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PROBLEMS OF FUNCTIONAL DETERMINANT AND FICTITIOUS LAGRANGIAN IN GAUGE THEORIES *

J.P. HSU and E.C.G. SUDARSHAN
Center for Particle Theory, The University of Texas at Austin, Austin, Texas 78712

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Using the method of Lagrange multiplier fields the fictitious Lagrangian of gauge-fixed gauge theories can be constructed. We obtain different but equivalent forms depending on the extent to which constraint equations are used. These constructions yield fictitious Lagrangians which restore unitarity and gauge invariance. The usual method of constructing fictitious Lagrangians using the change of gauge conditions under gauge transformations does not always lead to the same fictitious Lagrangian constructed by the above method. They are, however, related by an interchange of fictitious fields and lead to the same amplitudes. We also discuss a scale transformation of the fictitious fields, which changes the explicit form of the fictitious Lagrangian but not its amplitudes. In the course of this study we display certain technical problems in the definition of the functional determinants related to a fictitious Lagrangian.

1. Introduction

In a previous paper [1], we formulated the Weinberg unified theory [2] within the Lagrange multiplier formalism and obtained a new fictitious Lagrangian (f-Lagrangian, for short), which differs from the usual f-Lagrangian [3, 4]. Nevertheless, explicit calculations show that it restores unitarity and gauge independence, as the usual f-Lagrangian did. The usual f-Lagrangian (or gauge compensating term) is obtained by considering the change of gauge conditions under a general gauge transformation. We construct a new f-Lagrangian by using field equations for the negative metric Lagrange multiplier fields. The source terms in these Lagrange multiplier field equations produce extra unwanted amplitudes which upset unitarity. Once this dynamical origin of the extra amplitudes is known, we can construct the f-Lagrangian required to cancel the extra amplitudes and restore unitarity (and gauge invariance). Within the Lagrange multiplier formalism, the f-Lagrangian could be obtained in different but "equivalent" forms, depending on how one uses the constraint equations.

We discuss a different prescription for constructing the f-Lagrangian based on

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gauge-condition considerations. The f-Lagrangian obtained by this "new" prescription is apparently different from the usual one in Weinberg's unified theory. It is, however, identical with the new f-Lagrangian based on unitarity considerations. This is expected because in other simpler cases (e.g. the Georgi-Glashow theory and the massless Yang-Mills theory) the f-Lagrangian constructed in these two different ways are identical. This indicates that the dynamical origins for the violations of unitarity and for that of gauge invariance are, in fact, the same. We may remark that the possibility of different prescriptions arises only in theories with a complicated group structure (e.g. $SU(2) \times U(1)$, etc.), where one has to mix fields belonging to different irreducible representations.

We note that this method of constructing the f-Lagrangian based on unitarity consideration has little to do with the usual gauge symmetry and its spontaneous breaking. The significance of this method lies in the fact that it enables us to construct a renormalizable and unitary theory for massive charged vector bosons without introducing a "quartic potential" of scalar fields and without using the concept of spontaneously broken gauge symmetry (cf. sect. 7). We also note that the functional determinant defined by the usual power expansion

$$\det(1+M) = \exp\{\mathrm{Tr}\ln(1+M) = \exp\left\{\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\mathrm{Tr}\,(M^n)\right\}$$

does not give the same amplitudes as that of the corresponding f-Lagrangian. We suggest that the functional determinant in gauge theories should be defined and understood in terms of the f-Lagrangian which, if correct, could restore unitarity and gauge independence.

2. Lagrange multiplier fields and the f-Lagrangian

We first consider the construction of the f-Lagrangian on the basis of unitarity consideration within the Lagrange multiplier formalism [5]. A simple Lagrangian to illustrate the problem is that in Weinberg's unified theory [2]. The non-uniqueness of the usual prescription can be traced back to the mixture of fields (which are self-coupled) belonging to different irreducible representations. We have found that the addition of leptons as sources only makes additional algebraic complications but does not lead to any new physical problems. In fact, the resultant f-Lagrangian is not at all affected by leptons. For clarity of presentation we shall not consider leptons in this paper.

