PHYSICAL REVIEW D VOLUME 10, NUMBER 10 15 NOVEMBER 1974

Poincaré invariance of the spin-$\frac{3}{2}$ field in the presence of a minimal external electromagnetic interaction

G. B. Mainland
Dublin Institute for Advanced Studies, Dublin 4, Ireland

E. C. G. Sudarshan
Center for Particle Theory, The University of Texas, Austin, Texas 78712
(Received 19 February 1974)

Using the action principle, we determine the Poincaré transformation properties of second-quantized fields in the presence of external fields. We then construct expressions for the generators of the Poincaré group and explicitly verify that a second-quantized spin-$\frac{3}{2}$ field interacting with a minimal external electromagnetic field transforms covariantly. We thus conclude that the negative metric which appears in the quantization of the spin-$\frac{3}{2}$ field is not the result of a noncovariant quantization as has been claimed.

When a minimal external electromagnetic interaction is introduced into the spin-$\frac{3}{2}$ Rarita-Schwinger[1] equation, the anticommutator of the second-quantized spin-$\frac{3}{2}$ field should be positive-definite in all Lorentz frames, but it is not.[2] This appearance of a negative metric has been attributed to an incorrect quantization of the spin-$\frac{3}{2}$ field which leads to noncovariant transformation properties for the field variables.[3] But, as we shall demonstrate, the spin-$\frac{3}{2}$ field does transform covariantly.

The meaning of the statement "a system of second-quantized fields transforms covariantly" is clear: The generators of Poincaré transformations transform *all* the field variables in such a way that the equations of motion remain form-invariant. If an external field is present, this is not what is meant. The (second-quantized) generators of the Poincaré group act only on the second-quantized fields and commute with the external fields. Thus, only the second-quantized fields are transformed. The situation is further complicated when, as in the case of the quantized spin-$\frac{3}{2}$ field interacting with an external electromagnetic field, constraints exist between the quantized and external fields. Then only those parts of the second-quantized fields are transformed which do not depend explicitly on the external fields. Knowing that the Poincaré generators transform only those parts of the fields which are independent of the external fields, it is possible to use the action principle[4] to determine what the commutators of the Poincaré generators with the second-quantized fields should be. Using the anticommutation relation for the spin-$\frac{3}{2}$ field as given in Ref. 2, we explicitly calculate the commutators of the spin-$\frac{3}{2}$ field with the Poincaré generators and verify that they are identical with the relations demanded by the action principle. Thus, we conclude that the quantization given in Ref. 2 leads to covariant transformation properties for the spin-$\frac{3}{2}$ field.

The organization of the paper is as follows: First, for an arbitrary Lagrangian containing second-quantized and external fields, we use the action principle[4] to determine the correct Lorentz transformation properties of the second-quantized fields. We then briefly review the field and constraint equations for a Rarita-Schwinger spin-$\frac{3}{2}$ field interacting minimally with an external electromagnetic field. By determining the explicit coordinate dependence of the spin-$\frac{3}{2}$ field from the constraint equations, we are able to use the general results derived from the action principle and determine the correct Lorentz transformation properties for the spin-$\frac{3}{2}$ field. Finally, we explicitly calculate the commutator of the spin-$\frac{3}{2}$ field with the generators of translations and homogeneous Lorentz transformations and verify that the fields transform covariantly. The appearance of a negative metric in the anticommutator of the spin-$\frac{3}{2}$ field apparently results from a more fundamental flaw in the theory and is not the result of a noncovariant quantization.

Because an external field $\xi^{\hat{\beta}}$ is an explicit function of space-time, we must use a slightly different notation for derivatives than is customary in quantum field theory. We use $d_\mu = d/dx^\mu$ to indicate a total derivative and $\partial_\mu = \partial/\partial x^\mu$ to designate a partial derivative with respect to the explicit coordinate dependence. Hamilton's equation, for example, is then written

$$\dot{\phi}^{\tilde{\alpha}} = d_0\phi^{\tilde{\alpha}} = i[H, \phi^{\tilde{\alpha}}] + \partial_0\phi^{\tilde{\alpha}} ,$$

where $[A,B]$ is the commutator and H is the Hamiltonian.

With the exception of derivatives, our notation is that of Bjorken and Drell.[5] The space-time coordinates are denoted by $x^\mu = (t, x^1, x^2, x^3)$ and we use the metric tensor $g^{\mu\nu}$ where $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$. The Dirac gamma matrices γ^μ satisfy $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$. Greek indices range from 0 through 3, Roman indices range from 1 through 3, and all repeated indices are summed over the range of the index.

