On the determination of the relativistic wave equations associated with a given representation of $SL(2,C)$

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Straightforward algebraic techniques are presented and used to determine the structure of wave equations whose relativistic covariance is governed by two representations of $SL(2,C)$, $S_0(\Lambda) = (1,1/2) \oplus (1/2,1) \oplus (1/2,0) \oplus (0,1/2)$ and $S_1(\Lambda) = S_0(\Lambda) \oplus (1/2,0) \oplus (0,1/2)$, subject to the requirements that the equations should be parity preserving, admit an invariant Hermitian bilinear form realized by a numerical matrix η, and that they should describe a particle with a unique mass and spin. It is shown that $S_0(\Lambda)$ leads to a unique algebraic structure, that of the Rarita–Schwinger equation, whereas $S_1(\Lambda)$ leads either to a trivial extension of the former case or to a family of equations whose matrices have a minimal algebra with degree one higher than that of the former case. One such example reproduces the equation presented by Glass. When, contrary to custom, a singular η matrix is considered, it is shown that $S_1(\Lambda)$ allows for equations whose coefficient matrices are reducible but indecomposable. These equations are completely equivalent to the Rarita–Schwinger equation in the free case, but the added components may enter the dynamics in the presence of certain interactions. The present examples serve to illustrate techniques which may be applied in the study of any relativistic wave equation.

I. INTRODUCTION

A relativistic wave equation describing free, massive $(m>0)$ particles may be written in the general form

$$(i\beta\cdot\partial - m)\,\phi(x) = 0 \tag{I. 1}$$

where β_μ represents four $N\times N$ numerical matrices with the property that

$$S(\Lambda)\beta_\mu\, S(\Lambda^{-1}) = (\Lambda\beta)_\mu \tag{I. 2}$$

where $S(\Lambda)$ is the N-dimensional representation of $SL(2,C)$ which governs the index transformation of ϕ under the Poincaré group $\mathbf{P}$,

$$\phi'_\alpha(x) = S_{\alpha\beta}(\Lambda)\,\phi_\beta(\Lambda^{-1}x + a), \quad \alpha,\beta = 1,\ldots,N, \tag{I. 3}$$

Λ being the homogeneous Lorentz transformation and a the space–time translation.

In a recent study [1] straightforward constructive algebraic techniques [2] were used in order to classify under very general conditions the possible $S(\Lambda)$ which may be associated with β_μ which satisfied a given algebraic condition. A large class of such $S(\Lambda)$ was thereby enumerated. In the present note we shall employ these same constructive techniques in order to determine all possible β_μ associated with a given $S(\Lambda)$. More specifically, we shall examine the representations

$$S_0(\Lambda) \equiv (1,\tfrac{1}{2}) \oplus (0,\tfrac{1}{2}) \oplus (\tfrac{1}{2},1) \oplus (\tfrac{1}{2},0) \tag{I. 4}$$

and

$$S_1(\Lambda) \equiv S_0(\Lambda) \oplus (\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2}) \tag{I. 5}$$

and seek all possible β_μ which are covariant, parity symmetric, yield a unique spin-3/2, a unique mass $m>0$, and permit the existence of a Hermitianizing matrix η.

We shall find that these general restrictions determine a unique β-algebra in the case of $S_0(\Lambda)$ and a very limited number of possibilities for $S_1(\Lambda)$. The Rarita–Schwinger (R–S) equation [3] and the Glass equation [4,5] are recovered in addition to other related possibilities. By relaxing the requirement that η be nonsingular we are led to a new class of equations for $S_1(\Lambda)$ which have reducible but indecomposable β matrices. These equations are completely equivalent to the R–S theory in the free case but may deviate from the latter's predictions in the presence of interactions. We refer to the additional components as "barnacles."

In the next section we shall present our basic assumptions and apply them to $S_0(\Lambda)$ and $S_1(\Lambda)$ in Secs. III and IV, respectively, giving a complete listing of the possible wave equations associated with each representation. In Sec. V we shall briefly discuss the physical significance of the various possible wave equations and of the new barnacled equations.

