THE FAILURE OF THE USUAL GAUGE FORMALISM IN A CLASS OF GAUGE CONDITIONS *

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For a class of gauge conditions the usual formalism for non-Abelian gauge theories leads to violation of unitarity. In contrast, the Lagrange multiplier formalism allows us to construct an adequate fictitious Lagrangian. The theory so obtained satisfies unitarity.

People have developed two different but related formalisms for the treatment of non-Abelian gauge field theories. In the first method one uses the gauge conditions and their variations under general gauge transformations to introduce certain fictitious fields and their interactions. The theory so obtained is expected to be gauge-invariant and unitary [1]. In fact formal proofs have been given establishing these results [e.g. 2]. The second method uses the method of Lagrange multiplier fields: one uses the equations of motion of the Lagrange multiplier fields to isolate the nonunitary contributions and to introduce fictitious fields along with their interactions to restore unitarity [3]. In many cases these two methods yield identical fictitious Lagrangians which restore both unitarity and gauge invariance [3, 4]. In the course of a systematic study of gauge field theories we have come across a simple model for which the usual formalism leads to a fictitious Lagrangian which fails to satisfy unitarity [5], despite formal unitarity proofs. Intrigued by this we have explored this question further.

We find that there is a general class of gauge conditions in a non-Abelian gauge theory for which the usual formalism will not be unitary; in contrast the Lagrange multiplier field formalism leads to a satisfactory theory even for these gauge conditions. The reason for this discrepancy and modified rules for the first method are outlined.

For simplicity, let us consider the theory of the (massless) Yang-Mills field [6] $\boldsymbol{B}_\mu$ in the "Feynman gauge". Let us choose the gauge condition

$$\partial_\mu B^\mu - g(B^\mu \times \boldsymbol{\eta})(B_\mu \cdot \boldsymbol{\eta}) = \boldsymbol{F}(x)\,, \tag{1}$$

where $\boldsymbol{\eta}$ is a constant unit vector along the third axis in the isospin space and $\boldsymbol{F}(x)$ is some suitable function. The (gauge-fixed) Lagrangian density of the Yang–Mills field is

$$\mathcal{L} = -\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} - \tfrac{1}{2}\,[\partial_\mu B^\mu - g(B^\mu \times \boldsymbol{\eta}) B_\mu \cdot \boldsymbol{\eta}]^2, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g B_\mu \times B_\nu\,. \tag{2}$$

In terms of $W_\mu^\pm = (B_{1\mu} \mp \mathrm{i} B_{2\mu})/\sqrt{2}$ and $A_\mu = B_{3\mu}$, we can write (2) as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = -\tfrac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} - \tfrac{1}{4} |F_{\mu\nu} - \mathrm{i}g(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+)|^2, \quad \mathcal{L}_2 = -\tfrac{1}{2}(\partial_\mu A^\mu)^2 - |(\partial^\mu - \mathrm{i}gA^\mu) W_\mu^+|^2, \tag{3}$$

where $W_{\mu\nu}^+ = \partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + \mathrm{i}g(W_\mu^+ A_\nu - W_\nu^+ A_\mu)$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge-fixing terms $\partial_\mu A^\mu$, $(\partial^\mu - \mathrm{i}gA^\mu) W_\mu^+$ transform under infinitesimal gauge transformation as follows:

$$[\partial_\mu A^\mu]^\Lambda = \partial_\mu A^\mu - \mathrm{i}\partial_\mu(W^{+\mu}\Lambda^-) + \mathrm{i}\partial_\mu(W^{-\mu}\Lambda^+) + \frac{1}{g}\Box\,\Lambda^\circ, \qquad \Box = \partial_\mu \partial^\mu\,, \tag{4a}$$

$$\begin{aligned}[(\partial^\mu - \mathrm{i}gA^\mu) W_\mu^+]^\Lambda = {} & (\partial^\mu - \mathrm{i}gA^\mu) W_\mu^+ + \frac{1}{g}\Box\,\Lambda^+ + \mathrm{i}\partial^\mu(W_\mu^+ \Lambda^\circ) - \mathrm{i}\partial^\mu(A_\mu \Lambda^+) + gA^\mu(W_\mu^+ \Lambda^\circ - A_\mu \Lambda^+) \\ & - \mathrm{i}A_\mu \partial^\mu \Lambda^+ - gW_\mu^+(W^{+\mu}\Lambda^- - \Lambda^+ W^{-\mu}) - \mathrm{i}\,W_\mu^+ \partial^\mu \Lambda^\circ\,,\end{aligned} \tag{4b}$$

