems governed by increasingly complicated representations of \( SL(2, \mathbb{C}) \). These methods parallel those constructive techniques which have been used for a long time in the study of relativistic wave equations [16–18].
3. Representations of the Form $A \oplus B$

We seek all possible $\beta$-matrices such that (13), (14), and (15) hold where $J_{i}$ and $N_{r}$ are the generators of an $SL(2, C)$ representation of the form

$$(n, m) \oplus (n \pm \frac{1}{2}, m \pm \frac{1}{2}).$$

(23)

A. Classification of Representations

There are four possible choices of sign combinations in (23): (1) $(-, +)$; (2) $(+, +)$; (3) $(+, -)$; and (4) $(-, -)$. But (4) is equivalent to (2) (let $n \rightarrow n + \frac{1}{2}$ and $m \rightarrow m + \frac{1}{2}$ in (4)) and each member of (3) is a conjugate of a member of (1). Thus we need only consider the classes:

1. $(n, m) \oplus (n - \frac{1}{2}, m + \frac{1}{2})$
2. $(n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2})$.

B. Construction of the $\beta$-Matrices

We consider each class in order of increasing complexity.

(i) Class (1), $(n \geq \frac{1}{2}, m = 0)$; $(n, 0) \oplus (n - \frac{1}{2}, \frac{1}{2})$, $n \geq \frac{1}{2}$

From the discussion of the last section, $\beta_{0}$ is $6n + 1$ dimensional and must have the form

$$\begin{array}{c|ccc|c}
(n, 0) & (n - \frac{1}{2}, \frac{1}{2}) & & & \\
s = n & s = n & s = n - 1 & \\
\beta_{0} & \beta & & & \\
\beta' & & & & \\
\frac{1}{2} & s = n & (n, 0) & \\
\beta & & & & \\
\beta' & s = n & (n - \frac{1}{2}, \frac{1}{2}) & \beta, \beta' \in \mathcal{C}.
\end{array}$$

(24)

(14) imposes no additional conditions on $\beta_{0}$ ($N_{3}$ is given by Eqs. (A-4) and A-6) and (15) will be satisfied for nontrivial $\beta_{0}$ if and only if $\beta \beta' = 1$. Thus the algebraic condition (12) may be satisfied for the $SL(4)$ representation (i). Making suitable rescaling of the two representations amounts to taking the simple Hermitian choice $\beta = 1 = \beta'$ so that $\beta_{0}$ has $2n + 1$ eigenvalues $+1$, $2n + 1$ eigenvalues $-1$ (both sets of components transforming under rotations according to $D_{4 \leftrightarrow -4}(R)$) and $2n - 1$ eigenvalues $0$ (with the components transforming via $D_{4 \leftrightarrow +4}(R)$). Thus the equation will describe the positive and negative energy states of a spin-$n$
particle with $2n - 1$ dependent components which vanish in the rest frame. For $n = \frac{3}{2}$, we recover the Dirac equation, and for $n > \frac{3}{2}$, we recover some previously studied arbitrary spin wave equations [12].

(ii) **Class (1),** ($n = \frac{3}{2}, m \geq 0$): $(\frac{3}{2}, m) \oplus (0, m + \frac{1}{2})$

Since each of these cases is the conjugate of one of the previous cases, we again find the same results. (Since $(m, n)$ is obtained from $(n, m)$ by space reflection, the new $\beta'$ is obtained by space reflection, i.e., $\beta_n \rightarrow \beta_n', \beta \rightarrow -\beta$. This is an outer automorphism except for the $n = \frac{3}{2}$ case where it becomes an inner automorphism.) Thus these representations also satisfy (12) and describe a unique spin.

(iii) **Class (1),** ($n = 1, m = \frac{3}{2}$): $(1, \frac{3}{2}) \oplus (\frac{1}{2}, 1)$

In this case we will see that even if the invariance requirements ((13) and (14)) are met, then the algebraic requirement (15) may not be met. $\beta_n$ has the form (using (13))

$$
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
$$

and (14) implies (from Eq. A-6 and B-16) that $\beta = -\frac{1}{2}\alpha$ and $\beta' = -\frac{1}{2}\alpha'$. If we now demand that $\beta_n^0 = \beta_n$, we find that

$$\alpha = \alpha^2\alpha' \quad \quad \quad \quad \quad \alpha' = \alpha^2\alpha$$

$$-\frac{1}{2}\alpha = -\frac{1}{2}\alpha^2\alpha' \quad \quad \quad \quad \quad -\frac{1}{2}\alpha' = -\frac{1}{2}\alpha^2\alpha$$

which admit no nontrivial solution. Thus the equation whose wave function transforms according to (iii) is covariant but cannot satisfy the condition $\beta_n^0 = \beta_n$. It is also easy to see that the matrix (25) with conditions (26) can satisfy no algebraic condition of the form $\beta_n^0(\beta_n^0 - 1) = 0$ and thus the wave equation

$$(i\beta \cdot \partial - m) \phi_{\alpha, \psi}(\frac{1}{2}, 0)(\alpha) = 0$$

cannot describe a particle with a unique mass. It may also be seen from the spin structure of (25) that there is a spin-$\frac{1}{2}$ particle and a spin-$\frac{3}{2}$ particle entering on an
equal footing. Choosing \( \alpha = \alpha' \) and going to the rest frame, one sees that the mass of the spin-\( \frac{1}{2} \) particle is going to be twice the mass of the spin-\( \frac{3}{2} \) particle.

Such nonunique mass equations have been frequently considered in the past either from a general point of view or in an attempt to describe multiplets of particles [6, 19]. The above is the simplest Bhabha [16] equation with \([\beta^0, \beta^i] = \text{const.} \ S^{ij} \), the six \( SL(2, C) \) generators, for which we have a descending mass spectrum \( m(s) = \text{const.} \ (s + \frac{1}{2})^{-1} \). Since the algebraic condition we are imposing here precludes the occurrence of such equations, we shall not consider them further. We will see later, however, that there are many possible wave equations which satisfy the unique mass algebraic condition yet describe more than one spin. In a sense these too may be prejudiced against if we believe that an elementary system must have a unique mass and spin.

(iv) **Class (1), \( n \geq \frac{3}{2}, m = \frac{1}{2} \):** \( (n, \frac{1}{2}) \oplus (n - \frac{1}{2}, 1) \)

Here \( \beta_0 \) is \( 10n + 2 \) dimensional and must be written

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where the interlocking argument and (13) have been used. In this case as in the previous case, the imposition of (14) is nontrivial and relates the elements within an \( SL(2, C) \) rectangular block, i.e., in this case \( \beta_{15}(\beta_{31}) \) is related to \( \beta_{24}(\beta_{42}) \). This will be the case whenever we have a representation entering \( S(\Lambda) \) of the form \( (n, m) \) where both \( n \) and \( m \) are greater than \( \frac{1}{2} \). The relationship between \( \beta_{15} \) and \( \beta_{31} \) due to (14) is given in Appendix B. The result is

\[
\frac{\beta_{15}}{\beta_{31}} = \left( \frac{2n}{n - \frac{1}{2}} \right)^{1/2} = \frac{\beta_{24}}{\beta_{42}}. \tag{28}
\]

From (27) we see that the algebraic condition \( \beta_0^3 = \beta_0 \) will be satisfied if and only if

\[
\begin{align*}
\beta_{15}^2 \beta_{31} &= \beta_{15} & \beta_{24}^2 \beta_{42} &= \beta_{42} \\
\beta_{15} \beta_{12} &= \beta_{12} & \beta_{24} \beta_{24} &= \beta_{24} \\
\beta_{24} \beta_{24} &= \beta_{24} & \beta_{24} \beta_{24} &= \beta_{24}.
\end{align*}
\tag{29}
\]
Now, if, e.g., \( \beta_{13} = 0 \), then \( \beta_{21} = 0 \) and by (28) \( \beta_{24} = 0 = \beta_{42} \), and we have a trivial solution. This is true as well for the other entries so assume all are nonzero. We then have

\[
\beta_{13}^1 \beta_{21}^1 = 1 = \beta_{24}^1 \beta_{42}^1. \tag{30}
\]

But from (28) we have

\[
\beta_{24} \beta_{42} = \beta_{13} \beta_{21} \left( \frac{n - \frac{1}{2}}{2n} \right), \tag{31}
\]

and these last two conditions imply that \( n = -\frac{1}{2} \), which is impossible. Therefore, for the representation (iv), it is not possible to satisfy the algebraic condition (15).

It is clear from the above examples (iii) and (iv) that the reason why (15) cannot be satisfied is that the relation between the various entries of each \( SL(2, C) \) rectangular block is incompatible with the requirements on these entries from \( \beta_0^{2} = \beta_0 \). This situation will clearly worsen as the representation \((n, m)\) entering \( SL(2) \) reach higher and higher values of \( n \) and \( m \). Thus all of the remaining representations in Class (1) will not admit \( \beta \)-matrices which satisfy (15). The results for Class (1) have been tabulated in Table I.

Class (2) is symmetric under the interchange \( n \leftrightarrow m \) and so we need only consider \( n > m \).