In the Lagrange multiplier formalism, the boson Lagrangian in Weinberg's theory is [2] *

* This is to be contrasted with the cases discussed in ref. [5] where constraints must be used. See also Hsu [6].

$$\mathcal{L}_{\mathrm{W}}^{\mathrm{L}} = \mathcal{L}_{\mathrm{W1}} + M\{\chi_{\mathrm{A}}\partial_\mu A^\mu + \chi_{\mathrm{Z}}\,[\partial_\mu Z^\mu - iM\sec\theta\,(S^0 - \bar{S}^0)/\sqrt{2}\eta]$$
$$+ \chi^-(\partial^\mu W_\mu^+ + iMS^+/\xi) + \chi^+(\partial^\mu W_\mu^- - iMS^-/\xi)\}$$
$$+ M^2\,\{\tfrac{1}{2}\,\alpha\chi_{\mathrm{A}}^2 + \chi_{\mathrm{Z}}^2/(2\eta) + \chi^+\chi^-/\xi\}\ , \tag{1}$$

$$\mathcal{L}_{\mathrm{W1}} = -\tfrac{1}{2}\,|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + ie\,(W_\mu^+ A_\nu - W_\nu^+ A_\mu) - iG\cos^2\theta\,(W_\mu^+ Z_\nu - W_\nu^+ Z_\mu)|^2$$
$$-\tfrac{1}{4}\,|\partial_\mu A_\nu - \partial_\nu A_\mu - ie\,(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)|^2$$
$$-\tfrac{1}{4}\,|\partial_\mu Z_\nu - \partial_\nu Z_\mu + iG\cos^2\theta\,(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)|^2$$
$$+|\partial_\mu S^+ + \sqrt{\tfrac{1}{2}}\,iG\cos\theta\,W_\mu^+ S^0 + i\,(-eA_\mu + \tfrac{1}{2}\,G\cos 2\theta\,Z_\mu)\,S^+ + iMW_\mu^+|^2$$
$$+|\partial_\mu S^0 - \tfrac{1}{2}\,iGZ_\mu S^0 + \sqrt{\tfrac{1}{2}}\,iG\cos\theta\,W_\mu^- S^+ - \sqrt{\tfrac{1}{2}}\,iM\sec\theta Z_\mu|^2$$
$$-\lambda\,[S^+S^- + (S^0 + \sqrt{\tfrac{1}{2}}\,v)\,(S^0 + v/\sqrt{2}) - \tfrac{1}{2}\,v^2]^2\ , \tag{2}$$

$$e = -G\cos\theta\,\sin\theta\ , \qquad S^0 = \sqrt{\tfrac{1}{2}}\,(\psi + i\chi)\ ,$$

where we have chosen the gauge conditions $\partial_\mu A^\mu = Q$, $\partial^\mu W_\mu^+ + iMS^+/\xi = Q'$, etc, and $\chi^\pm, \chi_{\mathrm{A}}, \chi_{\mathrm{Z}}$ are the Lagrange multiplier fields (Q and Q' are some fixed functions).

The Lagrangian (1) leads to the constraints

$$\partial_\mu A^\mu + \alpha M\chi_{\mathrm{A}} = 0\ , \tag{3}$$

$$\partial_\mu Z^\mu - iM\sec\theta\,(S^0 - \bar{S}^0)/(\sqrt{2}\eta) + M\chi_{\mathrm{Z}}/\eta = 0\ , \tag{4}$$

$$\partial^\mu W_\mu^+ + iMS^+/\xi + M\chi^+/\xi = 0\ , \tag{5}$$

$$\partial^\mu W_\mu^- - iMS^-/\xi + M\chi^-/\xi = 0\ . \tag{6}$$

and the field equations for $W_\mu^\pm, Z_\mu, A_\mu, S^\pm, S^0$ and $\bar{S}^0$. The divergences of the field equations for the vector fields $W_\mu^\pm, Z_\mu$ and A_μ together with these field equations themselves lead to the equations for the Lagrange multiplier fields:

$$\Box\,\chi_{\mathrm{A}} + t_{\mathrm{A}} = 0\ , \qquad t_{\mathrm{A}} \equiv ie\,(W_\mu^-\partial^\mu\chi^+ - W_\mu^+\partial^\mu\chi^-) - eM(S^-\chi^+ + S^+\chi^-)/\xi\ , \tag{7}$$

$$(\Box + M_{\mathrm{Z}}^2/\eta)\,\chi_{\mathrm{Z}} + t_{\mathrm{Z}} = 0\ , \qquad M_{\mathrm{Z}} \equiv M\sec\theta\ , \tag{8}$$