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where $\Lambda^{\pm}(x)$ and $\Lambda^{o}(x)$ are the infinitesimal gauge functions. We also have the adjoint equation of (4b). According to the prescription in the usual formalism [1], we obtain the following fictitious Lagrangian involving scalar-fermions C and $\bar{C}$:

$$\begin{aligned}\mathcal{L}_{\mathrm{ff}}(C,\bar{C}) = {} & \partial_\mu \overline{C^{o}}\partial^\mu C^{o} + \mathrm{i}g\overline{C^{o}}\partial^\mu(W_\mu^+ C^- - W_\mu^- C^+) + \partial_\mu \overline{C^+}\partial^\mu C^+ - \mathrm{i}g\overline{C^+}\partial^\mu(W_\mu^+ C^{o} - A_\mu C^+) \\ & - g^2\overline{C^+}A^\mu(W_\mu^+ C^{o} - A_\mu C^+) + \mathrm{i}g\overline{C^+}A_\mu\partial^\mu C^+ + g^2\overline{C^+}W_\mu^+(W^{+\mu}C^- - C^+W^{-\mu}) + \mathrm{i}g\overline{C^+}W_\mu^+\partial^\mu C^{o} + \partial_\mu\overline{C^-}\partial^\mu C^- \\ & + \mathrm{i}g\overline{C^-}\partial^\mu(W_\mu^- C^{o} - A_\mu C^-) - g^2\overline{C^-}A^\mu(W_\mu^- C^{o} - A_\mu(C^-) - \mathrm{i}g\overline{C^-}A_\mu\partial^\mu C^- \\ & + g^2\overline{C^-}W_\mu^-(W^{-\mu}C^+ - C^-W^{+\mu}) - \mathrm{i}g\overline{C^-}W_\mu^-\partial^\mu C^{o} .\end{aligned} \tag{5}$$

According to the usual formalism, the effective Lagrangian of the theory is

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L} + \mathcal{L}_{\mathrm{ff}} , \tag{6}$$

where $\mathcal{L}$ is given by (3). The fictitious $\mathcal{L}_{\mathrm{ff}}$ is the same as that obtained from the functional determinant $\Delta(B_\mu)$ of Faddeev and Popov [1]:

$$\Delta(B_\mu)\int\delta^3([\partial_\mu B^\mu - g(B^\mu\times\boldsymbol{\eta})B_\mu\cdot\boldsymbol{\eta}]^\Lambda - F)\,\mathrm{d}[\Lambda] = \text{const.} \tag{7}$$

In gauge theory with spontaneously broken symmetry, let us choose the gauge condition

$$(\partial_\mu - \mathrm{i}gA_\mu)W^{+\mu} + \mathrm{i}MS^+/\xi = \text{fixed function } F^+(x) , \tag{8}$$

in the 't Hooft Lagrangian [1], i.e.

$$\begin{aligned}\mathcal{L}_{\mathrm{sp}} = {} & \mathcal{L}_1 + \tfrac{1}{2}|\partial_\mu\psi + \mathrm{i}g(W_\mu^- S^+ - W_\mu^+ S^-)|^2 + |\partial_\mu S^+ + \mathrm{i}MW_\mu^+ - \mathrm{i}gA_\mu S^+ + \mathrm{i}g\psi W_\mu^+|^2 + \tfrac{1}{2}(\mu_0^2 + 12\lambda v^2)\psi^2 \\ & + \lambda[4v\psi(2S^+S^- + \psi^2) + (2S^+S^- + \psi^2)^2] - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \\ & + (\mu_0^2 + 4v^2\lambda)(S^+S^- + v\psi) - \xi|(\partial_\mu - \mathrm{i}gA_\mu)W^{+\mu} + \mathrm{i}MS^+/\xi|^2 .\end{aligned} \tag{9}$$