(v) Class (2), \( (n \geq 0, m = 0), (n, 0) \oplus (n + \frac{1}{2}, \frac{1}{2}) \)

\( \beta_0 \) is \( 6n + 5 \) dimensional and its decomposition is

\[
\beta_0 = \begin{array}{c|c|c}
\text{n} & \text{n+1} & \text{n} \\
\hline
\text{n} & \beta & n \\
\hline
\beta' & n+1 \beta, \beta' \in \mathcal{C}, \\
\hline
\text{n} & n \\
\end{array} \tag{32}
\]

where (13) has been used. (14) imposes no new condition and (15) is satisfied for nontrivial solutions if \( \beta' \beta = 1 \). The Hermitian choice \( \beta = 1 = \beta' \) may be taken and we find that a spin-\( n \) particle-antiparticle pair is described. There are \( 2n + 3 \) dependent components which do not enter the time derivative term and thus vanish in the rest frame. For \( n = 0 \), this case reduces to the 5-component, spin-0 P-D-K formalism.

(vi) Class (2), \( (n \geq \frac{1}{2}, m = \frac{1}{2}): (n, \frac{1}{2}) \oplus (n + \frac{1}{2}, 1) \)

This case is similar to case (iv) above and cannot satisfy both the invariance and algebraic requirements simultaneously.
The results for the two-representation case are tabulated in Tables I and II. We see that the algebraic requirements may be satisfied only for the representations \((n, 0) \oplus (n + \frac{1}{2}, m + \frac{1}{2})\) and its conjugates. Each case leads to the description of a particle with unique spin, \(n\). The only self-conjugate cases correspond to \(n = 0\) and \(n = \frac{1}{2}\), i.e., the P-D-K spin-0 equation and the Dirac equation, respectively.

We wish also to emphasize that the spin described is unbounded despite the fact that the algebra remains \(\beta_0(\beta_0^2 - 1) = 0\). Thus the result that higher spin implies higher-order algebras depends upon assumptions which go beyond the demands of covariance.
4. REPRESENTATIONS OF THE FORM $A \oplus B \oplus C$

In this section we seek all possible $\bar{\beta}$-matrices such that (13), (14), and (15) are satisfied where $J_i$ and $N_i$ are the generators of $SL(2, C)$ in a representation of the form

$$(n_1, m_1) \oplus (n_2, m_2) \oplus (n_3, m_3),$$

i.e., a "3-representation."

A. Classification of Representations

In order to get a nontrivial 3-representation equation, each representation $(n_i, m_i)$ must interlock with another one. Thus we have the two possibilities:

1. $O-O-O$, i.e., $A-B-C$,

2. $O\rightarrow O$, i.e., $A\rightarrow B$,

Two interlocking representations must differ in their labels by $\frac{1}{2}$, i.e., $(n, m) \leftrightarrow (n + \frac{1}{2}, m \pm \frac{1}{2})$. Thus case (2) is not allowed since two representations whose labels differ by $\frac{1}{2}$ from a common third representation must themselves have labels which differ by an integer or zero. Thus there exists only one possibility, case (1).

Within case (1) we may further separate the representations into a class which contains no repeated irreducible representations and a class of the type $A \oplus B \oplus A$. We demonstrate, however, in Appendix C that representations of this type are reducible to the 2-representation case and hence we are left with a linear non-repeating chain as our only possibility,

$$(n \pm \frac{1}{2}, m \pm \frac{1}{2}) \oplus (n, m) \oplus (n \pm \frac{1}{2}, m \pm \frac{1}{2}).$$

(34)

Scanning the $+$ and $-$ signs in (34) and throwing out repeating cases, we may enumerate the possible representations as:

1. $++$, $+$
2. $++$, $-$
3. $++$, $-$
4. $--$, $+$
5. $--$, $-$
6. $--$, $+$
7. $+-$, $+$
8. $+-$, $-$
9. $+-$, $-$
10. $-+$, $+$
11. $-+$, $-$
12. $-+$, $-$

Now clearly the representations (1) and (7), (2) and (10), (3) and (6), (4) and (9), (5) and (12), and (8) and (11) are equivalent. Each representation of the form (4)
may be expressed as a conjugate of a representation of the form (5) and similarly for (1) and (2). We may thus reduce our attention to only four classes of 3-representations (1), (3), (4), and (8):

1. \((n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})\),
2. \((n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n, m) \oplus (n - \frac{1}{2}, m - \frac{1}{2})\),
3. \((n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})\),
4. \((n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n, m) \oplus (n - \frac{1}{2}, m + \frac{1}{2})\).

B. Construction of the \(\beta\)-Matrices

We know that covariant wave equations exist for all of the above representations. We now seek those representations whose \(\beta\)-matrices can satisfy (1.5). We again start from the lowest values of \(n\) and \(m\) and work our way up. We shall find in this case that there are very few representations which cannot satisfy the algebraic condition. We shall work out in detail some illustrative examples. The results are systematically tabulated in Tables III–VI.

**TABLE III**

\((n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})\) (3-Representation Class (1))

<table>
<thead>
<tr>
<th>(m)</th>
<th>(n)</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
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<td>(H_1)</td>
<td>(H_5)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(H_1)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
<td>(H_1)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(H_1)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{5}{2})</td>
<td>(H_1)</td>
<td>(H_1)</td>
<td>(H_5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

(i) Class (1), \((n \geq \frac{1}{2}, m = \frac{1}{2}); (n + \frac{1}{2}, 1) \oplus (n, \frac{1}{2}) \oplus (n + \frac{1}{2}, 0)\)

3-representation equations with the linear coupling scheme have the general \(SL(2, C)\) structure such that

\[D\phi(x) = \frac{i}{m} \beta \cdot \dot{\phi}(x) = \phi(x),\]  

(35)
which becomes

\[
\begin{array}{cccccc}
D_{12} & \quad & \quad & \quad & \quad & \quad \\
D_{21} & D_{23} & \quad & \quad & \quad & \quad \\
D_{32} & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

\[
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
= \begin{pmatrix}
A \\
B \\
C
\end{pmatrix}.
\]

(36)

For the case under consideration here, upon imposing (13) we find that \( \beta_0 \) has the decomposition

\[
\beta_0 =
\begin{array}{ccccccc}
& n + \frac{3}{4} & n + \frac{1}{2} & n - \frac{1}{2} & n + \frac{1}{2} & n - \frac{1}{2} & n + \frac{1}{2} \\
\beta_{34} & & & & & & \\
\beta_{35} & & & & & & \\
\beta_{42} & & & & & & \\
\beta_{53} & & & & & & \\
\beta_{64} & & & & & & \\
\end{array}
\]

(37)

It is shown in Appendix B, Eq. B-12, that the condition (14) imposes the restrictions

\[
\beta_{34} = \beta_{35} \left( \frac{n + \frac{3}{4}}{2n + 2} \right)^{1/2},
\]

(38a)

and

\[
\beta_{42} = \beta_{53} \left( \frac{n + \frac{1}{2}}{2n + 2} \right)^{1/2}.
\]

(38b)

If we now demand that \( \beta_0^2 = \beta_0 \), we find the additional conditions

\[
x(\beta_{34} \beta_{42} + \beta_{35} \beta_{53}) = x,
\]

(39a)

\[
\beta_{35} \beta_{53} = \beta_{33}.
\]

(39b)

and

\[
\beta_{33} \beta_{53} = \beta_{53}.
\]

(39c)

where \( x = \beta_{31}, \beta_{42}, \beta_{56}, \) or \( \beta_{64} \).
There are several possible solutions to (38) and (39).

(1) \( \beta_{a4} = 0 \iff \beta_{a5} = 0 \iff \beta_{a5} = 0 \iff \beta_{a5} = 0 \) and \( \beta_{a4}\beta_{a5} = 1 \). This solution reduces to the 2-representation case since the upper \((n + \frac{1}{2}, m + \frac{1}{2})\) components are identically zero.

(2) \( \beta_{a5} = 0 \) with the other entries nonzero yields \( \beta_{a5}\beta_{a5} = 1 \iff \beta_{a5}\beta_{a5} \) which is incompatible with (38). Choosing \( \beta_{a5} = 0 \) leads to the same conclusion.

(3) Hence we must assume that all \( \beta \)'s are nonzero. Conditions (39) then reduce to

\[
\beta_{a4}\beta_{a4} + \beta_{a5}\beta_{a5} = 1
\]

(40a)

and

\[
\beta_{a5}\beta_{a5} = 1.
\]

(40b)

These conditions and (38) have the symmetric, real solution

\[
\beta_{a4} = \left( \frac{n + \frac{1}{2}}{2n + 2} \right)^{1/2} = \beta_{a4},
\]

(41a)

\[
\beta_{a5} = 1 = \beta_{a5},
\]

(41b)

and

\[
\beta_{a5} = \left( \frac{n + \frac{1}{2}}{2n + 2} \right)^{1/2} = \beta_{a5}.
\]

(41c)

Thus the representation (i) does admit a covariant wave equation whose matrices satisfy (15).

The system described by this representation, however, while describing a unique mass, does not describe a unique spin. It can be seen by inspection (or, more carefully, by solving for the eigenvalues of \( \beta_0 \) and examining their multiplicity) that a wave equation with (37) and (41) as its \( \beta_0 \) will describe a particle-antiparticle pair with spin \( n + \frac{1}{2} \) and an independent particle-antiparticle with spin \( n - \frac{1}{2} \) both with mass \( m \). Thus although the algebra may be realized by matrices covariant with respect to the representation (i), this cannot be done in terms of a unique spin.