$$t_Z = iG\cos^2\theta\,(W_\mu^+\partial^\mu\chi^- - W_\mu^-\partial^\mu\chi^+) + GM_Z\psi\chi_Z/2\eta$$
$$+ GM\cos 2\theta\,(\chi^+S^- + \chi^-S^+)/2\xi\ , \qquad \psi = \sqrt{\tfrac{1}{2}}\,(S^0 + \overline{S}^0)\ ,$$
$$(\Box + M^2/\xi)\,\chi^+ + t^+ = 0\ , \tag{9}$$
$$t^+ = GM\cos\theta\,(\sqrt{2}\,\overline{S}^0)\,\chi^+/2\xi - \partial_\mu\chi^+(ieA^\mu - iG\cos^2\theta Z^\mu)$$
$$+ ieW_\mu^+\partial^\mu\chi_A - iG\cos^2\theta\,W_\mu^+\partial^\mu\chi_Z - GM_Z\cos\theta\,S^+\chi_Z/2\eta\ ,$$

and the adjoint of (9). These equations are obtained by tedious but straightforward calculations. We emphasize that (7)–(9) are derived without using the constraints (3)–(6) [6]. The source terms in (7)–(9) and the adjoint of (9) determine the interactions of the unphysical fields in the theory [5, 7].

The Lagrangian (1) could be written in the usual form [5]

$$\mathcal{L}_W = \mathcal{L}_{W1} - (\partial_\mu A^\mu)^2/2\alpha - \xi|\partial^\mu W_\mu^+ + iMS^+/\xi|^2 - (\tfrac{1}{2}\eta)$$
$$\times\,|\partial^\mu Z_\mu - iM\sec\theta\,(S^0 - \overline{S}^0)/(\sqrt{2}\eta)|^2\ , \tag{10}$$

which is usually used for writing down rules for the Feynman diagrams. (cf. eq. (15) below). In this Lagrangian, for each of the masses 0, $M_Z\eta^{-\frac{1}{2}}$, $M\xi^{-\frac{1}{2}}$ and $M\xi^{-\frac{1}{2}}$ there are two unphysical degrees of freedom corresponding to each of A_μ, Z_μ, W_μ^-, and W_μ^+. For A_μ these are the longitudinal and timelike components. For the massive field Z_μ they are the timelike component plus the scalar field $(S^0 - \overline{S}^0)/\sqrt{2}\,i$. For $W_\mu^\pm$ it is again the timelike components and the scalar quanta of $S^\pm$ that make up the two unphysical degrees of freedom. Therefore, (for the purpose of discussions below) in analogy with the massless Yang-Mills theory [5] the effective Lagrangian $\mathcal{L}_{\text{eff}}$ for the unphysical fields in (10) can be written in the form:

$$\mathcal{L}_{\text{eff}} = -\overline{\chi^+}\,(\Box + M^2/\xi)\,\chi^+ - \overline{\chi}^+t^+ - \overline{\chi}^-(\Box + M^2/\xi)\,\chi^- - \overline{\chi}^-t^-$$
$$- \overline{\chi}_A\Box\chi_A - \overline{\chi}_At_A - \overline{\chi}_Z(\Box + M_Z^2/\eta)\,\chi_Z - \overline{\chi}_Zt_Z\ , \tag{11}$$

where each of χ^+, χ^-, χ_A, and χ_Z should be understood as a *complex* scalar field, and the extra amplitude due to the production of these unphysical fields in the intermediate states is

$$X_1 = \int \exp\{\,i\int d^4x\ \mathcal{L}_{\text{eff}}\}\,d\,[\overline{\chi}^\pm, \chi^\pm, \overline{\chi}_A, \chi_A, \overline{\chi}_Z, \chi_Z] = \text{const.}/\det(1 + \mathcal{G}\cdot\gamma_1)\ , \tag{12}$$

where the matrices $\mathcal{G}$ and γ_1 are defined by

$$\begin{bmatrix} -(\Box+M^2/\xi-i\epsilon) & 0 & 0 & 0 \\ 0 & -(\Box+M^2/\xi-i\epsilon) & 0 & 0 \\ 0 & 0 & -(\Box-i\epsilon) & 0 \\ 0 & 0 & 0 & -(\Box+M_Z^2/\eta-i\epsilon) \end{bmatrix} \mathcal{G} = \delta(x-y) \ , \tag{13}$$

$$[\gamma_1] \begin{bmatrix} \chi^+ \\ \chi^- \\ \chi_A \\ \chi_Z \end{bmatrix} = \begin{bmatrix} t^+ \\ t^- \\ t_A \\ t_Z \end{bmatrix} . \tag{14}$$