II. ASSUMPTIONS

We shall impose the following restrictions upon the wave equation (I. 1):

(1) Equation (I. 1) is covariant under the proper Poincaré group and hence the $S(\Lambda)$ and β_μ must satisfy Eq. (I. 2). In terms of the generators of rotations and boosts, J_i and N_i, in the representation $S(\Lambda)$ this requirement may be stated entirely in terms of β_0 as

$$[J_i, \beta_0] = 0, \tag{II. 1a}$$

$$[[N_3, \beta_0], N_3] = \beta_0. \tag{II. 1b}$$

(2) Equation (I. 1) is covariant under space reflection, i. e., there exists a numerical matrix P such that $[x' = (x_0, -\mathbf{x})]$

$$\phi'(x') = \alpha_p P\phi(x) \tag{II. 2}$$

where α_p is a phase, $P^2 = I$, and

$$[\beta_0, P]_- = 0, \tag{II. 3a}$$

$$[\beta_i, P]_+ = 0, \quad i = 1, 2, 3. \tag{II. 3b}$$

Such a matrix will exist only if $S(\Lambda)$ is self-conjugate, i. e., composed of representations of the form (n,n) or $(n,m)\oplus(m,n)$. It will have matrix elements which are nonvanishing only in those blocks which connect conjugate representations and there it will be a multiple of the identity. For the two representations $S_0(\Lambda)$ and $S_1(\Lambda)$ we fix the relative scale of the conjugate components such that

$$P = \begin{array}{cc} A_i & A_i^c \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} A_i \\ A_i^c \end{array}, \qquad i=0,1, \tag{II. 4}$$

where, for $S_0 = A_0 \oplus A_0^c$,

$$A_0 \equiv (1, \tfrac{1}{2}) \oplus (0, \tfrac{1}{2}), \tag{II. 5a}$$

$$A_0^c \equiv (\tfrac{1}{2}, 1) \oplus (\tfrac{1}{2}, 0) \tag{II. 5b}$$

and, for $S_1 = A_1 \oplus A_1^c$,

$$A_1 = A_0 \oplus (0, \tfrac{1}{2}), \tag{II. 6a}$$

$$A_1^c = A_0^c \oplus (\tfrac{1}{2}, 0). \tag{II. 6b}$$

(3) We assume that there exists an invariant, Hermitian bilinear form realized by a numerical matrix η satisfying

$$S^\dagger(\Lambda)\eta S(\Lambda) = \eta \quad \text{for all } \Lambda \in \mathbf{P}, \tag{II. 7a}$$

$$P\eta P^{-1} = \eta, \tag{II. 7b}$$

$$\eta = \eta^\dagger, \tag{II. 7c}$$

and

$$\eta\beta_\mu = \beta_\mu^\dagger \eta. \tag{II. 7d}$$

Note that we have *not* made the usual assumption that η is nonsingular.

(4) Equation (I. 1) is assumed to describe a particle of unique mass m. This means[6] that β_μ must satisfy the algebraic relation

$$(\beta \cdot p)^n = (\beta \cdot p)^{n-2} p^2 \tag{II. 8}$$

for some n where p is any 4-momentum. In terms of β_0 this becomes

$$\beta_0^{n-2}(\beta_0^2 - 1) = 0. \tag{II. 9}$$

(5) Finally, we assume that Eq. (I. 1) describes a particle with a unique spin which we take for the cases considered to be $s = 3/2$. This means that the solutions of Eq. (I. 1) when written in momentum space and taken to the rest frame will have only eight nonvanishing components corresponding to the four spin degrees of freedom for $s = 3/2$ and the two signs of the energy.

The representations considered here contain only two spins, 3/2 and 1/2, the $s = 3/2$ representation occurring twice and the $s = 1/2$ representation occurring four and six times for the $S_0(\Lambda)$ and $S_1(\Lambda)$ representations, respectively. We may accordingly decompose β_0 into two submatrices which act only between components of the same spin:

$$\beta_0 = [\beta_0]_{3/2} \oplus [\beta_0]_{1/2} \tag{II. 10}$$

and from assumption (5) we have the characteristic equations for these submatrices

$$[\beta_0]_{3/2}^2 - 1 = 0 \tag{II. 11a}$$

and

$$[\beta_0]_{1/2}^N = 0 \tag{II. 11b}$$

for some N. From this viewpoint we see that it is the degree of nilpotency of the auxiliary spins which will determine the order of the algebra (II. 8).[7]

In the next two sections we shall seek the most general wave equations associated with the particular representations $S_0(\Lambda)$ and $S_1(\Lambda)$ when subject to the above restrictions.