This phenomenon occurs frequently in the present work and is due to the fact that the nonvanishing elements of the \( SL(2, C) \) rectangular blocks are proportional and hence may not independently vanish. Since these elements occupy \( SU(2) \) blocks of different dimension, multiple spins are unavoidable. This will happen whenever the \( SL(2, C) \) representation contains an irreducible component \((n, m)\) with both \( n \geq 1 \) and \( m \geq 1 \).

(ii) Class (1), \((n = 0, m \geq \frac{1}{2})\): \( (\frac{1}{2}, m + \frac{1}{2}) \oplus (0, m) \oplus (\frac{1}{2}, m - \frac{1}{2}) \)

This 10\( m + 5 \) dimensional \( \beta_0 \) will satisfy (13) and (14) if it has the form
If $\beta_0^a = \beta_0$, then we have either that some entries are zero and the system reduces to a 2-representation case or all entries are nonzero and

$$\beta_{22}\beta_{02} + \beta_{34}\beta_{44} = 1. \quad (43)$$

The relative magnitudes of $\beta_{22}$ and $\beta_{34}$ are at our disposal (as long as they are nonzero) and simply correspond to scale changes in the representations $A$, $B$, and $C$. Take, e.g., all entries $=1/\sqrt{2}$. Now it is easy to see that $\beta_0$ has $2m + 1$ eigenvalues +1, $2m + 1$ eigenvalues $-1$, and the rest zero. Thus a unique spin $m$ is described by this case.

(iii) Class (1), $(n = \frac{3}{2}, m \geq \frac{1}{2})$: $(1, m + \frac{1}{2}) \oplus (1, m) \oplus (1, m - \frac{1}{2})$

These cases also allow an Hermitian choice for $\beta_0$ and describe two spins, $m + \frac{1}{2}$ and $m - \frac{1}{2}$. $\beta_0$ takes the form

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& m + 1 & m & m & m & m - 1 & m + 1 & m \\
\hline
\beta_0 = & \beta_{22} & \beta_{34} & \beta_{43} & \beta_{64} & \beta_{67} & \beta_{75} & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& m + \frac{1}{2} & m & m - \frac{1}{2} & m + \frac{1}{2} & m - \frac{1}{2} & m + \frac{1}{2} & m - \frac{1}{2} \\
\hline
\beta_{22} & & & & & & & \\
\beta_{34} & & & & & & & \\
\beta_{43} & & & & & & & \\
\beta_{64} & & & & & & & \\
\beta_{67} & & & & & & & \\
\beta_{75} & & & & & & & \\
\hline
\end{array}
\]
with the coefficients related by Eqs. B-11 and B-12:

\[
\begin{align*}
\beta_{24} &= \beta_{35} \left( \frac{m + \frac{3}{2}}{2m + 2} \right)^{1/2}, \\
\beta_{43} &= \beta_{25} \left( \frac{m + \frac{3}{2}}{2m + 2} \right)^{1/2}, \\
\beta_{45} &= \beta_{36} \left( \frac{2m}{m - \frac{1}{2}} \right)^{1/2}, \\
\beta_{46} &= \beta_{37} \left( \frac{2m}{m - \frac{1}{2}} \right)^{1/2}.
\end{align*}
\]  
(45)

The algebraic requirements for nonzero \( \beta \):

\[
\begin{align*}
\beta_{24} \beta_{43} + \beta_{64} \beta_{64} &= 1, \\
\beta_{26} \beta_{36} + \beta_{56} \beta_{76} &= 1,
\end{align*}
\]  
(46)

may be simultaneously satisfied by the choice

\[
\beta_{36} = \left( \frac{2m + 1}{3} m + \frac{1}{2} \right)^{1/2} = \beta_{23},
\]  
(47)

which implies

\[
\begin{align*}
\beta_{35} &= \left( \frac{1}{3} m - \frac{1}{2} \right)^{1/2} = \beta_{15}, \\
\beta_{34} &= \left( \frac{1}{3} m + \frac{1}{2} \right)^{1/2} = \beta_{43}, \\
\beta_{46} &= \left( \frac{1}{3} \frac{2m}{m - \frac{1}{2}} \right)^{1/2} = \beta_{64},
\end{align*}
\]  
(48)

thus satisfying (45) and (46).

The remaining class (1) examples studied describe three spins and the results are tabulated in Table III. We thus have found no class (1) representation which cannot satisfy the algebraic conditions.

(iv) Class (2), \( (n = \frac{1}{2}, m = \frac{3}{2}); (1, 1) + (\frac{3}{2}, \frac{3}{2}) + (0, 0) \)

We have

\[
\begin{array}{cccccc}
2 & 1 & 0 & 1 & 0 & 0 \\
\beta_{0} & & & & \beta_{24} & \\
& \beta_{35} & & & & \\
& & \beta_{43} & & & \\
& & & \beta_{56} & & \\
& & & & \beta_{65} & \\
& & & & & \beta_{ij} \in \mathcal{E}
\end{array}
\]  
(49)
with the conditions (Eq. B-12)

\[ \beta_{34} = \beta_{23}(\frac{3}{2})^{1/2}, \quad \beta_{43} = \beta_{32}(\frac{3}{2})^{1/2}, \quad \beta_{41} = 1, \quad \beta_{23}\beta_{41} + \beta_{34}\beta_{43} = 1. \] (50)

Thus we find

\[ 1 = \beta_{23}\beta_{42} = \beta_{23}\beta_{42}^{2}, \] (51)

which implies that

\[ \beta_{23}\beta_{42} = -\frac{1}{2}. \] (52)

We therefore encounter a new phenomenon here. The invariance and algebraic requirements may be met but not by an Hermitian \( \beta_{0} \). Two spins are described, 1 and 0.

The other cases of class (2) with either \( n \) or \( m = \frac{1}{2} \) are identical to the above and yield a non-Hermitian \( \beta_{0} \) describing two spins. The remaining cases which we studied also permit \( \beta \)-matrices which satisfy the algebra and describe three spins. The results are presented in Table IV.

The results for class (3) are given in Table V. This class is similar to class (1) in that it allows an Hermitian choice for \( \beta_{0} \). The cases studied describe from two to four different spins.

Class (4) has an exceptional case with \( n = \frac{1}{2}, \ m = \frac{1}{2} \), i.e., \( (1, 0) \oplus (\frac{1}{2}, \frac{1}{2}) = (0, 1) \). This of course permits an Hermitian \( \beta_{0} \) and describes a unique spin, 1. It leads to the P–D–K spin-1 equation and is the only self-conjugate 3-representation which describes a single spin. It was our example of Section 2.

(v) Class (4), \( (n \gg 0, \ m = \frac{1}{2}) \}: \ (n + \frac{1}{2}, 0) \oplus (n, \frac{1}{2}) \oplus (n - \frac{1}{2}, 1) \)

Here \( \beta_{0} \) takes the form

\[ \beta_{0} = \begin{array}{|c|c|c|c|c|c|} \hline
n + \frac{1}{2} & n + \frac{1}{2} & n - \frac{1}{2} & n + \frac{1}{2} & n - \frac{1}{2} & n - \frac{1}{2} \\
\hline
\beta_{23} & \beta_{24} & \beta_{34} & \beta_{35} & \beta_{41} \in \mathcal{C} \\
\hline
\beta_{32} & \beta_{35} & \beta_{45} & \beta_{41} \in \mathcal{C} \\
\hline
\beta_{43} & \beta_{45} & \beta_{41} \in \mathcal{C} \\
\hline
\beta_{43} & \beta_{41} \in \mathcal{C} \\
\hline
\end{array} \] (53)
### TABLE IV

\((n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n, m) \oplus (n - \frac{1}{2}, m - \frac{1}{2})\) (3-Representation. Class (2))

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
<th>(\ldots)</th>
</tr>
</thead>
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<tr>
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<td>(N_0)</td>
<td>(N_3)</td>
<td>(N_6)</td>
<td>(N_9)</td>
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<td>(N_8)</td>
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<td>(N_4)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
<td>(N_3)</td>
<td>(N_6)</td>
<td>(\ldots)</td>
<td></td>
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<td></td>
</tr>
<tr>
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<td>(N_4)</td>
<td>(N_7)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{5}{2})</td>
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<td>(N_8)</td>
<td>(\ldots)</td>
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<td></td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE V

\((n - \frac{1}{2}, m - \frac{1}{2}) + (n, m) + (n + \frac{1}{2}, m - \frac{1}{2})\) (3-Representation. Class (3))

<table>
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<tr>
<th>(m)</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
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</thead>
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<td>(H_4)</td>
<td>(H_7)</td>
<td>(H_2)</td>
<td>(H_5)</td>
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</tr>
<tr>
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<td>(H_5)</td>
<td>(H_8)</td>
<td>(H_3)</td>
<td>(H_6)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
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<td>(H_6)</td>
<td>(H_9)</td>
<td>(H_4)</td>
<td>(H_7)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>2</td>
<td>(H_4)</td>
<td>(H_7)</td>
<td>(H_{10})</td>
<td>(H_5)</td>
<td>(H_8)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(\frac{5}{2})</td>
<td>(H_5)</td>
<td>(H_8)</td>
<td>(\ldots)</td>
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<tr>
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<td>(\ldots)</td>
<td>(\ldots)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

with

\[
\beta_{12}\beta_{21} + \beta_{43}\beta_{34} = 1, \quad \beta_{30}\beta_{03} = 1, \quad \beta_{04} = \beta_{30} \left( \frac{2n}{n - \frac{1}{2}} \right)^{1/2}, \quad \beta_{42} = \beta_{40} \left( \frac{2n}{n - \frac{1}{2}} \right)^{1/2}. \quad (54)
\]

and (Eq. B-11)
We find
\[ \beta_{12} \beta_{21} = -(n + \frac{1}{2})(n - \frac{1}{2}), \] (56)
so the algebra can be satisfied but not by an Hermitian $\beta_0$. Two spins are described, $n + \frac{1}{2}$ and $n - \frac{1}{2}$. The remaining class (4) representations with either $n$ or $m = \frac{1}{2}$ also lead to algebraic realizations by non-Hermitian $\beta_0$ with two spins.