The extra amplitude (12) upsets unitarity and gauge invariance. We remove it from the theory by dividing the integrand of the path integral defining the amplitude coming from the Lagrangian (1) or (10) by X_1. The theory is completely defined by

$$\begin{aligned} A_1 &\equiv \int (1/X_1) \exp\{ i \int d^4x \, (\mathcal{L}_W^L + \mathcal{L}_s) \} d\,[F, \chi^\pm, \chi_A, \chi_Z] \\ &= \text{const.} \int (1/X_1) \exp\{ i \int d^4x \, (\mathcal{L}_W + \mathcal{L}_s) \} d\,[F] \ , \end{aligned} \tag{15}$$

where $\mathcal{L}_s$ is the external source terms for physical particles and F stands for a set of fields:

$$F \equiv \{ W_\mu^\pm, Z_\mu, A_\nu, S^\pm, S^0, \bar{S}^0 \} \ . \tag{16}$$

In perturbation calculations, the factor $1/X_1$ in (15) is usually replaced by the f-Lagrangian $\mathcal{L}_1(\bar{D}, D)$

$$\mathcal{L}_1(\bar{D}, D) = \mathcal{L}_{\text{eff}} \Big|_{\chi^\pm, \chi_A, \chi_Z \to D^\pm, D_A, D_Z} \ , \tag{17}$$

where D^+, D^-, D_A, and D_Z are the complex formion-scalar fields. It restores unitarity and gauge independence of the theory (cf. sect. 6).

If one uses constraints (5) and (6) in eq. (7) and (8), one gets

$$\Box \chi_A + u_A = 0 \ , \qquad u_A \equiv ie\, \partial^\mu (W_\mu^- \chi^+ - W_\mu^+ \chi^-) \ , \tag{18}$$

$$(\Box + M_Z^2/\eta)\, \chi_Z + u_Z = 0 \ , \tag{19}$$

$$u_Z \equiv iG \cos^2\theta \, \partial^\mu (W_\mu^+ \chi^- - W_\mu^- \chi^+) - \frac{GM}{2\xi} (\chi^- S^+ + \chi^+ S^-) + \frac{GM_Z}{2\eta} \psi \chi_Z \ .$$

$$\begin{bmatrix} -(\Box+M^2/\xi-i\epsilon) & 0 & 0 & 0 \\ 0 & -(\Box+M^2/\xi-i\epsilon) & 0 & 0 \\ 0 & 0 & -(\Box-i\epsilon) & 0 \\ 0 & 0 & 0 & -(\Box+M_Z^2/\eta-i\epsilon) \end{bmatrix} \mathcal{G} = \delta(x-y) \ , \tag{13}$$

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Repeating steps (7)–(17), the source terms in (18), (19), and (9) give the following f-Lagrangian $\mathcal{L}_2$ involving complex fictious fields D, $\bar{D}$:

$$\mathcal{L}_2(\bar{D}, D) = \mathcal{L}_{\text{eff}}^{(2)}\Big|_{\chi^{\pm},\, \chi_A,\, \chi_Z \to D^{\pm},\, D_A,\, D_Z} , \tag{20}$$

$$\mathcal{L}_{\text{eff}}^{(2)} = -\bar{\chi}^{+}(\Box + M^2/\xi)\,\chi^{+} - \overline{\chi^{+}}\, t^{+} - \overline{\chi^{-}}(\Box + M^2/\xi)\,\chi^{-} - \overline{\chi^{-}}\, t^{-}$$

$$- \bar{\chi}_A \Box \chi_A - \bar{\chi}_A u_A - \bar{\chi}_Z(\Box + M_Z^2/\eta)\,\chi_Z - \bar{\chi}_Z u_Z ,$$

which has been discussed in ref. [1]. Although (20) is different from (17), explicit calculations up to and including the two-loop level show that the f-Lagrangian (17) also restores unitarity and gauge independence of the theory.