Then the usual gauge formalism leads to the effective Lagrangian:

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{sp}} = \mathcal{L}_{\mathrm{sp}} + \mathcal{L}_{\mathrm{ff}}(C,\bar{C}) - M^2\overline{C^+}C^+/\xi - M^2\overline{C^-}C^-/\xi + (gM/\xi)[\overline{C^+}(S^+C^{o} - \psi C^+) + \overline{C^-}(S^-C^{o} - \psi C^-)] . \tag{10}$$

There are formal proofs that $\mathcal{L}_{\mathrm{eff}}$ in (6) and $\mathcal{L}_{\mathrm{eff}}^{\mathrm{sp}}$ in (10) lead to unitary and gauge invariant amplitudes to any order [2].

However, one can show that (10) violates unitarity by calculating the imaginary amplitudes $\mathrm{Im}\,B$ of the fourth order scattering process (with one-loop) $\mathrm{W}^+(p)\gamma(q) \to \mathrm{W}^+(\bar{p})\gamma(\bar{q})$ due to the unphysical intermediate states $\mathrm{W}_s^+(t)\gamma(k)$, $\overline{C^-}(t)C^{o}(k)$ and $C^+(t)\overline{C^{o}}(k)$. The unphysical particles W_s^+, $\overline{C^-}$ and C^+ have the same mass $M\xi^{-1/2}$ and W_s^+ is the negative metric particle associated with the 4-vector field $W_\mu^+(x)$. There are nine diagrams due to $W_s^+\gamma$ in the intermediate states and four diagrams due to $\overline{C^-}C^{o}$ and $C^+\overline{C^{o}}$ in the intermediate states.and four diagrams and phase space considerations, the net contribution due to these intermediate states, i.e. $\mathrm{Im}\,B = \Sigma_{i=1}^9 \mathrm{Im}\,B_i(\mathrm{W}_s^+\gamma) + \Sigma_{i=1}^4 \mathrm{Im}\,B_i'(\overline{C^-}C^{o}, C^+\overline{C^{o}})$, must vanish if the theory is correct. The formal proof of unitarity not-withstanding, we find that

$$\sum_{i=1}^{9}\mathrm{Im}\,B_i(\mathrm{W}_s^+\gamma) = 0 , \tag{11}$$

$$\begin{aligned}\sum_{i=1}^{9}\mathrm{Im}\,B_i'(\overline{C^-}(t)C^{o}(k), C^+(t)\overline{C^{o}}(k)) \propto {} & g^4\int\mathrm{d}^4t\,\mathrm{d}^4k\,\delta(k^2)\,\theta(k_0)\,\delta(t^2 - M^2)\,\theta(t_0)\,\delta^4(p+q-t-k) \\ & \times\{\bar{e}\cdot\bar{\epsilon}[-e\cdot tk\cdot\epsilon/p\cdot k + (e\cdot p\epsilon\cdot k - \epsilon\cdot qe\cdot k + e\cdot\epsilon q\cdot k)/p\cdot q] + (\epsilon, e, p, q \leftrightarrow \bar{\epsilon}, \bar{e}, \bar{p}, \bar{q})\} \neq 0 ,\end{aligned} \tag{12}$$

where we have set $\xi = \alpha = 1$ for simplicity, and ϵ_μ, e_μ, $\bar{\epsilon}_\mu$ and $\bar{e}_\mu$ are respectively the polarization vectors of the physical vector bosons $W^+(p)$, $\gamma(q)$, $W^+(\bar{p})$ and $\gamma(\bar{q})$.