Let us consider the case

(vi) Class (4), $(n = 1, m = 1)$: $(\frac{1}{2}, \frac{1}{2}) \oplus (1, 1) \oplus (\frac{3}{2}, \frac{3}{2})$

\[
\begin{array}{cccccc}
2 & 1 & 2 & 1 & 0 & 2 & 1 \\
\beta_{13} & & & & & & 2 \\
& \beta_{34} & & & & & 1 \\
\beta_{21} & & \beta_{45} & & \beta_{56} & & 2 \\
& & & \beta_{67} & & 1 & \beta_{47} \in \mathcal{C} \quad (57) \\
\beta_{0} = & & & & & & 0 \\
& & \beta_{43} & & \beta_{54} & & 2 \\
& & & \beta_{34} & \beta_{56} & \beta_{67} & 1 \\
& & & & & & 1 \\
\end{array}
\]

with the conditions
\[ \beta_{13} \beta_{21} + \beta_{56} \beta_{65} = 1, \] (58a)
\[ \beta_{45} \beta_{56} + \beta_{47} \beta_{74} = 1, \] (58b)

and
\[ \beta_{13} = \beta_{24} \sqrt{3}, \]
\[ \beta_{31} = \beta_{45} \sqrt{3}, \]
\[ \beta_{43} = \beta_{74} \sqrt{3}, \]
\[ \beta_{56} = \beta_{47} \sqrt{3}. \] (59)

(59) and (58b) imply
\[ \beta_{13} \beta_{21} + \beta_{34} \beta_{43} = 3, \] (60)

which contradicts (58a). Thus the algebra may not be satisfied.
On the other hand, if we increase the value of $n$ by $\frac{1}{2}$, we get

(vii) \textit{Class (4), (n = \frac{3}{2}, m = 1): (2, \frac{1}{2}) \oplus (\frac{3}{2}, 1) \oplus (1, \frac{3}{2})}

\begin{align*}
\begin{array}{cccccccc}
\beta_{13} & \beta_{24} & & & & & \beta_{47} & \beta_{58} \\
\beta_{31} & & \beta_{36} & & & & \beta_{47} & \beta_{58} \\
\beta_{42} & & \beta_{42} & & \beta_{47} & & \beta_{58} \\
\beta_{53} & & & \beta_{53} & & \beta_{58} & & \\
\beta_{64} & & & & & & \beta_{58} & \\
\beta_{75} & & & & & & & \\
\beta_{86} & & & & & & & \\
\beta_{97} & & & & & & & \\
\beta_{108} & & & & & & & \\
\end{array}
\end{align*}

\[ \beta_{67} \in \mathcal{C} \] (61)

with (Eqs. B-11 and B-17)

\[ \begin{align*}
\beta_{13} \beta_{31} + \beta_{26} \beta_{63} &= 1, \\
\beta_{24} \beta_{42} + \beta_{75} \beta_{57} &= 1, \\
\beta_{36} \beta_{65} &= 1, \\
\beta_{13} &= \beta_{24} (\frac{3}{2})^{1/2}, \\
\beta_{58} &= \frac{1}{3} \beta_{24}, \\
\beta_{47} &= -\frac{2}{3} \beta_{36},
\end{align*} \] (62)

and similarly for the index-switched terms. These relations reduce to

\[ \beta_{13} \beta_{31} = -8 \]

and

\[ \frac{1}{3} \beta_{13} \beta_{31} + 4 = 1, \] (63)

which are consistent. Thus the algebra may be satisfied for non-Hermitian $\beta_0$ and three spins, $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$, are described.
The class (4) \((n = 1, m = \frac{3}{2})\) case is identical to the above and for all other values studied the results are tabulated in Table VI.

**TABLE VI**

\((n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n, m) \oplus (n - \frac{1}{2}, m + \frac{1}{2})\) (3 Representation, Class (4))

<table>
<thead>
<tr>
<th>(m)</th>
<th>(n)</th>
<th>(1)</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
<td>(H_1)</td>
<td>(N_4)</td>
<td>(N_2)</td>
<td>(N_2)</td>
<td>(N_4)</td>
<td>\ldots</td>
</tr>
<tr>
<td>1</td>
<td>(N_2)</td>
<td>(X)</td>
<td>(N_3)</td>
<td>(N_2)</td>
<td>(N_3)</td>
<td>\ldots</td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
<td>(N_8)</td>
<td>(N_4)</td>
<td>(X)</td>
<td>(N_4)</td>
<td>\ldots</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(N_2)</td>
<td>(N_4)</td>
<td>(N_4)</td>
<td>\ldots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{5}{2})</td>
<td>(N_2)</td>
<td>(N_4)</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We see from the tables that except for a couple of isolated representations, the algebra may always be satisfied. If the spin is unique, however, we are again left with only the class (1), \(n = 0\) family and class (4), \(n = \frac{1}{2} = m\).

5. **Representations of the Type \( A \oplus B \oplus C \oplus D \)**

Since the techniques have been adequately described in the previous sections, we shall, for the case of the 4-representation equations, emphasize mostly the results.

A. Classification of Representations

Consider all of the possible interlocking schemes for four irreducible representations of \(SL(2, C)\). Since we do not wish to reconsider the 3- and 2-representation classes, we will throw out those representations which reduce to these, e.g., the disconnected diagram

\[
\begin{align*}
A \leftrightarrow B \\
C \leftrightarrow D
\end{align*}
\]

(64)

corresponds to a (reducible) direct sum of two 2-representation cases [20]. If we further recall that triangle coupling is forbidden, then we are left with the following possible schemes:
Neglecting representations which are related to others by conjugation, i.e., 
\((n, m) \rightarrow (m, n)\) for each of the four components [21], we have for class (a) the following possibilities: \((A \oplus B \oplus C \oplus D)\)

(a-1), \((n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n + \frac{1}{2}, m - \frac{1}{2}),\)

(a-2), \((n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n - \frac{1}{2}, m + \frac{1}{2}) \oplus (n + \frac{1}{2}, m - \frac{1}{2}),\)

and

(a-3), \((n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n - \frac{1}{2}, m + \frac{1}{2}).\)

We have also neglected those cases with repeated representations since, as shown in Appendix C, such representations are equivalent to a 3-representation case.

We classify types (b) and (c) at the same time. Consider the general representation

\((n_1, m_1) \oplus (n_2, m_2) \oplus (n_3, m_3) \oplus (n_4, m_4)\)

with the interlocking scheme

\[A \leftrightarrow B \leftrightarrow C \leftrightarrow D,\]

where \(D\) may or may not interlock (i.e., differ in its labels by \(\pm \frac{1}{2}\)) with \(A\). We may pictorially illustrate the possible sequences of \(n\)'s and \(m\)'s as follows:
where the diagram indicates whether each succeeding \( n \) (or \( m \)) value (starting with the representation \( A \), \( (n, m) \)) increases or decreases by \( \frac{1}{2} \). Since these 8 sequences exhaust the possibilities, we have a total of \( 8 \times 8 = 64 \) possible sequences for representations of the form (68), and thus each sequence may be labeled by a pair \( [q, p] \), where \( q, p = 1, \ldots, 8 \) according to (69). For example, the \([2, 6]\) representation would be

\[
(n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})
\]

according to this scheme.

We now limit these 64 possibilities as follows:

(A) Since \([p, q]\) is the conjugate of \([q, p]\), we need only consider cases \([p, q]\) where \( p \leq q \). This eliminates 28 of the original 64.

(B) By writing the representations \( A \oplus B \oplus C \oplus D \) in the reverse order, we see that \([8, 8] = [1, 1]\), \([7, 7] = [5, 5]\), \([6, 6] = [3, 3]\), \([4, 4] = [2, 2]\), \([6, 7] = [3, 5]\), \([4, 7] = [2, 5]\), \([4, 6] = [2, 3]\) and \([4, 5] = [2, 7]\).