Now, if one further uses all the constraints (3)–(6), eq. (9) could be written as

$$(\Box + M^2/\xi)\,\chi^{+} + ieM\chi_A\chi^{+}(\xi^{-1} - \alpha) + iG\cos^2\theta M\chi_Z\chi^{+}(\eta^{-1} - \xi^{-1}) + v^{+} = 0 , \tag{21}$$

$$v^{+} = \frac{GM\cos\theta}{2\xi}\,\chi^{+}(\psi + i\chi) + iG\cos^2\theta\,\partial_\mu(Z^\mu\chi^{+}) - ie\partial_\mu(A^\mu\chi^{+})$$

$$+ ie\partial^\mu(W_\mu^{+}\chi_A) - iG\cos^2\theta\,\partial^\mu(W_\mu^{+}\chi_Z) - \frac{eM}{\xi}\,S^{+}\chi_A \tag{22}$$

$$+ \frac{GM}{2\xi}\cos 2\theta\, S^{+}\chi_Z .$$

There is no obvious way to construct the f-Lagrangian from eqs. (7), (8), (21) or (18), (19), (21) unless $\alpha = \xi^{-1} = \eta^{-1}$. If one set $\alpha = \xi^{-1} = \eta^{-1}$, there is a simple f-Lagrangian for these equations. For example, the corresponding extra amplitude due to the sources terms in (18), (19), and (21) is

$$X_3 = \text{const}/\det(1 + \mathcal{G}\cdot\gamma_3) , \qquad (\alpha = \xi^{-1} = \eta^{-1}) , \tag{23}$$

where γ_3 is defined by

$$[\gamma_3]\begin{bmatrix}\chi^{+}\\ \chi^{-}\\ \chi_A\\ \chi_Z\end{bmatrix} = \begin{bmatrix} v^{+}\\ v^{-}\\ u_A\\ u_Z\end{bmatrix} , \tag{24}$$

and v^{-} is the adjoint of v^{+}. The Lagrangian corresponding to $1/X_3$ is

$$\mathcal{L}_3(\bar{D}, D) = \mathcal{L}_{\text{eff}}^{(3)}\Big|_{\chi^{\pm},\, \chi_A,\, \chi_Z \to D^{\pm},\, D_A,\, D_Z} , \qquad (\alpha = \xi^{-1} = \eta^{-1}) , \tag{25}$$

$$\mathcal{L}^{(3)}_{\text{eff}} = -\overline{\chi^+}\,(\Box + M^2/\xi)\,\chi^+ - \overline{\chi^+}v^+ - \overline{\chi^-}\,(\Box + M^2/\xi)\,\chi^- - \overline{\chi^-}v^- \\ -\overline{\chi}_A\,\Box\,\chi_A - \overline{\chi}_A u_A - \overline{\chi}_Z(\Box + M_Z^2/\eta)\,\chi_Z u_Z\ . \tag{26}$$

Thus, we see that one may obtain different forms of the fictitious Lagrangian (17), (25), or (20). By explicit calculations at the two-loop level, we have that all these three f-Lagrangians given the same net results for arbitrary α and η with $\xi = 1$. It appears puzzling because (25) is obtained under the condition $\alpha = \xi^{-1} = \eta^{-1}$. This will be discussed in sect. 6.

3. Gauge conditions and the usual f-Lagrangian

The gauge compensating term on the usual f-Lagrangian is, according to the ideas of Faddeev and Popov [8, 3] obtained by considering the change of gauge conditions under the infinitesimal gauge transformations. The gauge conditions in the Lagrangian (1) are

$$\partial_\mu A^\mu = 0\ , \quad \partial_\mu Z^\mu - \frac{iM\sec\theta}{\sqrt{2}\,\eta}\,(S^0 - \overline{S}^0) = 0\ , \tag{27}$$

$$\partial^\mu W^+_\mu + \frac{iM}{\xi}\,S^+ = 0\,, \qquad \partial^\mu W^-_\mu - \frac{iM}{\xi}\,S^- = 0\ . \tag{28}$$