To derive the Poincaré transformation properties of second-quantized fields in the presence of external fields, we will apply the action principle[4] to the Lagrangian density $\mathcal{L}(\phi^{\tilde\alpha}, d_\mu \phi^{\tilde\alpha}, \xi^{\hat\beta}, d_\mu \xi^{\hat\beta})$ where $\phi^{\tilde\alpha}$ and $\xi^{\hat\beta}$ are respectively the second-quantized and external fields. (Indices with a tilde or a caret range respectively over the number of second-quantized or external fields.) Any explicit coordinate dependence of $\phi^{\tilde\alpha}$ results from $\phi^{\tilde\alpha}$'s dependence on $\xi^{\hat\beta}$ and derivatives of $\xi^{\hat\beta}$. To distinguish between implicit and explicit coordinate dependence in the argument of $\phi^{\tilde\alpha}$, we write it as

$$\phi^{\tilde\alpha} = \phi^{\tilde\alpha}(x, \xi^{\hat\beta}(x), \partial_\gamma \xi^{\hat\beta}(x), \ldots, [\partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)]),$$

where the first x in the argument indicates implicit dependence, and the remaining terms represent the possible explicit dependence. For convenience, the above relation will be abbreviated as

$$\phi^{\tilde\alpha} = \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)). \tag{1}$$

From the principles of quantum theory, the total variation $\delta_0 \phi^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]]$ of the field $\phi^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]]$ on the spacelike surface σ under an infinitesimal canonical transformation is given by

$$\begin{aligned}\delta_0 \phi^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]] &\equiv \phi'^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]] - \phi^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]] \\ &= i[F[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]], \phi^{\tilde\alpha}[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]]],\end{aligned} \tag{2}$$

where $F[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]]$ is the generator of the canonical transformation.

To formulate the action principle we consider a region bounded by two parallel spacelike planes σ_1 and σ_2. The action W is defined by

$$W = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathcal{L}.$$

The action principle is the statement that the variation of the action δW as a result of the variation of the fields,

$$\begin{aligned}\phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) &\to \phi'^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) \\ &= \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) + \delta_0 \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)),\end{aligned}$$

and the variation of the surfaces, $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$, is equal to the difference of the generator of the transformation evaluated at σ_1 and σ_2. That is,

$$\delta W = F[\sigma_2, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma_2]] - F[\sigma_1, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma_1]]. \tag{3}$$

The variation in the action is given by

$$\delta W = \int_{\sigma_1}^{\sigma_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^{\tilde\alpha}} \delta_0 \phi^{\tilde\alpha} + \frac{\partial \mathcal{L}}{\partial (d_\mu \phi^{\tilde\alpha})} d_\mu \delta_0 \phi^{\tilde\alpha} \right] + \left(\int_{\sigma_2} - \int_{\sigma_1} \right) d\sigma_\nu \delta x^\nu \mathcal{L}. \tag{4}$$

The first term results from varying the field components by an amount $\delta_0 \phi^{\tilde\alpha}$ at each space-time point and the second term comes from the shift in the boundary planes by an amount δx^μ. The external fields $\xi^{\hat\beta}$, or course, are not varied. In order to obtain the usual Lagrange equations of motion, we have assumed that $\delta_0 \phi^{\tilde\alpha}$ and $d_\mu \delta_0 \phi^{\tilde\alpha}$ commute with certain field variables so that (4) is correct. Integrating the second quantity in the first term of (4) by parts and converting the four-divergence into a surface term, one obtains

$$\delta W = \int_{\sigma_1}^{\sigma_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^{\tilde\alpha}} - d_\mu \frac{\partial \mathcal{L}}{\partial (d_\mu \phi^{\tilde\alpha})} \right] \delta_0 \phi^{\tilde\alpha} + \left(\int_{\sigma_2} - \int_{\sigma_1} \right) d\sigma_\rho \left[\frac{\partial \mathcal{L}}{\partial (d_\rho \phi^{\tilde\alpha})} \delta_0 \phi^{\tilde\alpha} + g^{\rho\mu} \mathcal{L} \delta x_\mu \right].$$

The requirement (3) is then satisfied if

$$\frac{\partial \mathcal{L}}{\partial \phi^{\tilde\alpha}} - d_\mu \frac{\partial \mathcal{L}}{\partial (d_\mu \phi^{\tilde\alpha})} = 0$$

and

$$F[\sigma, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}[\sigma]] = \int_\sigma d\sigma_\rho \left[\frac{\partial \mathcal{L}}{\partial (d_\rho \phi^{\tilde\alpha})} \delta_0 \phi^{\tilde\alpha} + g^{\rho\mu} \mathcal{L} \delta x_\mu \right]. \tag{5}$$

At this point we introduce the local variation for second-quantized fields in the presence of external fields

$$\delta \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) = \phi'^{\tilde\alpha}(x', \partial'_\gamma \cdots \partial'_\delta \xi^{\hat\beta}(x')) - \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) \tag{6a}$$

$$= \delta_0 \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) + d^\mu \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) \delta x_\mu, \tag{6b}$$

which is so named because the geometrical point denoted x in the old coordinate system is denoted x' in the new coordinate system. Note that under a local variation the second-quantized fields and coordinates are transformed, but the external field itself is not. If the transformation $x \to x'$ implicit in (6a) were a Poincaré transformation, then (6a) would be the local variation of the field under a Poincaré transformation. Thus, we conclude that under the Poincaré transformation

$$x^\mu \to x'^\mu = (g^{\mu\nu} + \epsilon^{\mu\nu})x_\nu + \epsilon^\mu \ , \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \tag{7a}$$

the fields are transformed in the following manner:

$$\phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) \to \phi'^{\tilde\alpha}(x', \partial'_\gamma \cdots \partial'_\delta \xi^{\hat\beta}(x')) = \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) + \delta\phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)), \tag{7b}$$