(C) A combination of the order reversal and conjugation relates:

\([6, 8] = [1, 3]\), \([7, 8] = [1, 5]\), \([5, 8] = [1, 7]\), \([5, 6] = [3, 7]\), \([4, 8] = [1, 2]\), \([3, 8] = [1, 6]\), \([3, 4] = [2, 6]\) and \([2, 8] = [1, 4]\).
We are thus left with 20 families of representations which we group as follows:

(b) (A interlocking with D)

\[
\begin{align*}
[2, 2] R & 
\quad [2, 6] R \\
[2, 3] R & 
\quad [2, 7] R \\
[2, 4] & 
\quad [3, 3] R^2 \\
[2, 5] & 
\quad [3, 5] R \\
[3, 6] R^2 & \\
[3, 7] R & \\
[5, 5] R & \\
[5, 7],
\end{align*}
\]

where \( R(R^2) \) signifies the presence of repeated (twice) representations; and

(c) (A not interlocking with D)

\[
[1, n] \quad \text{where } n = 1, ..., 8.
\]

Now class (b) permits further reduction since, by relabeling, its four representations are cyclically symmetric. In this way we relate \([2, 4] \equiv [2, 5] \equiv [5, 7]\), thus finding only one type of nonrepeating class (b) representation:

(b-1), \((n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})\).

In Appendix C we show that if a class (b) repeating representation is used and the algebraic condition (12) is satisfied, then the equation reduces to a 3-representation case. Thus we have only one type of class (b) representation, (b-1).

Class (c) representations are all inequivalent and contain no repetitions. We must consider each case separately. They are

\[
\begin{align*}
\text{(c-1)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m + 1) \oplus (n + \frac{3}{2}, m + \frac{3}{2}), \\
\text{(c-2)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m + 1) \oplus (n + \frac{3}{2}, m + \frac{1}{2}), \\
\text{(c-3)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m + \frac{1}{2}), \\
\text{(c-4)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m - \frac{1}{2}), \\
\text{(c-5)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m + \frac{1}{2}), \\
\text{(c-6)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m - \frac{1}{2}), \\
\text{(c-7)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m - 1) \oplus (n + \frac{3}{2}, m - \frac{1}{2}), \\
\text{(c-8)}, & \quad (n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m - 1) \oplus (n + \frac{3}{2}, m - \frac{3}{2}).
\end{align*}
\]

B. Construction of the \( \beta \)-Matrices

The results for the 4-representation \( \beta \)-matrices are presented in Tables VII-XIII. Since the techniques used here have been previously explained, we shall only comment on some of the results.
(a) Class (a)

\[ \begin{array}{c}
A \leftrightarrow B \\
\downarrow \quad C \\
\quad D \\
\end{array} \]

(71)

These representations lead to wave equations with the $SL(2, C)$-structure $D\phi = \phi$ where

\[
D = \begin{array}{cccc}
A & B & C & D \\
D_{12} & D_{13} & D_{14} & A \\
D_{21} & & & B \\
D_{31} & & & C \\
D_{41} & & & D \\
\end{array}
\]

(72)

The results for the two classes (a-1) and (a-2) are presented in Tables VII and VIII respectively. The calculations are very similar to the previous ones and will not be given. For each class the invariance and algebraic requirements may be met by a Hermitian $\beta_n$ for all $m$ and $n$ values considered. In all such cases, two or more spins are described. There are no unique spin realizations either in this group or, as we shall see, for any 4-representation system.

**TABLE VII**

$(n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n + \frac{1}{2}, m - \frac{1}{2})$ (4-Representation. Class (a-1))

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{3}{2}$</th>
<th>2</th>
<th>$\frac{5}{2}$</th>
<th>...</th>
</tr>
</thead>
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</tr>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
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<td>$H_3$</td>
<td>$H_3$</td>
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<td></td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE VIII

\((n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n - \frac{1}{2}, m + \frac{1}{2}) \oplus (n + \frac{1}{2}, m - \frac{1}{2})\) (4-Representation.
Class (a-2))

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
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<td>(H_{1})</td>
<td>(H_{\frac{3}{2}})</td>
<td>(H_{2})</td>
<td>(H_{\frac{5}{2}})</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>(H_{\frac{1}{2}})</td>
<td>(H_{1})</td>
<td>(H_{\frac{3}{2}})</td>
<td>(H_{2})</td>
<td>(H_{\frac{5}{2}})</td>
<td>...</td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
<td>(H_{\frac{1}{2}})</td>
<td>(H_{1})</td>
<td>(H_{\frac{3}{2}})</td>
<td>(H_{2})</td>
<td>(H_{\frac{5}{2}})</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>(H_{\frac{1}{2}})</td>
<td>(H_{1})</td>
<td>(H_{\frac{3}{2}})</td>
<td>(H_{2})</td>
<td>(H_{\frac{5}{2}})</td>
<td>...</td>
</tr>
<tr>
<td>(\frac{5}{2})</td>
<td>(H_{\frac{1}{2}})</td>
<td>(H_{1})</td>
<td>(H_{\frac{3}{2}})</td>
<td>(H_{2})</td>
<td>(H_{\frac{5}{2}})</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Class (a-3) is tabulated in Table IX. Satisfactory Hermitian \(\beta_\theta\) describing two or more spins exist for all considered values of \(n\) and \(m\). We shall illustrate this class by considering in more detail the case \((n = 1, m = 1)\), i.e., \((1, 1) + (\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2}) + (\frac{1}{2}, \frac{1}{2})\). Here \(\beta_\theta\) has the form

\[
\beta_\theta = \begin{array}{cccccc}
2 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 1 \\
\beta_{14} & \beta_{15} & \beta_{16} & \beta_{17} & \beta_{18} & \beta_{19} & \beta_{20} & \beta_{21} & \beta_{22}
\end{array}
\]

(73)
TABLE IX

\[(n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n - \frac{1}{2}, m + \frac{1}{2})\]

(4-Representation, Class (a-3))

<table>
<thead>
<tr>
<th>m (\bar{m})</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
<th>(\frac{5}{2})</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
<td>(H_5)</td>
<td>(H_5)</td>
<td>(H_5)</td>
<td>(H_5)</td>
<td>(H_5)</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>(H_6)</td>
<td>(H_6)</td>
<td>(H_6)</td>
<td>(H_6)</td>
<td>(H_6)</td>
<td>...</td>
</tr>
<tr>
<td>(\frac{3}{2})</td>
<td>(H_6)</td>
<td>(H_6)</td>
<td>(H_4)</td>
<td>(H_4)</td>
<td>(H_4)</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>(H_5)</td>
<td>(H_5)</td>
<td>(H_4)</td>
<td>(H_4)</td>
<td>(H_4)</td>
<td>...</td>
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<tr>
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<td></td>
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<td>...</td>
</tr>
</tbody>
</table>

with the algebraic requirements for nonzero \(\beta_{ij}\):

\[
\beta_{14}\beta_{41} + \beta_{18}\beta_{81} = 1,
\beta_{26}\beta_{62} + \beta_{28}\beta_{82} + \beta_{25}\beta_{52} = 1,
\beta_{37}\beta_{73} = 1,
\]

(74)

and invariance requirements from Appendix B:

\[
\beta_{14} = \beta_{28} \sqrt{3},
\beta_{26} = \beta_{27}(\frac{1}{2})^{1/2},
\beta_{18} = \beta_{29} \sqrt{3},
\]

(75)

with similar conditions on the index-reversed coefficients.

Using (75) to eliminate terms in (74), we get

\[
\beta_{26}\beta_{62} + \beta_{28}\beta_{82} = \frac{1}{3},
\beta_{25}\beta_{52} + \frac{1}{2}\beta_{26}\beta_{62} + \beta_{28}\beta_{82} = 1,
\]

(76)

which are consistent. Thus we may take as a solution

\[
\beta_{27} = 1 = \beta_{73},
\beta_{26} = (\frac{1}{2})^{1/2} = \beta_{62},
\beta_{25} = (\frac{1}{2})^{1/2} \cos \theta = \beta_{52},
\beta_{18} = \cos \theta = \beta_{61},
\beta_{26} = (\frac{1}{2})^{1/2} \sin \theta = \beta_{62},
\]

(77)

and

\[
\beta_{14} = \sin \theta = \beta_{41},
\]
where $\Theta$ is any angle. By suitable relative sign changes of the various representations, we may, without any essential loss of generality, put $0 \leq \Theta \leq \pi/2$. Thus $\beta_\Theta$ may be chosen Hermitian. It is easy to see from the structure of $\beta_\Theta$ that three spins are described: 2, 1 and 0.

(b) Class (b)

These representations have a somewhat more complicated structure than the previous case due to the additional interlocking. Its $SL(2, C)$ structure is $D\phi = \phi$

\[
D = \begin{array}{cccc}
A & B & C & D \\
D_{12} & D_{14} & & \\
D_{21} & D_{23} & & \\
D_{32} & D_{34} & & \\
D_{41} & D_{43} & & \\
& & \end{array}
\]

the $D_{ij}$ again representing nonvanishing $SL(2, C)$ rectangular subblocks.

**TABLE X**

\[
(n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2})
\]

(4-Representation, Class (b-1))

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>1</th>
<th>$\frac{3}{2}$</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
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</tr>
<tr>
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<td>$H_\frac{1}{2}$</td>
<td>$H_1$</td>
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<tr>
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<td>$H_\frac{3}{2}$</td>
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</tbody>
</table>
As shown above, there is essentially only one family of representations of this class, viz. (b-1). The results for this family are listed in Table X. The calculations follow the usual scheme but in this case are slightly more complicated than the previous cases and we shall omit them. In every case studied, the algebraic conditions may be satisfied with two or more spin values and a single spin may have more than the usual 2(2s + 1) components associated with it.