The infinitesimal local gauge transformations for the SU(2) × U(1) group associated with the Weinberg Lagrangian $\mathcal{L}_{\text{W1}}$ in (2) are

$$S^+ \to S^+ - \tfrac{1}{2}i\,S^+\Lambda_{3+} - \sqrt{\tfrac{1}{2}}i\,S^0\Lambda^+ - \frac{iM}{g}\Lambda^+\ , \tag{29}$$

$$S^0 \to S^0 + \tfrac{1}{2}i\,S^0\,\Lambda_{3-} - \sqrt{\tfrac{1}{2}}i\,S^+\Lambda^- + \frac{iM_z}{\sqrt{2}G}\,\Lambda_{3-}\ , \tag{30}$$

$$W^+_\mu \to W^+_\mu - iW^+_\mu\Lambda_3 + iW^3_\mu\Lambda^+ + \frac{1}{g}\,\partial_\mu\,\Lambda^+\ , \tag{31}$$

$$W^3_\mu \to W^3_\mu + i\,(W^+_\mu\Lambda^- - W^-_\mu\Lambda^+) - \frac{1}{g}\,\partial_\mu\Lambda_3\ , \tag{32}$$

$$B_\mu \to B_\mu - \frac{1}{g'}\,\partial_\mu(2\Lambda_0)\,, \qquad g = G\cos\theta\,, \qquad g' = G\sin\theta\ , \tag{33}$$

where

$$\Lambda_{3-} = \Lambda_3 - 2\Lambda_0\,, \quad \Lambda_{3+} = \Lambda_3 + 2\Lambda_0\,, \Lambda^\pm = \sqrt{\tfrac{1}{2}}(\Lambda_1 \mp i\Lambda_2)\,, \quad W^\pm_\mu = \sqrt{\tfrac{1}{2}}(W_{1\mu} \mp iW_{2\mu})\,.$$

It follows that

$$\delta(\partial_\mu A^\mu) = \frac{1}{G}\Box\Lambda_A + \frac{u_A}{G\cos\theta}\bigg|_{\chi^\pm\to\Lambda^\pm} \equiv \frac{1}{G}\Box\Lambda_A + \frac{Q_A}{G\cos\theta}\ , \tag{34}$$

$$\delta\left[\partial_\mu Z^\mu - \frac{iM\sec\theta}{\sqrt{2}\,\eta}(S^0 - \bar{S}^0)\right] = \frac{1}{G}(\Box + M_Z^2/\eta)\Lambda_Z + \frac{u_Z}{G\cos\theta}\bigg|_{\chi^\pm,\chi_A\to\Lambda^\pm,\Lambda_A}$$

$$\equiv \frac{1}{G}(\Box + M_Z^2/\eta)\Lambda_Z + \frac{Q_Z}{G\cos\theta}\ , \tag{35}$$

$$\delta\left[\partial^\mu W_\mu^+ + \frac{iM}{\xi}S^+\right] = \frac{1}{G}(\Box + M^2/\xi)\Lambda^+ + \frac{v^+}{G\cos\theta}\bigg|_{\chi^\pm,\chi_A,\chi_Z\to\Lambda^\pm,\Lambda_A,\Lambda_Z}$$

$$\equiv \frac{1}{G\cos\theta}(\Box + M^2/\xi)\Lambda^+ + \frac{Q^+}{G\cos\theta}\ , \tag{36}$$

where u_A, u_Z and v^+ are given by (18), (19), and (22) respectively. According to the *usual prescription,* the expressions (34)–(36) suggest the f-Lagrangian [9]:

$$\mathcal{L}(\bar{\phi},\phi) = \{-\bar{\Lambda}^+(\Box + M^2/\xi)\Lambda^+ - \bar{\Lambda}^+Q^+$$

$$-\bar{\Lambda}^-(\Box + M^2/\xi)\Lambda^- - \bar{\Lambda}^-Q^- - \bar{\Lambda}_A(\Box)\Lambda_A - \bar{\Lambda}_A Q_A$$

$$-\bar{\Lambda}_Z(\Box + M_Z^2/\eta)\Lambda_Z - \bar{\Lambda}_Z Q_Z\}_{\Lambda^\pm,\Lambda_A,\Lambda_Z\to\phi^\pm,\phi_A,\phi_Z}\ , \tag{37}$$

where $\phi^\pm$, ϕ_A and ϕ_Z are complex fictitious fields and Q^- is the adjoint of Q^+. The functional determinant J_ϕ corresponding to (37) is