III. THE STRUCTURE OF WAVE EQUATIONS TRANSFORMING VIA $S_0(\Lambda)$

Using the generators of $S_0(\Lambda)$[8] and invoking Eqs. (II. 1) and (II. 3a) we may write the general form of β_0 in this representation as

$$\beta_0 = \begin{array}{cc} A_0 & A_0^c \\ \begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix} \end{array} \begin{array}{c} A_0 \\ A_0^c \end{array} \tag{III. 1}$$

where

$$M = \begin{array}{ccc} (\tfrac{1}{2},1) & \multicolumn{2}{c}{(\tfrac{1}{2},0)} \\ 3/2 & 1/2 & 1/2 \\ \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & -\tfrac{1}{2}m_{11} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix} \end{array} \begin{array}{cc} 3/2 & (1,\tfrac{1}{2}) \\ 1/2 & \\ 1/2 & (0,\tfrac{1}{2}) \end{array}$$

$$= \begin{bmatrix} M_{3/2} & 0 \\ 0 & M_{1/2} \end{bmatrix}, \qquad m_{ij} \in C, \tag{III. 2}$$

which satisfies assumptions (1) and (2).

Let us now invoke assumptions (4) and (5). We have from Eqs. (II. 11) that

$$M_{3/2}^2 = m_{11}^2 I = I \tag{III. 3}$$

and

$$M_{1/2}^N = 0 \quad \text{for some } N. \tag{III. 4}$$

Equation (III. 3) implies that $m_{11}^2 = 1$ and we take $m_{11} = +1$. This involves no loss of generality and corresponds to the freedom of picking the overall sign of β_μ. Equation (III. 4) then implies that

$$\begin{bmatrix} -1/2 & m_{23} \\ m_{32} & m_{33} \end{bmatrix}^N = 0 \tag{III. 5}$$

for some N.

Since the cases $N = 0$ and $N = 1$ are not possible and since $\dim(M_{1/2})=2$, $N=2$ is the only possible exponent in Eq. (III. 5). This leads to the conditions

$$m_{33}=\tfrac{1}{2} \tag{III. 6a}$$

and

$$m_{23}m_{32}=-\tfrac{1}{4}. \tag{III. 6b}$$

We shall now consider assumption (3). The most general η matrix for $S_0(\Lambda)$ satisfying Eqs. (II. 7a)—(II. 7c) may be written as

$$\eta=\begin{array}{cc} A_0 & A_0^c \\ \left[\begin{array}{c|c} 0 & \hat{\eta} \\ \hline \hat{\eta} & 0 \end{array}\right] \end{array}\begin{array}{l} \\ A_0 \\ A_0^c \end{array} \tag{III. 7}$$

with

$$\hat{\eta}=\begin{array}{ccc} 3/2 & 1/2 & 1/2 \\ \left[\begin{array}{c|c|c} \eta_1 & 0 & 0 \\ \hline 0 & \eta_1 & 0 \\ \hline 0 & 0 & \eta_2 \end{array}\right] \end{array}\begin{array}{l} \\ 3/2 \\ 1/2\,, \\ 1/2 \end{array} \qquad \eta_i\in R. \tag{III. 8}$$

Equation (II. 7d) tells us that

$$\hat{\eta}M=M^{\dagger}\hat{\eta} \tag{III. 9}$$

or that m_{11} and m_{33} are real and

$$\eta_1\overline{m}_{23}=m_{32}\eta_2. \tag{III. 10}$$

Note that in view of Eq. (III. 6b) we must have η_1 and η_2 nonzero in order to avoid a trivial theory. We may also fix the sign and normalization of the rest frame solutions by taking $\eta_1=1$.

The most general β_0 satisfying assumptions (1)—(5) for $S_0(\Lambda)$ is thus found from Eqs. (III. 6) and (III. 9) and is given by Eq. (III. 1) with

$$M=\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & -\tfrac{1}{2} & \alpha \\ \hline 0 & -\overline{\alpha} & \tfrac{1}{2} \end{array}\right] \tag{III. 11}$$

where $|\alpha|^2=\tfrac{1}{4}$. The elements of η in Eq. (III. 8) are $\eta_1=1$, $\eta_2=-1$.