(c) Class (c)

\[ A \leftrightarrow B \leftrightarrow C \leftrightarrow D. \]

The \( SL(2, C) \) structure here is

\[
D = \begin{array}{cccc}
A & B & C & D \\
D_{12} & & & \\
& D_{23} & & \\
& & D_{34} & \\
& & & D_{43} \\
\end{array}
\]

(80)

A common result for class (c) is that the requirements upon the coefficients force one representation to decouple, hence reducing the system to a 3-representation case. In fact, of the eight families, only three permit the satisfaction of the invariance

**Table XI**

\((n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m + \frac{3}{2})\)

(4-Representation. Class (c-3))

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>(\frac{3}{2})</th>
<th>2</th>
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<tbody>
<tr>
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<tr>
<td>(\frac{3}{2})</td>
<td>(X)</td>
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<td>2</td>
<td>(X)</td>
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</tbody>
</table>
and algebraic requirements for certain values of \( n \) and \( m \). All of these realizations can be in terms of Hermitian \( \beta_0 \) and describe two spins. The results are given in Tables XI–XIII.

Thus we see that each class of 4-representation system contains families which will satisfy our requirements. None of these, however, has a representation which will describe a unique spin.

**TABLE XII**

\[(n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m - \frac{1}{2})\]

(4-Representation. Class (c-4))

<table>
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<th>( n )</th>
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<th>( \frac{1}{2} )</th>
<th>1</th>
<th>( \frac{3}{2} )</th>
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<td></td>
</tr>
<tr>
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<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( X )</td>
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</tbody>
</table>

**TABLE XIII**

\[(n, m) \oplus (n + \frac{1}{2}, m - \frac{1}{2}) \oplus (n + 1, m) \oplus (n + \frac{3}{2}, m - \frac{1}{2})\]

(4-Representation. Class (c-6))

<table>
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<tr>
<th>( m )</th>
<th>( n )</th>
<th>0</th>
<th>( \frac{1}{2} )</th>
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<th>( \frac{3}{2} )</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
<tbody>
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<td>( H_8 )</td>
<td>( H_8 )</td>
<td>( H_8 )</td>
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</tr>
<tr>
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<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
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<td></td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( X )</td>
<td>( X )</td>
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</tbody>
</table>
6. SUMMARY AND DISCUSSION

We have systematically sought first-order relativistic wave equations whose coefficient matrices satisfy the simplest (aside from the Dirac algebra), unique mass condition, \((\beta \cdot p)^2 = p^\mu \beta_\mu \cdot p\), which also guarantees causality in a minimally coupled external electromagnetic field. A complete list of results for representations of \(SL(2, C)\) with less than five irreducible components \((n, m)\) with either \(n\) or \(m\) less than two is presented in Tables I-XIII. We have found that there exist many families of representations of this algebra most of which lead to wave equations whose solutions describe more than one spin state. The representations

\[
\begin{align*}
(1) \quad & (n, 0) \oplus (n - \frac{1}{2}, \frac{1}{2}), \\
(2) \quad & (n, 0) \oplus (n + \frac{1}{2}, \frac{1}{2}), \\
(3) \quad & (n + \frac{1}{2}, \frac{1}{2}) \oplus (n, 0) \oplus (n - \frac{1}{2}, \frac{1}{2}), \\
(4) \quad & (1, 0) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (0, 1),
\end{align*}
\]

and their conjugates are the only representations which describe a unique spin. The only self-conjugate, unique spin representations among these lead to the Dirac \((s = \frac{1}{2})\) and the P-D-K \((s = 0, 1)\) equations both of which satisfy a more restrictive algebra than that considered.

Going to higher numbers of irreducible components, it is easy to extrapolate from the present considerations that one may always find realizations of this algebra no matter how many irreducible components there are but that none of these representations will describe a unique spin with multiplicity \(2(2s + 1)\). For example, the representation

\[
\mathcal{S} = (0, 0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (1, 0) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus \cdots \oplus (N/2, 0)
\]

will lead to a wave equation whose coefficients can satisfy all of the algebraic conditions but the number of spins described increases with \(N\).

Representations which are not self-conjugate have been prejudiced against in the past since they do not admit a Hermitianizing matrix which provides in a canonical way for the construction of a scalar product and, in field theory, bilinear field densities. Nevertheless, it is still possible to have a scalar product and the associated field densities without a Hermitianizing matrix and so the representations which are not self-conjugate cannot be eliminated from consideration for this reason alone [22].

In addition to providing a study of the realizations of the algebra,

\[
\sum_p (\beta_\mu \beta_{\mu} - g_{\mu\nu} \beta_{\nu}) = 0,
\]

we also hope that we have provided an exposition of those straightforward and simple algebraic techniques which may be fruitfully applied in the determination of the structure and content of any relativistic wave equation.
APPENDIX A: SOME REPRESENTATIONS OF THE LIE ALGEBRA OF $SL(2, \mathbb{C})$

Consider the Lie algebra of $SL(2, \mathbb{C})$ in the form

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad (A-1)$$
$$[J_i, N_j] = i\epsilon_{ijk} N_k, \quad (A-2)$$
$$[N_i, N_j] = -i\epsilon_{ijk} J_k, \quad (A-3)$$

where the matrices $J_i$ and $N_i$ represent the generators of rotations and boosts respectively. It may be verified that the following matrices realize these relations according to the indicated irreducible representation of $SL(2, \mathbb{C})$. We list only the representation and not its conjugate which may be obtained by changing the sign of $N_i$, i.e., $N_i^{(n,m)} = -N_i^{(m,n)}$.

(1) $(n, 0)$

$$J_i = S_i^{(n)} \quad N_i = -iS_i^{(n)}, \quad (A-4)$$

where $S_i^{(n)}$ is the generator of $D^{(n)}(R)$, the $2n + 1$ dimensional irreducible unitary representation of $SU(2)$, i.e., $S_i^{(n)}$ represents the three "spin-$n$" matrices. The verification of this case is trivial.

(2) $(n, \frac{1}{2}), n \geq \frac{1}{2}$

$$J_i = \begin{bmatrix} S_i^{(n+1)} & S_i^{(n-1)} \\ S_i^{(n-1)} & S_i^{(n+1)} \end{bmatrix}, \quad (A-5)$$

$$N_i = -i \begin{bmatrix} n - \frac{1}{2} & S_i^{(n+1)} \\ 1 & n + \frac{1}{2} - K_i^{(n+1)} \\ \frac{1}{n + \frac{1}{2}} K_i^{(n+1)} & 1 \\ n + \frac{1}{2} & S_i^{(n-1)} \end{bmatrix}, \quad (A-6)$$

where $K_i^{(n)}$ represents three rectangular matrices with the property that

$$D^{(n-1)}(R^{-1}) K_i^{(n)} D^{(n)}(R) = D^{(n)}(R) K_i^{(n)} D^{(n-1)}(R^{-1}), \quad (A-7)$$

where $i, j = 1, 2, 3; \alpha, \gamma' = 1, \ldots, 2n - 1; \beta, \delta = 1, \ldots, 2n + 1$, and $R$ is a rotation in 3-space. These matrices are a convenient vehicle for handling the relevant
Clebsch–Gordon coefficients which enter the calculations. They have been discussed previously [23] and the reader is referred to this reference for the relations, proofs, and explicit representation of these matrices. The only relation needed below which is not stated in [23] is

$$K_i^{(n-1)}K_j^{(n)} - K_i^{(n-1)}K_j^{(n)} = 0,$$  \hspace{1cm} (A-8)

which may be demonstrated in a straightforward manner.

The proof that A-5 and A-6 satisfy A-1, A-2, and A-3 in the manner indicated is given in great detail in [23]. The following realizations may be similarly verified.

---

(3) \((n, 1), \quad n \geq 1\).

$$J_i = \begin{array}{|c|c|c|}
\hline
S_i^{(n+1)} & S_i^{(n)} & S_i^{(n-1)} \\
\hline
\end{array},$$  \hspace{1cm} (A-9)

$$N_i = -i \begin{array}{|c|c|c|}
\hline
\frac{n-1}{n+1} S_i^{(n+1)} & \frac{2}{n+1} \left( \frac{n}{2n+1} \right)^{1/2} K_i^{(n+1)} & \frac{2}{n} \left( \frac{n+1}{2n+1} \right)^{1/2} K_i^{(n)} \\
\hline
\frac{2}{n+1} \left( \frac{n}{2n+1} \right)^{1/2} K_i^{(n+1)} & \frac{n-1}{n+1} \frac{n+2}{n} S_i^{(n)} & \frac{2}{n} \left( \frac{n+1}{2n+1} \right)^{1/2} K_i^{(n)} \\
\hline
\frac{2}{n} \left( \frac{n+1}{2n+1} \right)^{1/2} K_i^{(n)} & \frac{n+2}{n} S_i^{(n-1)} & \frac{n+2}{n} S_i^{(n-1)} \\
\hline
\end{array}.$$  \hspace{1cm} (A-10)

---

(4) \((n, \frac{2}{3}), \quad n \geq \frac{2}{3}\).

$$J_i = \begin{array}{|c|c|c|}
\hline
S_i^{(n+1)} & S_i^{(n+1)} & S_i^{(n-1)} \\
\hline
S_i^{(n+1)} & S_i^{(n-1)} & S_i^{(n-1)} \\
\hline
\end{array}.$$  \hspace{1cm} (A-11)
These representations will be used in Appendix B in order to determine the relations imposed upon the coefficients of $\beta_0$ by the invariance requirements.