The unitarity-violating result (12) is related to the nontrivial form of the amplitude Z_0 due to C^0 and $\overline{C^0}$ in the intermediate states, which can be formally obtained by evaluating the integral over path corresponding to the Lagrangian (10). Making a change of variables, we obtain

$$Z_0 = [\det(\Box + M^2/\xi)] \exp\left\{ -\mathrm{i} \int \mathrm{d}^4x (\Box + M^2/\xi)^{-1}_{xy} \bar{B}(y) A(x) \right\} ,$$

where $A(x) = \mathrm{i}g\,\partial^\mu(W^+_\mu C^- - W^-_\mu C^+)$ and $\bar{B}(x) = -\mathrm{i}g\,[\overline{C^+}(\partial^\mu W^+_\mu) - \overline{C^-}(\partial^\mu W^-_\mu)] - g^2(\overline{C^+}W^+_\mu + \overline{C^-}W^-_\mu)A^\mu + (gM/\xi)(\overline{C^+}S^+ + \overline{C^-}S^-)$. We observe that Z_0 does not contribute to the 1-loop self-energy of $W^-(p)$ because of the factor $(\partial^\mu W^\pm_\mu)$ in $\bar{B}(x)$ and calculations show that there is no trouble at the 1-loop level (order g^2). We may remark that the result (11) is closely related to the fact that the Lagrangian (9) is, aside from the gauge-fixing term $-(\partial^\mu A_\mu)^2/2$, Abelian gauge invariant. Also, the scalar-fermion fields C^0 and $\overline{C^0}$ in (5) lead to the nontrivial amplitude $Z_0^{YM} = Z_0|_{M=0, S^\pm \equiv 0, \xi=1}$.

We now show that the correct effective Lagrangian can be obtained in the Lagrangian multiplier formalism [3]. In this formalism, we introduce the Lagrangian multiplier field $X(x)$ and rewrite (3) as

$$\begin{aligned} \mathcal{L}_X = & -\tfrac{1}{2} W^+_{\mu\nu} W^{-\mu\nu} - \tfrac{1}{4} |F_{\mu\nu} - \mathrm{i}g(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu)|^2 \\ & + GX_0 \partial_\mu A^\mu + \tfrac{1}{2} G^2 X_0^2 + GX^+(\partial^\mu + \mathrm{i}gA^\mu) W^-_\mu + GX^-(\partial^\mu - \mathrm{i}gA^\mu) W^+_\mu + G^2 X^+ X^- , \end{aligned} \tag{13}$$

where G is an arbitrary parameter with the dimension of mass. From (13), we can derive the field equations for $W^\pm_\mu(x)$, $A_\nu(x)$ and the constraint equations for $X^\pm$ and X_0. Taking the divergences of the field equations and using these equations, we obtain

$$\Box X_0 = 0, \qquad \Box = \partial_\mu \partial^\mu , \tag{14}$$

$$(\Box - 2\mathrm{i}gA_\nu \partial^\nu - g^2 A_\nu A^\nu) X^+ + g^2 W^{+\mu} [W^-_\mu X^+ - W^+_\mu X^-] = 0 , \tag{15}$$

and the adjoint equation of (15). We have set $X_0 = 0$ in (15) which is consistent with (14).

The result (14) is not surprising because the Lagrangian (3) aside from the gauge-fixing term $-(\partial_\mu A^\mu)^2/2$, is strictly invariant under the infinitesimal Abelian gauge transformation):

$$W^+_\mu \to W^+_\mu(1 - \mathrm{i}\omega(x)), \qquad W^-_\mu \to W^-_\mu(1 + \mathrm{i}\omega(x)), \qquad A_\mu \to A_\mu - \partial_\mu \omega(x)/g , \tag{16}$$

where $\omega(x)$ is an arbitrary function. The equation (14) implies that the unphysical components of the 4-vector field A_μ are "free" and, therefore, we can simply ignore them. The situation is exactly the same as that in quantum electrodynamics. However, the presence of source terms in (15) implies that the two unphysical components of W^+_μ *do interact* and contribute extra unwanted amplitudes which upset unitarity. Therefore, we must introduce complex fictitious scalar-fermions D and $\bar{\mathrm{D}}$ with the same couplings as indicated in (15) to remove the unwanted amplitudes and to restore unitarity [3]. The fictitious Lagrangian must be constructed according to (15) and its adjoint equation. We obtain [3]

$$\begin{aligned} \bar{\mathcal{L}}_{\mathrm{ff}}(\mathrm{D}, \bar{\mathrm{D}}) = & -\overline{D^+}(\Box - 2\mathrm{i}gA_\nu \partial^\nu - g^2 A_\nu A^\nu) D^+ - g^2 \overline{D^+} W^{+\mu}(W^-_\mu D^+ - W^+_\mu D^-) \\ & - \overline{D^-}(\Box + 2\mathrm{i}gA_\nu \partial^\nu - g^2 A_\nu A^\nu) D^- - g^2 \overline{D^-} W^{-\mu}(W^+_\mu D^- - W^-_\mu D^+) . \end{aligned}$$