Subject to the above assumptions there is therefore only one algebraic structure possible for $S_0(\Lambda)$, namely,

$$(\beta\cdot p)^2[(\beta\cdot p)^2-1]=0. \tag{III. 12}$$

This is the algebra which characterizes the R—S system.[9]

IV. THE STRUCTURE OF WAVE EQUATIONS TRANSFORMING VIA $S_1(\Lambda)$

Again by constructing the generators of $S_1(\Lambda)$ (Ref. 8) and imposing assumptions (1) and (2), we are led to the general form

$$\beta_0=\begin{array}{cc} A_1 & A_1^c \\ \left[\begin{array}{c|c} 0 & N \\ \hline N & 0 \end{array}\right] \end{array}\begin{array}{l} \\ A_1 \\ A_1^c \end{array} \tag{IV. 1}$$

where

$$N=\left[\begin{array}{c|c} N_{3/2} & 0 \\ \hline 0 & N_{1/2} \end{array}\right], \tag{IV. 2}$$

$N_{3/2}$ is a multiple of the (four-dimensional) identity, $N_{3/2}=\lambda I$, and

$$N_{1/2}=\left[\begin{array}{c|c|c} -\tfrac{1}{2}\lambda & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right], \quad a_{ij}\in C. \tag{IV. 3}$$

The assumptions (4) and (5) will again be satisfied if and only if $\lambda^2=1$ (we again take $\lambda=+1$ with no loss of generality) and either

$$N_{1/2}^2=0, \quad N_{1/2}\neq 0 \tag{IV. 4}$$

or

$$N_{1/2}^3=0, \quad N_{1/2}^2\neq 0, \tag{IV. 5}$$

the degree being bounded by the dimensionality of the matrix. There are thus two distinct algebraic possibilities.

Consider now the most general η matrix satisfying Eqs. (II. 7a)—(11. 7c):

$$\eta=\begin{array}{cc} A_1 & A_1^c \\ \left[\begin{array}{c|c} 0 & \hat{\eta} \\ \hline \hat{\eta} & 0 \end{array}\right] \end{array}\begin{array}{l} \\ A_1 \\ A_1^c \end{array} \tag{IV. 6}$$

with

$$\hat{\eta}=\begin{array}{cccc} 3/2 & 1/2 & 1/2 & 1/2 \\ \left[\begin{array}{c|c|c|c} \eta_1 & & & \\ \hline & \eta_1 & & \\ \hline & & \eta_2 & \eta_3 \\ \hline & & \overline{\eta}_3 & \eta_4 \end{array}\right] \end{array}\begin{array}{l} \\ 3/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{array}. \tag{IV. 7}$$

We may again fix the normalization of the $(1,\tfrac{1}{2})$ and $(\tfrac{1}{2},1)$ components such that $\eta_1=1$ and since $\hat{\eta}$ is Hermitian we may diagonalize it by taking a suitable linear combination of the identical $SL(2,C)$ representations, thus setting $\eta_3=0$ with no loss of generality. Finally, we may set the scale of the lower components such that we have

$$\hat{\eta}=\begin{array}{cccc} 3/2 & 1/2 & 1/2 & 1/2 \\ \left[\begin{array}{c|c|c|c} 1 & & & \\ \hline & 1 & & \\ \hline & & \mu & \\ \hline & & & \nu \end{array}\right] \end{array}\begin{array}{l} \\ 3/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{array} = \left[\begin{array}{c|c} \hat{\eta}_{3/2} & \\ \hline & \hat{\eta}_{1/2} \end{array}\right] \tag{IV. 8}$$

where $\mu,\nu=+1,-1$ or 0.

Equation (II.7d) will be satisfied for β_0 if and only if

$$\hat{\eta}_{1/2} N_{1/2} = N_{1/2}^{+} \hat{\eta}_{1/2} \tag{IV.9}$$

where the matrices are given by Eqs. (IV.3) and (IV.8). The conditions on the elements of $N_{1/2}$ are thus

$$\mu a_{21} = \bar{a}_{12}, \tag{IV.10a}$$

$$\nu a_{31} = \bar{a}_{13}, \tag{IV.10b}$$

$$\nu a_{32} = \mu \bar{a}_{23}, \tag{IV.10c}$$

and a_{22} and a_{33} must be real.