$$\xi^{\hat\beta}(x) \to \xi^{\hat\beta}(x') . \tag{7c}$$

Our objective now is to determine the total variation $\delta_0\phi^{\tilde\alpha}$ under the infinitesimal Lorentz transformation (7), so that we can use (2) to determine the commutators of the spin-$\frac{3}{2}$ field with the generators of the transformation. This is most easily accomplished by first determining the local variation and then using (6b) to obtain the total variation. To determine the local variation under (7) we will exploit our knowledge of the Poincaré transformation properties of the fields when all of the fields (external as well as internal) are transformed. Under such a transformation the field equations are invariant provided

$$\phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) \to \phi'^{\tilde\alpha}(x', \partial'_\gamma \cdots \partial'_\delta \xi'^{\hat\beta}(x')) = \phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) + \tfrac{1}{2}\tilde I^{\tilde\alpha\tilde\beta}_{\mu\nu}\epsilon^{\mu\nu}\phi^{\tilde\beta}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) , \tag{8}$$

$$\xi^{\hat\alpha}(x) \to \xi'^{\hat\alpha}(x') = \xi^{\hat\alpha}(x) + \tfrac{1}{2}\hat I^{\hat\alpha\hat\beta}_{\mu\nu}\epsilon^{\mu\nu}\xi^{\hat\beta}(x) . \tag{9}$$

Of course (8) and (9) are not the transformations generated by the Poincaré generators because the external field has been transformed. But, as we shall see, they will be useful in obtaining an explicit expression for the local variation generated by the Poincaré generators. To determine the local variation under the Poincaré transformation (7) we first use (9) to rewrite $\partial'_\gamma \cdots \partial'_\delta \xi^{\hat\beta}(x')$ as follows:

$$\begin{aligned}
\partial'_\gamma \cdots \partial'_\delta \xi^{\hat\beta}(x') &= (\partial'_\gamma \cdots \partial'_\delta)\{\xi'^{\hat\beta}(x') - [\xi'^{\hat\beta}(x') - \xi^{\hat\beta}(x')]\} \\
&= (\partial'_\gamma \cdots \partial'_\delta)\{\xi'^{\hat\beta}(x') - [\xi'^{\hat\beta}(x') - \xi^{\hat\beta}(x^\mu + \epsilon^{\mu\nu}x_\nu + \epsilon^\mu)]\} \\
&= (\partial'_\gamma \cdots \partial'_\delta)\left\{\xi'^{\hat\beta}(x') - [\xi'^{\hat\beta}(x') - \xi^{\hat\beta}(x)] + \frac{\partial\xi^{\hat\beta}}{\partial x^\mu}(\epsilon^{\mu\nu}x_\nu + \epsilon^\mu)\right\} \\
&= (\partial'_\gamma \cdots \partial'_\delta)\left[\xi'^{\hat\beta}(x') - \tfrac{1}{2}\hat I^{\hat\beta\hat\alpha}_{\mu\nu}\epsilon^{\mu\nu}\xi^{\hat\alpha} + \frac{\partial\xi^{\hat\beta}}{\partial x^\mu}(\epsilon^{\mu\nu}x_\nu + \epsilon^\mu)\right] .
\end{aligned}$$

Equation (6a) can then be written

$$\delta\phi^{\tilde\alpha}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) = \tfrac{1}{2}\tilde I^{\tilde\alpha\tilde\beta}_{\mu\nu}\epsilon^{\mu\nu}\phi^{\tilde\beta}(x, \partial_\gamma \cdots \partial_\delta \xi^{\hat\beta}(x)) + \left[\frac{\partial\phi^{\tilde\alpha}}{\partial\xi^{\hat\beta}} + \cdots + \frac{\partial\phi^{\tilde\alpha}}{\partial(\partial_\gamma \cdots \partial_\delta\xi^{\hat\beta})}\partial_\gamma \cdots \partial_\delta\right]\left[-\tfrac{1}{2}\hat I^{\hat\beta\hat\alpha}_{\mu\nu}\epsilon^{\mu\nu}\xi^{\hat\alpha} + \frac{\partial\xi^{\hat\beta}}{\partial x^\mu}(\epsilon^{\mu\nu}x_\nu + \epsilon^\mu)\right] . \tag{10}$$

Introducing the more compact notation

$$\frac{d\phi^{\tilde\alpha}}{d\xi^{\hat\beta}} \equiv \left[\frac{\partial\phi^{\tilde\alpha}}{\partial\xi^{\hat\beta}} + \cdots + \frac{\partial\phi^{\tilde\alpha}}{\partial(\partial_\gamma \cdots \partial_\delta\xi^{\hat\beta})}\partial_\gamma \cdots \partial_\delta\right] , \tag{11}$$

and suppressing the arguments of $\phi^{\tilde\alpha}$, (10) can be rewritten as

$$\delta\phi^{\tilde\alpha} = \tfrac{1}{2}\tilde I^{\tilde\alpha\tilde\beta}_{\mu\nu}\epsilon^{\mu\nu}\phi^{\tilde\beta} + \frac{d\phi^{\tilde\alpha}}{d\xi^{\hat\beta}}\left[-\tfrac{1}{2}\hat I^{\hat\beta\hat\alpha}_{\mu\nu}\epsilon^{\mu\nu}\xi^{\hat\alpha} + \frac{\partial\xi^{\hat\beta}}{\partial x^\mu}(\epsilon^{\mu\nu}x_\nu + \epsilon^\mu)\right] . \tag{12}$$