Thus, the effective Lagrangian for the massless Yang-Mills fields in the Lagrange multiplier formalism is

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L} + \bar{\mathcal{L}}_{\mathrm{ff}}(\mathrm{D}, \bar{\mathrm{D}}), \qquad D = \{D^+, D^-\} , \tag{18}$$

which is different from $\mathcal{L}_{\mathrm{eff}}$ given by (6).

Similarly, for the spontaneously broken symmetry case the Lagrangian multiplier formalism leads to the effec-

tive Lagrangian $\bar{\mathcal{L}}'_{\mathrm{eff}}$

$$\bar{\mathcal{L}}'_{\mathrm{eff}} = \mathcal{L}_{\mathrm{sp}} + \bar{\mathcal{L}}_{\mathrm{ff}}(\mathrm{D}, \bar{\mathrm{D}}) - M^2\overline{D^+}D^+/\xi - M^2\overline{D^-}D^-/\xi - \frac{qM}{\xi}\psi(\overline{D^+}D^+ + \overline{D^-}D^-) . \tag{19}$$

We have calculated the imaginary amplitudes of the 1-loop self-energy of W^- and the photon. The results verify that the effective Lagrangian (19) yields unitary and gauge independent amplitudes. Moreover, we have also checked unitarity as well as the α- and ξ-independence of the 1-loop scattering of $W^+\gamma$ by explicit calculations. In this case, we have the result (11) and we *do not* have C^o and $\overline{C^o}$ which lead to the non-vanishing amplitude (12) which upsets unitarity.

This situation is unexpected. We note that the purpose of gauge compensating terms is to restore the change of the gauge-fixing terms during the evolution of a physical system. But if that is the reason for the gauge compensating terms we must introduce them only for those gauge-fixing terms which are not maintained according to the equations of motion. But with the choice of gauge conditions we have equation (14) which tells us that the neutral component gauge fixing term does not need any gauge compensating term. But the standard method insists on using such a term and we can trace the violation of unitarity directly to this unwanted gauge compensating term.

We propose that, if the gauge conditions of the types shown in (1) (or (8)) are chosen, the corresponding Faddeev–Popov functional determinant should be determined by

$$\bar{\Delta}(B_\lambda)\int \delta^2([\partial_\mu B^\mu - g(B^\mu \times \boldsymbol{\eta})B_\mu \cdot \boldsymbol{\eta}]_t^\Lambda - F_t)\,\delta(\Lambda_3)\,\mathrm{d}[\boldsymbol{\Lambda}] = \text{const.}, \qquad F_t = (F, F, 0) . \tag{20}$$

This leads to the fictitious Lagrangian (17)[‡]. The validity of (20) has been checked in gauge theories (including massive vector boson) with the guage condition (8) and the usual electromagnetic gauge condition $\partial_\mu A^\mu = F^o(x)$. We may remark that the expression (20) has to be modified for the case of any non-linear electromagnetic gauge condition, for example, for the choice [e.g. 7] $\partial_\mu A^\mu + \beta A_\mu A^\mu = F^o(x)$.

In conclusion we remark that the usual method of constructing the fictitious Lagrangian must be modified by rules which are obtained from a careful examination of which gauge conditions can be preserved in course of time. In contrast, the method of Lagrangian multiplier fields yields the correct fictitious Lagrangian in all cases: we have verified this by direct calculation up to and including two-loop diagrams [5].

[‡] We have used dynamical equation $\Box\partial_\mu A^\mu = 0$, which can be derived from the Lagrangian (2).

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