We shall label the various cases according to the various possible values of (μ, ν). Noting that equivalent systems result if μ and ν are interchanged, there are six possibilities: (i) $(1,1)$, (ii) $(-1,-1)$, (iii) $(+1,-1)$, (iv) $(1,0)$, (v) $(-1,0)$, (iv) $(0,0)$.

(i) $(\mu,\nu)=(1,1)$. Here Eqs. (IV.10) imply that $N_{1/2}$ is Hermitian and hence diagonalizable by a unitary transformation which preserves $\hat{\eta}=I$. But Eqs. (IV.4) or (IV.5) imply that $N_{1/2}$ is nilpotent and so its eigenvalues must be zero. This implies that $N_{1/2}=0$ which contradicts Eq. (IV.3) if we are to have a nontrivial theory. Case (i) must therefore be abandoned.

(ii) $(\mu,\nu)=(-1,-1)$. Here Eqs. (IV.10) yield

$$N_{1/2} = \begin{array}{|c|c|c|} \hline -\frac{1}{2} & a & b \\ \hline -\bar{a} & c & e \\ \hline -\bar{b} & \bar{e} & d \\ \hline \end{array} \,. \tag{IV.11}$$

In this case e may be set equal to zero by a suitable $\hat{\eta}$-preserving unitary transformation which diagonalizes $N_{1/2}$ on the lower two components.

(iia) If $N_{1/2}^2=0$ then it is easy to verify that either $a=0=c$ or $b=0=d$ and we are left with the R–S system of Sec. III trivially extended by zero entries in the additional components.

(iib) On the other hand, Eq. (IV.5) may be satisfied if and only if

$$\det(N_{1/2}-\lambda) = -\lambda^3. \tag{IV.12}$$

This places the restriction on the elements of $N_{1/2}$;

$$c+d-\tfrac{1}{2}=0, \tag{IV.13a}$$

$$\tfrac{1}{4}-cd-|a|^2-|b|^2=0, \tag{IV.13b}$$

and

$$(-\tfrac{1}{2}c+|a|^2)d+c|b|^2=0. \tag{IV.13c}$$

If we choose any of the parameters in Eqs. (IV.13) to be zero, then we are reduced to the case (iia) and a trivial extension of the R–S equation. Avoiding these cases, we may solve Eqs. (IV.13) and find

$$d=\tfrac{1}{2}-c, \tag{IV.14a}$$

$$|a|^2 = c^3/(2c-\tfrac{1}{2}) \tag{IV.14b}$$

$$|b|^2 = (c-\tfrac{1}{2})^3/(2c-\tfrac{1}{2}), \tag{IV.14c}$$

thus parameterizing the solutions in terms of c. We see that consistent nontrivial solutions exist whenever $c \in \langle -\infty, 0\rangle \cup \langle \frac{1}{2}, \infty\rangle$. For the special choice $c=-\frac{1}{2}$ we get

$$N_{1/2} = \begin{array}{|c|c|c|} \hline -\dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & \dfrac{\sqrt{2}}{\sqrt{3}} \\ \hline \dfrac{-1}{2\sqrt{3}} & -\dfrac{1}{2} & 0 \\ \hline -\dfrac{\sqrt{2}}{\sqrt{3}} & 0 & 1 \\ \hline \end{array} \tag{IV.15}$$

which reproduces the β-matrices presented by Glass[4,5] in his counterexample to the relation of Umezawa and Visconti.[10] We see how the order of the minimal equation for β_0 is increased from the $\beta_0^2(\beta_0^2-1)=0$ of the R–S equation to $\beta_0^3(\beta_0^2-1)=0$ of the Glass equation without increasing the highest value of the spin. We may also note that any choice of c which leads to nonzero a and b must yield a system which satisfies the higher order algebra.