Note that $d\phi^{\tilde\alpha}/d\xi^{\hat\beta}$ does not commute with c-number functions of x because of the derivatives $\partial_\gamma \cdots \partial_\delta$ in the definition (11). The total variation is found by combining (6b) and (12):

$$\delta_0\phi^{\tilde\alpha} = \left(-d_\mu\phi^{\tilde\alpha} + \frac{d\phi^{\tilde\alpha}}{d\xi^{\hat\beta}}\frac{\partial\xi^{\hat\beta}}{\partial x^\mu}\right)(\epsilon^{\mu\nu}x_\nu + \epsilon^\mu) + \tfrac{1}{2}\tilde I^{\tilde\alpha\tilde\beta}_{\mu\nu}\epsilon^{\mu\nu}\phi^{\tilde\beta} - \tfrac{1}{2}\frac{d\phi^{\tilde\alpha}}{d\xi^{\hat\beta}}\hat I^{\hat\beta\hat\alpha}_{\mu\nu}\xi^{\hat\alpha} . \tag{13}$$

By writing $F[\sigma, \partial_\gamma \cdots \partial_\delta\xi^{\hat\beta}[\sigma]]$ in the form

$$F[\sigma, \partial_\gamma \cdots \partial_\delta\xi^{\hat\beta}[\sigma]] = -P^\mu\epsilon_\mu + \tfrac{1}{2}J_{\mu\nu}\epsilon^{\mu\nu} , \tag{14}$$

and using (13), (5) yields the following expressions for the generators of translations P^μ and homogeneous Lorentz transformations $J_{\mu\nu}$:

$$P^\mu = \int d\sigma_\rho \left[\frac{\partial \mathfrak{L}}{\partial(d_\rho \phi^{\tilde{\alpha}})} \left(d^\mu \phi^{\tilde{\alpha}} - \frac{\partial \phi^{\tilde{\alpha}}}{\partial x^\mu} \right) - g^{\rho\mu} \mathfrak{L} \right] , \tag{15}$$

$$J_{\mu\nu} = \int d\sigma_\rho \left\{ \frac{\partial \mathfrak{L}}{\partial(d_\rho \phi^{\tilde{\alpha}})} \left[\tilde{I}^{\tilde{\alpha}\tilde{\beta}}_{\mu\nu} \phi^{\tilde{\beta}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \hat{I}^{\hat{\beta}\hat{\alpha}}_{\mu\nu} \xi^{\hat{\alpha}} + \left(d_\nu \phi^{\tilde{\alpha}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\nu} \right) x_\mu - \left(d_\mu \phi^{\tilde{\alpha}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\mu} \right) x_\nu \right] + (g^\rho{}_\mu x_\nu - g^\rho{}_\nu x_\mu) \mathfrak{L} \right\} . \tag{16}$$

In obtaining (15) we have also used

$$\frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\mu} = \frac{\partial \phi^{\tilde{\alpha}}}{\partial x^\mu} .$$

According to (2), (13), and (14), P^μ and $J_{\mu\nu}$ satisfy the equation

$$i[-P^\mu \epsilon_\mu + \tfrac{1}{2} J_{\mu\nu} \epsilon^{\mu\nu}, \phi^{\tilde{\alpha}}] = \left(-d_\mu \phi^{\tilde{\alpha}} + \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\mu} \right) (\epsilon^{\mu\nu} x_\nu + \epsilon^\mu) + \tfrac{1}{2} \tilde{I}^{\tilde{\alpha}\tilde{\beta}}_{\mu\nu} \epsilon^{\mu\nu} \phi^{\tilde{\beta}} - \tfrac{1}{2} \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \hat{I}^{\hat{\beta}\hat{\alpha}}_{\mu\nu} \epsilon^{\mu\nu} \xi^{\hat{\alpha}} .$$

Since the above equation is valid for arbitrary values of the parameters ϵ_μ and $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, P^μ and $J_{\mu\nu}$ satisfy the equations

$$i[P^\mu, \phi^{\tilde{\alpha}}] = (d^\mu - \partial^\mu) \phi^{\tilde{\alpha}} , \tag{17}$$

$$i[J_{\mu\nu}, \phi^{\tilde{\alpha}}] = -\left(d_\mu \phi^{\tilde{\alpha}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\mu} \right) x_\nu + \left(d_\nu \phi^{\tilde{\alpha}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \frac{\partial \xi^{\hat{\beta}}}{\partial x^\nu} \right) x_\mu + \tilde{I}^{\tilde{\alpha}\tilde{\beta}}_{\mu\nu} \phi^{\tilde{\beta}} - \frac{d\phi^{\tilde{\alpha}}}{d\xi^{\hat{\beta}}} \hat{I}^{\hat{\beta}\hat{\alpha}}_{\mu\nu} \xi^{\hat{\alpha}} . \tag{18}$$