APPENDIX B: INVARIANCE CONDITIONS ON $\beta_0$

In this appendix we wish to determine the general structural requirements imposed upon $\beta_0$ by invariance. The conditions on $\beta_0$ as discussed in Section 2 are

$$[\beta_0, J_t] = 0$$

and

$$[[N_2, \beta_0], N_3] = \beta_0,$$

where $J_t$ and $N_t$ generate the (reducible) representation of $SL(2, \mathbb{C})$ according to which the wave function is to transform.
If we take \( J_i \) to be the completely reduced direct sum of irreducible spin matrices,

\[
J_i = \bigoplus_{j=1}^{N} S_j^{(n_j)},
\]

(B-3)

where \( n_j \) is the spin of the \( j \)th entry, then condition (B-1) with (B-3) implies that each (in general rectangular \( SU(2) \)-block) of \( \beta_0, \beta_{km}, k, m = 1, \ldots, N \), satisfies a condition of the form

\[
S_l^{(n_k)} \beta_{km} = \beta_{km} c_l^{(n_k)}, \quad i = 1, 2, 3,
\]

(B-4)

where \( S_l^{(n_k)} \) is \( 2n_k + 1 \) dimensional, \( S_l^{(n_m)} \) is \( 2n_m + 1 \) dimensional, and \( \beta_{km} \) has \( 2n_k + 1 \) rows and \( 2n_m + 1 \) columns.

We wish to show that the only nonvanishing \( \beta_{km} \) which satisfy (B-4) are square (i.e., \( n_k = n_m \)) and multiples of the identity: Apply relation (B-4) twice to get

\[
S_l^{(n_k)} S_l^{(n_m)} \beta_{km} = S_l^{(n_k)} \beta_{km} S_l^{(n_m)} = \beta_{km} c_l^{(n_k)} S_l^{(n_m)},
\]

(B-5)

where none of the explicit indices are summed over. Now sum over \( i = 1, 2, 3 \) and use the fact that \( S_l^{(n_k)} S_l^{(n_m)} = n_k(n_k + 1) \) to get

\[
n_k(n_k + 1) \beta_{km} = n_k(n_m + 1) \beta_{km}.
\]

(B-6)

Thus \( \beta_{km} \neq 0 \) if and only if \( n_k = n_m \). Furthermore, we have from the irreducibility of the set \( S_l^{(n_k)} \), \( i = 1, 2, 3 \), that \( \beta_{km} \) is a constant multiple of the \( 2n_k + 1 \) dimensional identity matrix \([15-17]\).

Thus \( \beta_0 \) may in general be decomposed into its \( SU(2) \)-blocks which vanish if nonsquare and are constant multiples of the identity if square.

Now consider the effect of condition (B-2) on the \( SL(2, C) \)-blocks. We may write \( \beta_0 \) in terms of its \( SL(2, C) \)-blocks \( \mathcal{A}_{ij} \), \( i, j = 1, \ldots, M \), and the boost generator in the 3-direction as

\[
N_3 = \bigoplus_{i=1}^{M} N_3^{(i)}.
\]

(B-7)

In this form condition (B-2) becomes

\[
2N_3^{(i)} \mathcal{A}_{ij} N_3^{(j)} - \mathcal{A}_{ij} [N_3^{(i)}]^2 - [N_3^{(i)}]^2 \mathcal{A}_{ij} = \mathcal{A}_{ij}.
\]

(B-8)

Now we know from the work of Bhabha \([15-17]\) that \( \mathcal{A}_{ij} \) is zero unless the \( i \)th and \( j \)th representation interlock, i.e., are of the form \((n, m)\) and \((n \pm \frac{1}{2}, m \pm \frac{1}{2})\). Let us consider the structure of \( \mathcal{A}_{ij} \) starting from the simplest such interlocking representation. We write out \( \mathcal{A}_{ij} \) in terms of its \( SU(2) \)-blocks and note the spin value. Using the representations of \( N_3^{(i)} \) given in Appendix A, we find that (B-4)
determines $\Delta_{ij}$ up to an arbitrary factor. Denote by $S(A)^{(i)}$ a representation of $SL(2, \mathbb{C})$ and all $\alpha$'s $\in \mathbb{C}$.

(i) $S(A)^{(i)} = (n, 0), S(A)^{(j)} = (n - \frac{1}{2}, \frac{1}{2}), n \geq 1$:

$$
\Delta_{ij} = \begin{bmatrix}
\alpha & n \\
 n - 1 & n
\end{bmatrix}
$$

and a blank entry = 0.

(ii) $S(A)^{(i)} = (n, 0), S(A)^{(j)} = (n + \frac{1}{2}, \frac{1}{2}), n \geq 0$:

$$
\Delta_{ij} = \begin{bmatrix}
n + 1 & n \\
 n & \alpha
\end{bmatrix}
$$

(iii) $S(A)^{(i)} = (n, \frac{1}{2}), S(A)^{(j)} = (n - \frac{1}{2}, 1), n \geq \frac{3}{2}$:

$$
\Delta_{ij} = \begin{bmatrix}
\alpha_0 & n + \frac{1}{2} \\
 n + \frac{1}{2} & \alpha_1 \\
 n - \frac{1}{2} & \alpha_1
\end{bmatrix}
$$

where

$$
\alpha_0 = \alpha_1 \left(\frac{2n}{n - \frac{1}{2}}\right)^{1/2}
$$

(iv) $S(A)^{(i)} = (n, \frac{1}{2}), S(A)^{(j)} = (n + \frac{1}{2}, 1), n \geq \frac{3}{2}$:

$$
\Delta_{ij} = \begin{bmatrix}
\alpha_0 & n + \frac{1}{2} \\
 n + \frac{1}{2} & \alpha_1 \\
 n - \frac{1}{2} & \alpha_1
\end{bmatrix}
$$

where

$$
\alpha_0 = \alpha_1 \left(\frac{n + \frac{3}{2}}{2n + 2}\right)^{1/2}
$$
(v) $S(A)^{(i)} = (n, 1), S(A)^{(ii)} = (n - \frac{1}{2}, \frac{3}{2}), n \geq 2$:

\[
\begin{array}{cccc}
& 1 & n - 1 & n - 2 \\
\alpha_0 & & & \\
\alpha_1 & & & \\
\alpha_2 & & & \\
\end{array}
\]

\[\Delta_{ij} = \]

\[\text{where} \]

\[\alpha_1 = \alpha_0 \left(\frac{2n - 1}{3n}\right)^{1/2}, \quad \alpha_2 = \alpha_0 \left(\frac{n - 1}{3n}\right)^{1/2}.\]

(vi) $S(A)^{(i)} = (n, 1), S(A)^{(ii)} = (n + \frac{1}{2}, \frac{3}{2}), n \geq 1$:

\[
\begin{array}{cccc}
& 1 & n & n - 1 \\
\alpha_0 & & & \\
\alpha_1 & & & \\
\alpha_2 & & & \\
\end{array}
\]

\[\Delta_{ij} = \]

\[\text{where} \]

\[\alpha_1 = \alpha_0 \left(\frac{2n + 3}{n + 2}\right)^{1/2}, \quad \alpha_2 = \alpha_0 \left(\frac{3n + 3}{n + 2}\right)^{1/2}.\]

Special cases missed by the above scheme (low $n$ values) are

(vii) $S(A)^{(i)} = (\frac{1}{2}, 0), S(A)^{(ii)} = (0, \frac{3}{2})$:

\[
\Delta_{ij} = \]

\[\text{with} \]

\[\alpha_1 = \alpha_0 .\]

(viii) $S(A)^{(i)} = (1, \frac{3}{2}), S(A)^{(ii)} = (\frac{3}{2}, 1)$:

\[
\Delta_{ij} = \]

\[\text{with} \]

\[\alpha_1 = -\frac{1}{2}\alpha_0 .\]
RELATIVISTIC WAVE EQUATIONS

(ix) \( S(\lambda)^{(i)} = (\frac{3}{2}, 1), S(\lambda)^{(j)} = (1, \frac{3}{2}) \):

\[
\begin{array}{ccc}
\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\
\alpha_0 & \alpha_1 & \frac{1}{2} \\
\frac{1}{2} & \alpha_2 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

\( \Delta_{ij} = \)

\( \alpha_1 = -\frac{2}{3}\alpha_0, \quad \alpha_2 = \frac{1}{3}\alpha_0. \)

(B-17)

(x) \( S(\lambda)^{(i)} = (2, \frac{3}{2}), S(\lambda)^{(j)} = (\frac{3}{2}, 2) \):

\[
\begin{array}{cccc}
\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\alpha_0 & \alpha_1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \alpha_2 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

\( \Delta_{ij} = \)

where

\( \alpha_1 = -\frac{2}{3}\alpha_0, \quad \alpha_2 = \frac{1}{3}\alpha_0, \quad \alpha_0 = -\frac{1}{3}\alpha_0. \)

(B-18)

It is straightforward to verify these coefficients. Start with Eq. (B-8), use the fact that \( \Delta_{ij} \) has nonvanishing elements only in its square blocks, write out \( N_i^{ij} \) and \( N_i^{ji} \) from Appendix A, and use the algebraic properties of the \( S_n \) and \( K_3 \) matrices to solve for the coefficients. Since one is guaranteed that interlocking representations will admit a nontrivial \( \Delta_{ij} \), one need only perform a few simple manipulations to determine these values.