(iii) $(\mu,\nu)=(1,-1)$.

$$N_{1/2} = \begin{array}{|c|c|c|} \hline -\frac{1}{2} & a & b \\ \hline \bar{a} & c & e \\ \hline -\bar{b} & -\bar{e} & d \\ \hline \end{array} \,, \tag{IV.16}$$

$$\hat{\eta} = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & -1 \\ \hline \end{array} \,. \tag{IV.17}$$

The metric operator $\hat{\eta}$ will be preserved by any transformation of the form

$$T(\theta) = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & \cosh\theta & \sinh\theta \\ \hline 0 & \sinh\theta & \cosh\theta \\ \hline \end{array} \,, \quad \theta \in R, \tag{IV.18}$$

since $T^{+}\hat{\eta}T=\hat{\eta}$. If we first transform the $N_{1/2}$ of Eq. (IV.16) by the unitary transformation

$$F = \begin{array}{|c|c|c|} \hline e^{i\alpha} & & \\ \hline & e^{i\beta} & \\ \hline & & e^{i\gamma} \\ \hline \end{array} \,, \tag{IV.19}$$

then we may fix α, β, and γ such that all of the elements of $N_{1/2}$ are real. We now transform $N_{1/2}$ by $T(\theta)$:

$$N'_{1/2} = T(\theta) N_{1/2} T^{-1}(\theta) = \begin{array}{|c|c|c|} \hline -\frac{1}{2} & a' & b' \\ \hline a' & c' & e' \\ \hline -b' & -e' & d' \\ \hline \end{array} \tag{IV.20}$$

where

$$a' = a\cosh\theta - b\sinh\theta,$$

$$b' = -a\sinh\theta + b\cosh\theta,$$

$$c' = c\cosh^2\theta - d\sinh^2\theta - 2e\cosh\theta\sinh\theta,$$

$$d' = -c\sinh^2\theta + d\cosh^2\theta + 2e\cosh\theta\sinh\theta,$$

and

$$e' = e(\cosh^2\theta + \sinh^2\theta) + (d-c)\cosh\theta\,\sinh\theta.$$

We may now consider the following cases,

(a) If $|b| > |a|$ then we may have $a' = 0$ by choosing $\coth\theta = b/a$.

(b) If $|a| > |b|$ then we may have $b' = 0$ by choosing $\coth\theta = a/b$.

(c) $|a| = |b|$.

Now $\det(N_{1/2} - \lambda) = -\lambda^3$ will hold if and only if

$$c + d - \tfrac{1}{2} = 0, \tag{IV.21a}$$

$$\tfrac{1}{4} - cd - e^2 + a^2 - b^2 = 0, \tag{IV.21b}$$

$$-\tfrac{1}{2}cd - \tfrac{1}{2}e^2 - da^2 + cb^2 - 2abe = 0. \tag{IV.21c}$$

(a) Choose $a = 0$. Then Eqs. (IV.21) admit the solutions

$$d = \tfrac{1}{2} - c, \tag{IV.22a}$$

$$b^2 = \tfrac{1}{8}[1/(c + \tfrac{1}{2})], \tag{IV.22b}$$

$$e^2 = c^3/(c + \tfrac{1}{2}) \tag{IV.22c}$$

which are consistent for any $c \geqslant 0$.

(b) Choose $b = 0$. Then we get

$$d = \tfrac{1}{2} - c \tag{IV.23a}$$

$$a^2 = \tfrac{1}{8}[1/(c-1)], \tag{IV.23b}$$

$$e^2 = (2c-1)^3/8(c-1) \tag{IV.23c}$$

which are consistent for any $c > 1$.

From Eqs. (IV.22) we see that if $c = 0$ then $e = 0$ and we recover the trivially extended R–S equation and $N_{1/2}^2 = 0$. For the remaining values of c we have $N_{1/2}^3 = 0$ but $N_{1/2}^2 \neq 0$.

(c) Choose $b = \epsilon a$, $\epsilon = \pm 1$. We then get from Eqs. (IV.21)

$$d = \tfrac{1}{2} - c, \tag{IV.24a}$$

$$e^2 = c^2 - \tfrac{1}{2}c + \tfrac{1}{4}, \tag{IV.24b}$$

$$a^2 = \tfrac{1}{8}(2c - \tfrac{1}{2} - 2\epsilon|e|)^{-1}. \tag{IV.24c}$$

If $\epsilon = -1$ then these conditions are consistent for all c, but if $\epsilon = +1$ they imply that $a^2 < 0$ for any c. One solution would be $c = 0$:

$$N_{1/2} = \tfrac{1}{2}\begin{array}{|c|c|c|}\hline -1 & 1 & -1 \\ \hline 1 & 0 & 1 \\ \hline 1 & -1 & 1 \\ \hline \end{array} \tag{IV.25}$$

which satisfies $N_{1/2}^3 = 0$, $N_{1/2}^2 \neq 0$. This is, in fact, again always the case for type (iii) matrices.