We would now like to obtain explicit expressions for (17) and (18) for the spin-$\frac{3}{2}$ field interacting with an external electromagnetic field. To accomplish this we first must determine the constraint equations from which the explicit space-time dependence can be found. In the presence of a minimal external electromagnetic field, the Rarita-Schwinger[1] vector spinor ψ^λ obeys field equations that can be obtained from the Lagrangian density[3] (we omit the tilde and caret signs on the vector indices from now on):

$$\mathfrak{L} = \bar{\psi}_\mu (D_\sigma \gamma^\sigma + m) \psi^\mu - \bar{\psi}_\mu (D^\nu \gamma^\mu + D^\mu \gamma^\nu) \psi_\nu + \bar{\psi}_\mu \gamma^\mu (D^\rho \gamma_\rho - m) \gamma^\nu \psi_\nu , \tag{19}$$

where $D_\mu = -i d_\mu + eA_\mu$ and e is the charge of the spin-$\frac{3}{2}$ field. The field equation obtained from (19) is

$$(D_\sigma \gamma^\sigma + m) \psi^\mu - (D^\nu \gamma^\mu + D^\mu \gamma^\nu) \psi_\nu + \gamma^\mu (D^\rho \gamma_\rho - m) \gamma^\nu \psi_\nu = 0 . \tag{20}$$

The primary constraint

$$-D^i \psi^i + D^i \gamma_i \gamma^j \psi_j - m\gamma^i \psi_i = 0 \tag{21}$$

results from setting $\mu = 0$ in (20). The secondary constraint

$$B^\mu \psi_\mu = 0 ;$$

$$B^\mu = \gamma^\mu + \frac{ie}{3m^2} F_{\rho\nu} \gamma^\rho \gamma^\mu \gamma^\nu , \quad F_{\rho\nu} = \partial_\nu A_\rho - \partial_\rho A_\nu \tag{22}$$

is found by left-multiplying (20) by γ_μ and D_μ and combining the two resulting equations.

To determine the explicit coordinate dependence of ψ^i, we examine the field equation (20) for one infinitesimal transformation and the primary constraint (21) for another. Exploiting the invariance of the field equations under these transformations is just a trick to determine the explicit coordinate dependence of the spin-$\frac{3}{2}$ field. If, in analogy to (8) and (9), we transform the external as well as the second-quantized fields,

$$x^\mu \to x'^\mu = (g^{\mu\nu} + \epsilon^{\mu\nu}) x_\nu + \epsilon^\mu , \quad d'^\mu = (g^{\mu\nu} + \epsilon^{\mu\nu}) d_\nu , \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} , \tag{23a}$$

$$A^\beta(x) \to A'^\beta(x') = (g^{\beta\alpha} + \tfrac{1}{2} \hat{I}^{\beta\alpha}_{\mu\nu} \epsilon^{\mu\nu}) A_\alpha(x) , \tag{23b}$$

$$\psi^\alpha(x, \partial_\gamma \cdots \partial_6 A^\beta(x)) \to \psi'^\alpha(x', \partial'_\gamma \cdots \partial'_6 A'^\beta(x')) = (g^{\alpha\beta} + \tfrac{1}{2} \tilde{I}^{\alpha\beta}_{\mu\nu} \epsilon^{\mu\nu}) \psi_\beta(x, \partial_\gamma \cdots \partial_6 A^\beta(x)) , \tag{23c}$$

the field equations remain invariant provided

$$\hat{I}^{\alpha\beta}_{\mu\nu} = g^\alpha{}_\mu g^\beta{}_\nu - g^\alpha{}_\nu g^\beta{}_\mu , \tag{24a}$$

$$\tilde{I}^{\alpha\beta}_{\mu\nu} = \tfrac{1}{4} [\gamma_\mu, \gamma_\nu] g^{\alpha\beta} + g^\alpha{}_\mu g^\beta{}_\nu - g^\alpha{}_\nu g^\beta{}_\mu . \tag{24b}$$

From (23) we have

$$\psi'^{\alpha}(x',\partial'_{\gamma}\cdots\partial'_{6}A'^{\beta}(x'))=\psi'^{\alpha}(x',\partial'_{\gamma}\cdots\partial'_{6}[A^{\beta}(x)+\tfrac{1}{2}\hat{I}^{\beta\sigma}_{\mu\nu}\epsilon^{\mu\nu}A_{\sigma}])$$
$$=\psi'^{\alpha}(x',\partial_{\gamma}\cdots\partial_{6}A^{\beta}(x))+\tfrac{1}{2}\frac{d\psi^{\alpha}}{dA^{\beta}}\hat{I}^{\beta\sigma}_{\mu\nu}\epsilon^{\mu\nu}A_{\sigma}.$$

Combining the above equation with (23c), we obtain

$$\frac{1}{2}\frac{d\psi^{\alpha}}{dA^{\beta}}\hat{I}^{\beta\sigma}_{\mu\nu}\epsilon^{\mu\nu}A_{\sigma}=(g^{\alpha\beta}+\tfrac{1}{2}\tilde{I}^{\alpha\beta}_{\mu\nu}\epsilon^{\mu\nu})\psi_{\beta}(x,\partial_{\gamma}\cdots\partial_{6}A^{\beta}(x))-\psi'^{\alpha}(x',\partial'_{\gamma}\cdots\partial'_{6}A^{\beta}(x)). \tag{25}$$