APPENDIX C: REPEATING REPRESENTATIONS

In this appendix we shall consider those representations of \( SL(2, C) \) which lead to \( \beta \)-matrices which satisfy all of our requirements but which contain repetitions of the same irreducible components.
Since a representation cannot interlock with itself, we can have no repeaters in the $A \oplus B$ case. The first repetitions may enter in the case

(i) $A \oplus B \oplus A$

The general structure of the equation

$$D\phi = \phi$$

with $D = (i\beta / m) \beta \cdot \partial$ for this representation can be written as

$$D\phi = \begin{bmatrix} D_{12} & D_{21} & D_{23} \\ D_{21} & D_{22} & 0 \\ D_{23} & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ A \end{bmatrix} = \begin{bmatrix} A \\ B \\ A \end{bmatrix}.$$  \hspace{1cm} (C-2)

Since they transform identically, we may always take linear combinations of the top and bottom sets of components of $\phi$ and map $D$ by the corresponding similarity transformation. Take

$$T = \begin{bmatrix} a & b \\ 1 & -a \end{bmatrix},$$  \hspace{1cm} (C-3)

where for $a^2 + b^2 = 1$, $T$ is unitary. Transforming $D$ we find

$$D' = TDT^{-1} = \begin{bmatrix} aD_{12} + bD_{23} \\ aD_{21} + bD_{23} \\ bD_{21} - aD_{23} \end{bmatrix},$$  \hspace{1cm} (C-4)

and since $D_{12}$ and $D_{23}$ both act between representations $A$ and $B$, they must, by the results of Appendix B, be proportional. Hence we may choose $b$ and $a$ such that the lowest line in $D'$ vanishes. Thus in this primed bases we see from Eq. (C-1) that the lower components of $\phi$ which transform under $A$ will be identically zero and separate from the system. Thus a repeating representation of the form $A \oplus B \oplus A$ is equivalent to the case $A \oplus B$.

Two possibilities for repeating representations within a direct sum of four irreducible representations are possible.
(ii) $A \oplus B \oplus B \oplus C$ with the coupling scheme
\[
\begin{array}{c}
A & \leftarrow & B \\
\uparrow & & \uparrow \\
B & \rightarrow & C.
\end{array}
\]
(C.5)

For this case
\[
D = \begin{array}{ccc}
D_{12} & D_{13} & D_{14} \\
\hline
D_{21} & & A \\
D_{31} & & B \\
D_{41} & & C
\end{array}
\]
(C.6)

and $D_{12}(D_{21})$ is proportional to $D_{13}(D_{31})$. In the same way as before, we may again make a unitary transformation between the two $B$ components such that in the primed basis one of the two sets vanishes identically. Thus a representation with the coupling scheme (ii) with repeaters is equivalent to the form $A \oplus B \oplus C$.

(iii) $A \oplus B \oplus B \oplus C$ with the coupling scheme
\[
\begin{array}{c}
A & \leftarrow & B \\
\uparrow & & \downarrow \\
B & \rightarrow & C.
\end{array}
\]
(C.7)

$D$ for this case takes the form
\[
D = \begin{array}{ccc}
\alpha_{11}D_1 & \alpha_{13}D_1 & A \\
\hline
\alpha_{21}D_1' & & \alpha_{24}D_2 \\
\alpha_{31}D_1' & & \alpha_{34}D_3 \\
\alpha_{41}D_1' & \alpha_{43}D_5 & C
\end{array}
\]
(C.8)

where $\alpha_{ij} \in \mathbb{C}$ and $D_1, D_2$ are the corresponding rectangular $SL(2, \mathbb{C})$-blocks of $D$. 
Now consider the general similarity transformation \( D' = ADA^{-1} \) where

\[
A = \begin{bmatrix}
1 & \alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{11} & \alpha_{12} \\
\alpha_{22} & \alpha_{21} & 1
\end{bmatrix}, \quad \alpha_{ij} \in \mathbb{C}, \quad (C-9)
\]

thus mixing the components with the same transformation properties. We find for \( D' \),

\[
D' = \frac{1}{\det A} \begin{bmatrix}
(a_{23} \alpha_{12} - a_{21} \alpha_{13}) D_1 \\
(a_{11} \alpha_{21} + a_{12} \alpha_{22}) D_1^\dagger \\
(a_{21} \alpha_{21} + a_{22} \alpha_{23}) D_1^\dagger \\
(a_{22} \alpha_{12} - a_{21} \alpha_{13}) D_2^\dagger
\end{bmatrix}
\]

\[
\begin{array}{c|c|c|c|c}
& A & B & C \\
\hline
(a_{11} \alpha_{13} - a_{12} \alpha_{12}) D_1 & (a_{11} \alpha_{24} + a_{12} \alpha_{23}) D_2 \\
& (a_{11} \alpha_{24} + a_{12} \alpha_{23}) D_2 & (a_{11} \alpha_{24} + a_{22} \alpha_{23}) D_2 \\
& (a_{11} \alpha_{24} + a_{22} \alpha_{23}) D_2 & (a_{11} \alpha_{24} + a_{22} \alpha_{23}) D_2 \\
& (a_{11} \alpha_{24} - a_{12} \alpha_{13}) D_2^\dagger & (a_{11} \alpha_{24} - a_{12} \alpha_{13}) D_2^\dagger & (a_{11} \alpha_{24} - a_{12} \alpha_{13}) D_2^\dagger
\end{array}
\]

We cannot immediately deduce that there exists a transformation such that one of the sets of middle components vanishes identically. But since for \( p^2 = m^2 \) the algebraic condition (12) requires \( D^a = D \), we may get further constraints on the coefficients \( \alpha_{ij} \). The (2, 1) and (3, 1) components of the equation \( D^a = D \) may be written

\[
\frac{\alpha_{11}}{\alpha_{21}} D_1^\dagger = \left[ \frac{\alpha_{21} \alpha_{12} \alpha_{23}}{\alpha_{24}} + \frac{\alpha_{21} \alpha_{12} \alpha_{23}}{\alpha_{24}} \right] D_1^\dagger D_1 D_1^\dagger + \left[ \alpha_{44} \alpha_{21} + \alpha_{45} \alpha_{21} \right] D_2 D_1 D_1^\dagger \quad (C-11)
\]
and

\[
\frac{\alpha_{31}}{\alpha_{34}} D_1^* = \left[ \frac{\alpha_{31}\alpha_{12}\alpha_{21}}{\alpha_{34}} + \frac{\alpha_{31}\alpha_{13}\alpha_{31}}{\alpha_{34}} \right] D_1^* D_1 D_1^* + \left[ \alpha_{45}^{\alpha_{31}} + \alpha_{43}^{\alpha_{31}} \right] D_2 D_3 D_1^*.
\] (C-12)

and subtracting we find

\[
\left[ \frac{\alpha_{31}}{\alpha_{34}} - \frac{\alpha_{31}}{\alpha_{34}} \right] D_1^* = \left[ \left( \alpha_{12}^{\alpha_{21}} + \alpha_{13}^{\alpha_{31}} \right) \left( \frac{\alpha_{31}}{\alpha_{34}} - \frac{\alpha_{31}}{\alpha_{34}} \right) \right] D_1^* D_1 D_1^*. \quad (C-13)
\]

If we now multiply by $D_4$, then since $D_4 D_1^* \neq 0$ and $(D_4 D_1^*)^2 \neq D_4 D_1^*$, we must have

\[
\alpha_{31}^{\alpha_{31}} = \frac{\alpha_{31}^{\alpha_{31}}}{\alpha_{34}^{\alpha_{34}}}, \quad (C-14)
\]

that is, the coupling of the two $B$ representations in (C-8) to the $A$ representation and $C$ representation are proportional. Thus we may choose, e.g., $\alpha_{31}^{\alpha_{31}}$ and $\alpha_{22}^{\alpha_{22}}$ in (C-10) such that if the algebraic requirements are met by $D$, then there exists a nonsingular matrix $A$ such that for $D' = A D A^{-1}$, we have $D' \phi' = \phi'$, and in this basis one set of $B$-components of $\phi'$ vanishes identically and the system reduces to a 3-representation case. We have assumed in the above that the coefficients of interest were nonzero. Similar arguments go through if this assumption is not made.

References

1. We shall use the metric $\gamma_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.


20. One may wish to consider this case for other reasons, e.g., if one insisted upon the existence of an Hermitianizing matrix which mixed components which are not mixed by the $\beta$-matrices such as in the parity doubled representation $(s, 0) \oplus (s - \frac{1}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, s - \frac{1}{2}) \oplus (0, s)$. We shall restrict our attention here, however, only to the case of irreducible $\beta$-matrices.

21. For example, each member of the class $(n, m) \oplus (n + \frac{1}{2}, m + \frac{1}{2}) \oplus (n - \frac{1}{2}, m - \frac{1}{2}) \oplus (n - \frac{1}{2}, m + \frac{1}{2})$ is equivalent by conjugation to a member of class $(a - 1)$ and is therefore neglected.