We now depart slightly from orthodoxy and consider singular η matrices. These will lead to new types of wave equations called "barnacled" equations.

(iv) and (v), $(\mu, \nu) = (\epsilon, 0)$, $\epsilon = \pm 1$. Here we have

$$N_{1/2} = \begin{array}{|c|c|c|}\hline -\tfrac{1}{2} & a & 0 \\ \hline \epsilon\bar{a} & c & 0 \\ \hline b & e & d \\ \hline \end{array} \tag{IV.26}$$

with $d = 0$ if we demand that $N_{1/2}$ be nilpotent. $\det(N_{1/2} - \lambda) = -\lambda^3$ implies that $c = \tfrac{1}{2}$ and $|a|^2 = -\epsilon\tfrac{1}{4}$. Thus case (iv) ($\epsilon = +1$) cannot be satisfied but case (v) does admit the solution

$$N_{1/2} = \begin{array}{|c|c|c|}\hline -\tfrac{1}{2} & \pm\tfrac{1}{2} & 0 \\ \hline \mp\tfrac{1}{2} & \tfrac{1}{2} & 0 \\ \hline b & e & 0 \\ \hline \end{array} \tag{IV.27}$$

which satisfies $N_{1/2}^2 = 0$ only if $b = \mp e$. Otherwise, we get $N_{1/2}^3 = 0$, $N_{1/2}^2 \neq 0$.

(vi) $(\mu, \nu) = (0, 0)$. Here

$$N_{1/2} = \begin{array}{|c|c|c|}\hline -\tfrac{1}{2} & 0 & 0 \\ \hline a & c & e \\ \hline b & e' & d \\ \hline \end{array} \tag{IV.28}$$

which can never be nilpotent and hence this case must be abandoned.

Thus there are three classes of equations which satisfy all of our requirements. These correspond to η matrices of the classes (ii), (iii), and (iv). For the case of nonsingular η[(ii) and (iii)] the formalism either reduces to the R–S equation trivially extended by zeroes or increases by one the order of the β-algebra by increasing the degree of nilpotency of the spin-$\tfrac{1}{2}$ submatrix. As we shall see in the next section for the case of singular η, the "barnacled" equation is completely equivalent to the R–S equation in that its solutions are identical but the order of the β_0's minimal equation may be increased.

V. SUMMARY AND DISCUSSION

We have examined all possible wave equations subject to general physical assumptions which transform according to a given representation of $SL(2, C)$ for two special choices of representation. The first choice, $S_0(\Lambda)$, led to a unique algebraic structure

$$(\beta\cdot p)^2[(\beta\cdot p)^2 - p^2] = 0 \tag{V.1}$$

and consequently to the R–S equation. The second choice, $S_1(\Lambda)$, either (1) collapsed to a trivial extension of the former choice obeying the same algebra or (2) led to an algebra which is of higher degree than Eq. (V.1):

$$(\beta\cdot p)^3[(\beta\cdot p)^2 - 1] = 0. \tag{V.2}$$

One such case reproduced the theory presented by Glass. Higher algebras were seen to be prohibited by the dimensions of the nilpotent submatrices involved.

Finally, we considered a class of equations which admit only a singular η-matrix. Let us consider this new class in more detail.