If we now consider the transformation

$$x^{\mu}\to x'^{\mu}=(g^{\mu\nu}+\epsilon^{\mu\nu})x_{\nu}+\epsilon^{\mu},$$
$$d'^{\mu}=(g^{\mu\nu}+\epsilon^{\mu\nu})d_{\nu}, \tag{26a}$$
$$\epsilon^{\mu\nu}=-\epsilon^{\nu\mu},$$
$$A^{\beta}(x)\to A^{\beta}(x), \tag{26b}$$

$$\psi^{i}(x,\partial_{\gamma}\cdots\partial_{6}A^{\beta}(x))\to\psi'^{i}(x',\partial'_{\gamma}\cdots\partial'_{6}A^{\beta}(x))$$
$$\equiv(g^{i\beta}+\tfrac{1}{2}\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu})\psi_{\beta}(x,\partial_{\gamma}\cdots\partial_{6}A^{\beta}(x))+\tfrac{1}{2}\tau^{i}_{\mu\nu}\epsilon^{\mu\nu}, \tag{26c}$$

we show in the Appendix that the primary constraint (21) is form-invariant provided

$$\tau^{i}_{\mu\nu}=e(D^{i}+\tfrac{1}{2}m\gamma^{i})(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}\hat{I}^{k\beta}_{\mu\nu}A_{\beta}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})$$
$$=\{\{e(D^{i}+\tfrac{1}{2}m\gamma^{i})(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})\}\}\hat{I}^{k\beta}_{\mu\nu}A_{\beta}-ie(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})\partial^{i}\hat{I}^{k\beta}_{\mu\nu}A_{\beta}. \tag{27}$$

The double brackets $\{\{\ \ \}\}$ indicate that the derivative in D^{i} acts only inside the brackets. Equation (27) permits (26c) to be rewritten as

$$0=-(g^{i\beta}+\tfrac{1}{2}\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu})\psi_{\beta}(x,\partial_{\gamma}\cdots\partial_{6}A^{\beta})+\psi'^{i}(x',\partial'_{\gamma}\cdots\partial'_{6}A^{\beta}(x))$$
$$+\{\{-\tfrac{1}{2}e(D^{i}+\tfrac{1}{2}m\gamma^{i})(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})\}\}\hat{I}^{k\beta}_{\mu\nu}\epsilon^{\mu\nu}A_{\beta}$$
$$+\tfrac{1}{2}ie(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})\partial^{i}\hat{I}^{k\beta}_{\mu\nu}\epsilon^{\mu\nu}A_{\beta}. \tag{28}$$

Taking $\alpha=i$ in (25), adding the result to (28), and using the fact that A_{β} is arbitrary yields the explicit coordinate dependence of ψ^{i}:

$$\frac{d\psi^{i}}{dA^{k}}=-e(D^{i}+\tfrac{1}{2}m\gamma^{i})(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})$$
$$+ie(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})\partial^{i} \tag{29a}$$
$$=\frac{\partial\psi^{i}}{\partial A^{k}}. \tag{29b}$$

Using the explicit expression (11) for $d\psi^{i}/dA^{k}$ and comparing the two sides of (29a), we see that ψ^{i} depends implicitly on A_{k}, but not on derivatives of A_{k}. Thus, (29b) immediately follows. By combining (29) with the secondary constraint (22), we determine the explicit coordinate dependence of ψ^{0}:

$$\frac{\partial\psi^{0}}{\partial A^{k}}=-B_{0}^{-1}B_{i}\frac{\partial\psi^{i}}{\partial A^{k}}, \tag{30a}$$

$$\frac{\partial\psi^{0}}{\partial(\partial_{k}A_{0})}=-\frac{\partial\psi^{0}}{\partial(\partial_{0}A_{k})}$$
$$=-\frac{2ie}{3m^{2}}B_{0}^{-1}\gamma_{0}(\psi^{k}-\gamma^{k}\gamma_{i}\psi^{i}), \tag{30b}$$

$$\frac{\partial\psi^{0}}{\partial(\partial_{k}A_{j})}=\frac{ie}{3m^{2}}B_{0}^{-1}\gamma_{0}[\gamma^{j},\gamma^{k}]\psi^{0}$$
$$-\frac{ie}{3m^{2}}B_{0}^{-1}(\gamma^{j}\gamma_{i}\gamma^{k}-\gamma^{k}\gamma_{i}\gamma^{j})\psi^{i}. \tag{30c}$$

Now that the explicit space-time dependence of the fields ψ^{μ} is known, the transformation properties of these fields under a translation or homogeneous Lorentz transformation can be determined from (17) and (18), respectively. For ψ^{i} the equations take the form

$$i[P_{\mu},\psi^{i}]=(d_{\mu}-\partial_{\mu})\psi^{i}, \tag{31}$$

$$i[J_{\mu\nu},\psi^{i}]=x_{\mu}(d_{\nu}-\partial_{\nu})\psi^{i}-x_{\nu}(d_{\mu}-\partial_{\mu})\psi^{i}+\tilde{I}^{i\beta}_{\mu\nu}\psi_{\beta}-ie[g^{i}{}_{\mu}(\partial_{\nu}A^{k})-g^{i}{}_{\nu}(\partial_{\mu}A^{k})](\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r})$$
$$+e(D^{i}+\tfrac{1}{2}m\gamma^{i})\hat{I}^{k\alpha}_{\mu\nu}A_{\alpha}(\tfrac{3}{2}m^{2}-\tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_{k}-\gamma_{k}\gamma_{r}\psi^{r}), \tag{32}$$

where

$$\partial_\mu \psi^i = -e(D^i + \tfrac{1}{2}m\gamma^i)(\partial_\mu A^k)(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r). \quad (33)$$

Using the secondary constraint (22), the corresponding equations for ψ^0 can be rewritten in the convenient form

$$-B_0{}^{-1}B_i i[P_\mu, \psi^i] = -B_0{}^{-1}B_i\,(d_\mu - \partial_\mu)\psi^i\,, \quad (34)$$

$$\begin{aligned}-B_0{}^{-1}B_i\, i[J_{\mu\nu}, \psi^i] = &-B_0{}^{-1}B_i(x_\mu(d_\nu - \partial_\nu)\psi^i - x_\nu(d_\mu - \partial_\mu)\psi^i + \tilde{I}^{i\beta}_{\mu\nu}\psi_\beta \\ &- e\{i[g^i{}_\mu(\partial_\nu A^k) - g^i{}_\nu(\partial_\mu A^k)] - (D^i + \tfrac{1}{2}m\gamma^i)\hat{I}^{k\alpha}_{\mu\nu}A_\alpha\}(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r)) \\ &- \frac{\partial\psi^0}{\partial(\partial^\delta A^\beta)}[g^\delta{}_\mu(\partial_\nu A^\beta) - g^\delta{}_\nu(\partial_\mu A^\beta) + \hat{I}^{\beta\alpha}_{\mu\nu}(\partial^\delta A_\alpha)] + B_0{}^{-1}B_\alpha I^{\alpha\beta}_{\mu\nu}\psi_\beta. \quad (35)\end{aligned}$$

The last two terms in (35) are equal to $B_0{}^{-1}\frac{1}{4}[\gamma_\mu, \gamma_\nu] \times B^\alpha\psi_\alpha = 0$ [from Eq. (22)]; thus, if ψ^i satisfies (31) and (32), ψ^0 automatically satisfies (34) and (35).

We now explicitly verify that ψ^i transforms according to (31) and (32). To accomplish this it is convenient to introduce the spin-$\frac{3}{2}$ field

$$\phi_i = P_{ik}\psi^k\,, \quad (36)$$

and the spin-$\frac{1}{2}$ field

$$\chi = \gamma_k\psi^k\,, \quad (37)$$

where

$$P_{jk} = g_{jk} - \tfrac{1}{3}\gamma_j\gamma_k\,. \quad (38)$$

In terms of ϕ_i and χ the primary constraint (21) becomes

$$\chi = -\tfrac{3}{2}(\tfrac{3}{2}m - D^i\gamma_i)^{-1}D^k\phi_k\,. \quad (39)$$

We have introduced the spin-$\frac{3}{2}$ field because the quantization conditions for ϕ_i are relatively simple as compared with those for ψ^i. The action principle leads to the quantization conditions[2]

$$\{\phi_i(x,t), \phi_j(x',t)\} = 0\,, \quad (40)$$

$$\{\phi_i(x,t), \phi_j^\dagger(x',t)\} = P_{ir}[g^{rk} - D^r(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ab}\sigma_{ab})^{-1}D^k] \times P_{kj}\delta^3(\vec{x} - \vec{x}')\,. \quad (41)$$

Using the anticommutators (40) and (41), the commutators of P_μ with ϕ_i and χ have been calculated in Ref. 6 and are given by

$$i[P_\mu, \phi^i] = d_\mu\phi^i + eP^i{}_m D^m(\partial_\mu A^k) \times (\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r)\,, \quad (42)$$

$$i[P_\mu, \chi] = d_\mu\chi + e(\tfrac{3}{2}m^2 + D^m\gamma_m)(\partial_\mu A^k) \times (\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r)\,. \quad (43)$$

Equation (36), $\psi^i = \phi^i + \frac{1}{3}\gamma^i\chi$, allows (42) and (43) to be combined so as to give an explicit expression for $i[P^\mu, \psi^i]$. A trivial calculation verifies that the resulting expression is identical with (31).

A direct calculation of $i[J_{\mu\nu}, \psi^i]$ is very similar to that of $i[P_\mu, \psi^i]$. We first calculate $J_{\mu\nu}$ from (16) and use the primary constraint (39) to express $J_{\mu\nu}$ in terms of ϕ^i and derivatives of ϕ^i. Using (40) and (41), the commutator $i[J_{\mu\nu}, \phi^i]$ is found to be

$$\begin{aligned}i[J_{\mu\nu}, \phi^i] = &x_\mu(d_\nu - \partial_\nu)\phi^i - x_\nu(d_\mu - \partial_\mu)\phi^i + P^i{}_r\tilde{I}^{r\beta}_{\mu\nu}\psi_\beta \\ &- eP^i{}_n\{i[g^n{}_\mu(\partial_\nu A^k) - g^n{}_\nu(\partial_\mu A^k)] - D^n\hat{I}^{k\beta}_{\mu\nu}A_\beta\}(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ab}\sigma_{ab})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r)\,, \quad (44)\end{aligned}$$

where, according to (29) and (36),

$$\partial_\nu\phi^i = -eP^i{}_m D^m(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\partial_\nu A^k)(\psi_k - \gamma_k\gamma_r\psi^r)\,. \quad (45)$$