We have from Eq. (IV. 27) (indicating the dimension of the submatrices)

$$\beta_0=\begin{array}{cc} \begin{array}{cc} 10 & 10 \end{array} & \\ \left[\begin{array}{c|c} 0 & N \\ \hline N & 0 \end{array}\right] & \begin{array}{c} 10 \\ 10 \end{array} \end{array}, \quad N=\begin{array}{cc} \begin{array}{cc} 4 & 6 \end{array} & \\ \left[\begin{array}{c|c} N_{3/2} & 0 \\ \hline 0 & N_{1/2} \end{array}\right] & \begin{array}{c} 4 \\ 6 \end{array} \end{array}, \tag{V.3}$$

and

$$N_{1/2}=\begin{array}{cc} \begin{array}{ccc} 2 & 2 & 2 \end{array} & \\ \left[\begin{array}{c|c|c} -\frac{1}{2} & \frac{1}{2} & 0 \\ \hline -\frac{1}{2} & \frac{1}{2} & 0 \\ \hline b & e & 0 \end{array}\right] & \begin{array}{c} 2 \\ 2 \\ 2 \end{array} \end{array}. \tag{V.4}$$

The matrices thus take the structure (diagonalizing P)

$$\beta_0=\begin{array}{cc} \begin{array}{cc} 10 & 10 \end{array} & \\ \left[\begin{array}{c|c} N & \\ \hline & -N \end{array}\right] & \begin{array}{c} 10 \\ 10 \end{array} \end{array} \tag{V.5}$$

which may be written in the form

$$\beta_0=\begin{array}{cc} \begin{array}{cc} 16 & 4 \end{array} & \\ \left[\begin{array}{c|c} \Gamma_0 & 0 \\ \hline B_0 & 0 \end{array}\right] & \begin{array}{c} 16 \\ 4 \end{array} \end{array} \tag{V.6}$$

where Γ_0 is the matrix for the R—S equation and B_0 represents the arbitrary coupling of the extra 4-spinor components. The wave equation thus takes the form

$$\left\{ i \left[\begin{array}{c|c} \Gamma_\mu & 0 \\ \hline B_\mu & 0 \end{array}\right] \partial^\mu - m \right\} \begin{pmatrix} \psi \\ \psi_B \end{pmatrix} = 0 \tag{V.7}$$

from which we recover the R—S equation

$$(i\Gamma \cdot \partial - m)\psi = 0 \tag{V.8}$$

for the 16 upper components and

$$iB \cdot \partial\psi = m\psi_B \tag{V.9}$$

for the four lower components. Once an R—S solution ψ is given, then ψ_B is completely determined. Hence the solutions to Eq. (V. 7) are the same as those of Eq. (V. 8). Furthermore, the components ψ_B do not enter the scalar product because the η matrix now acts as a projector onto the ψ components. Thus the system (V. 7) is completely equivalent to the R—S system. Hence we have called the additional components ψ_B which transform like a Dirac 4-spinor "barnacles."

Note, however, that although the barnacled equation defines the same set of solutions as the unbarnacled equation, the minimal equation of the β_0 matrix may be changed for certain choices of the barnacle coupling. Thus we have two equations with two different algebras but with the same solutions.

It is easy to see from Eq. (V. 7) that all of the above remarks remain valid even in the presence of a minimally coupled external electromagnetic field, $\partial_\mu \to \partial_\mu + ieA_\mu$. Indeed we see that a more general external field coupling formed from any product of β-matrices[5,11] could never involve the ψ_B components in the interaction, i.e., the equation for ψ is unchanged by the presence of ψ_B. Barnacles may, however, enter the dynamics in a nontrivial way for other couplings which are now possible since the β_μ are reducible (but indecomposable). They may therefore provide a convenient vehicle for modifying the dynamics of a physical process while being guaranteed to be absent in the asymptotic states which are governed by the free equation of motion. A more general and detailed discussion will be presented in a subsequent study.

We have seen that simple algebraic techniques may be applied to a given representation of $SL(2, C)$ in order to find the structure of all possible relativistic wave equations corresponding to that representation. This procedure may, of course, be applied to any representation. The present cases merely served (we hope) as illustrative examples.

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[7]That the role of subsidiary conditions is to eliminate the unwanted auxiliary spins in a given theory has been emphasized, e.g., by D.L. Pursey, Ann. Phys. (N.Y.) 32, 157 (1965) in his general study. Here we have first order wave equations without subsidiary conditions and we see that the elimination of the unwanted spins is accomplished by the nilpotency of β_0 on the given subspace thus guaranteeing that the auxiliary spins will vanish in the rest frame.

[8]See Ref. 1, Appendices A and B.

[9]For another approach to this classification see K. Johnson and E.C.G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).

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