Combining the constraint (39) with (44), we obtain

$$\begin{aligned}i[J_{\mu\nu}, \chi] &= -\tfrac{3}{2}(\tfrac{3}{2}m - D^j\gamma_j)^{-1}D_i i[J_{\mu\nu}, \phi^i] \\ &= x_\mu(d_\nu - \partial_\nu)\chi - x_\nu(d_\mu - \partial_\mu)\chi + \gamma_r\tilde{I}^{r\beta}_{\mu\nu}\psi_\beta \\ &\quad - e\{i\gamma_m[g^m{}_\mu(\partial_\nu A^k) - g^m{}_\nu(\partial_\mu A^k)] - (\tfrac{3}{2}m + D^i\gamma_i)\hat{I}^{k\beta}_{\mu\nu}A_\beta\}(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\psi_k - \gamma_k\gamma_r\psi^r)\,, \quad (46)\end{aligned}$$

where

$$\partial_\nu\chi = -e(\tfrac{3}{2}m + D^i\gamma_i)(\tfrac{3}{2}m^2 - \tfrac{1}{2}eF^{ns}\sigma_{ns})^{-1}(\partial_\nu A^k)(\psi_k - \gamma_k\gamma_r\psi^r)\,. \quad (47)$$

Combining (44) and (46), we find the calculated expression for $i[J_{\mu\nu}, \psi_i]$ agrees with (32).

We conclude that in the presence of a minimal external electromagnetic field, the second-quantized spin-$\frac{3}{2}$ field transforms covariantly. The negative metric which appears in the quantization[2] apparently results from some more fundamental flaw in the theory.

Helpful discussions with Professor J. T. Lewis and Professor L. O'Raifeartaigh are gratefully acknowledged.

APPENDIX

In this appendix we calculate the expression for $\tau^{i}_{\mu\nu}$ which appears in (27). If we demand that the primary constraint (21) remain form-invariant under the transformation (26), we must have

$$0=-(-i d'^{i}+eA^{i})\psi'_i+(-i d'^{i}+eA^{i})\gamma_i\gamma^{j}\psi'_j - m\gamma^{i}\psi'_i . \tag{A1}$$

Using (26) to write the primed variables in terms of the unprimed variables, we obtain

$$\begin{aligned}0=&-\tfrac{1}{2}D_i\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta+i\epsilon^{i\mu}d_\mu\psi_i+\tfrac{1}{2}D^{i}\gamma_i\gamma_j\tilde{I}^{j\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta\\&-i\epsilon^{i\mu}d_\mu\gamma_i\gamma_j\psi^{j}-\tfrac{1}{2}m\gamma_i\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta\\&+\tfrac{1}{2}(-D_i+D^{j}\gamma_j\gamma_i-m\gamma_i)\tau^{i}_{\mu\nu}\epsilon^{\mu\nu} .\end{aligned} \tag{A2}$$

(A2) is not in a form convenient for determining $\tau^{i}_{\mu\nu}$, but can be put in a convenient form by using the identity

$$\begin{aligned}0=&-\tfrac{1}{2}D_i\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta+i\epsilon^{i\mu}d_\mu\psi_i+\tfrac{1}{2}D^{i}\gamma_i\gamma_j\tilde{I}^{j\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta\\&-i\epsilon^{i\mu}d_\mu\gamma_i\gamma_j\psi^{j}-\tfrac{1}{2}m\gamma_i\tilde{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu}\psi_\beta\\&-\tfrac{1}{2}e\hat{I}^{i\beta}_{\mu\nu}\epsilon^{\mu\nu}A_\beta(\psi_i-\gamma_i\gamma_j\psi^{j}) .\end{aligned} \tag{A3}$$

(A3) is easily derived by exploiting the fact that the primary constraint is form-invariant under the transformation (23) and can be obtained in the same way as (A2). Its correctness can, of course, be checked explicitly using the expressions (24) for $\tilde{I}$ and $\hat{I}$. Subtracting (A2) from (A3) and remembering that $\epsilon^{\mu\nu}=-\epsilon^{\nu\mu}$ is an arbitrary parameter, we obtain

$$e\hat{I}^{i\beta}_{\mu\nu}A_\beta(\psi_i-\gamma_i\gamma_j\psi^{j})=(D_i-D^{j}\gamma_j\gamma_i+m\gamma_i)\tau^{i}_{\mu\nu} . \tag{A4}$$

From the identity

$$(D_i-D_j\gamma^{j}\gamma_i+m\gamma_i)(D^{i}+\tfrac{1}{2}m\gamma^{i})=\tfrac{3}{2}m^2-\tfrac{1}{2}eF^{ij}\sigma_{ij} \tag{A5}$$

we readily verify that (27) is a solution of (A4) as required.

[1]W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

[2]K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).

[3]S. N. Gupta and W. W. Repko, Phys. Rev. 177, 1921 (1969).

[4]J. Schwinger, Phys. Rev. 82, 914 (1951); 91, 713 (1953); 91, 728 (1953).

[5]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[6]G. B. Mainland and E. C. G. Sudarshan, Phys. Rev. D 8, 1088 (1973).