$$J_\phi = \int \exp\{i\int d^4x\,\mathcal{L}(\bar{\phi},\phi)\}\,d[\phi,\bar{\phi}], \qquad \phi \equiv \{\phi^\pm,\phi_A,\phi_Z\},$$

$$= \det(1 + \mathcal{G}\cdot\gamma_\phi)\ , \tag{38}$$

where we ignore an irrelevant constant factor and γ_ϕ is given by

$$[\gamma_\phi]\begin{bmatrix}\Lambda^+\\ \Lambda^-\\ \Lambda_A\\ \Lambda_Z\end{bmatrix} = \begin{bmatrix}Q^+\\ Q^-\\ Q_A\\ Q_Z\end{bmatrix}\ . \tag{39}$$

The results (37) and (38) with arbitrary $\xi > 0$ and arbitrary $\eta > 0$ are given in ref. [9].

4. Prescriptions for constructing gauge compensating terms

The idea behind gauge compensating terms is to introduce additional terms in the Lagrangian to compensate the change of gauge conditions during the evolution of the physical system. In so doing, the gauge condition could be consistently defined for all times and the theory remains gauge independent. The reason why one has to introduce complex (rather than real) fictitious fields can be understood from dynamical considerations: namely, the change of each of the gauge conditions in (27) and (28) are due to the intervention of two non-observable fields. For example, the interactions of the longitudinal and the time-like photons change the gauge condition $\partial_\mu A^\mu = 0$; while the interactions of $\partial^\mu W^+_\mu$ (the spin-0 part of the 4-vector field W^+_μ) and S^+ changes the gauge condition $\partial^\mu W^+_\mu + iMS^+/\xi = 0$. Thus, one must introduce two-component (i.e. complex) fictitious fields to compensate the change of each gauge conditions [5]. In quantum field theory, the two non-observable fields $\partial^\mu W^+_\mu$ and S^+ have the same mass $M\xi^{-\frac{1}{2}}$ but opposite metric; the fictitious particle and its anti-particle have also the same mass $M\xi^{-\frac{1}{2}}$ but opposite metric [10].

Before gauge symmetry is spontaneously broken in Weinberg's theory, we have a parameter $\bar{\alpha}$ associated with the gauge condition for the isoscalar vector field $B_\mu(x)$ and another independent parameter $\bar{\xi}$ for the isovector vector field $A_\mu(x)$. Thus, we have gauge-fixing terms $-(\partial_\mu B^\mu)^2/(2\bar{\alpha})$ and $-\frac{1}{2}\bar{\xi}(\partial_\mu A^\mu)^2$, for example, in the Lagrangian. After the symmetry is broken, we still have an arbitrary parameter α (independent of ξ) because there is a residual exact Abelian gauge symmetry for the interaction of the photon. However, there is no similar gauge symmetry for the interaction of the Z_μ field. This does not matter because any change of the gauge condition for $Z_\mu(x)$ is compensated by the f-Lagrangian. Therefore, the theory is independent of the parameter η associated with the gauge condition for $Z_\mu(x)$.

It is natural to have the following correspondence between $W^\pm_\mu(x)$, $\Lambda^\pm(x)$ and the fictitious field $\phi^\pm(x)$:

$$W^{\pm\mu} = \sqrt{\tfrac{1}{2}}(W^\mu_1 \mp iW^\mu_2) \leftrightarrow \Lambda^\pm = \sqrt{\tfrac{1}{2}}(\Lambda_1 \mp i\Lambda_2) \leftrightarrow \phi^\pm \quad . \tag{40}$$

Now since A_μ and Z_μ are given by

$$A_\mu = \sin\theta\ W^3_\mu + \cos\theta\ B_\mu \quad , \qquad Z_\mu = \cos\theta\ W^3_\mu - \sin\theta\ B_\mu \quad , \tag{41}$$

how shall we write down the corresponding linear combinations for Λ_3 and Λ_0? The problem arises because $\frac{1}{2}g' \neq g$, there is no relation at all between Λ_3 and Λ_0. Symmetry considerations and the structure of (32) and (33) suggest that we could write (33) as

$$B_\mu \to B_\mu - \frac{1}{g}\partial_\mu\left(\frac{g}{\frac{1}{2}g'}\Lambda_0\right) \quad , \tag{42}$$

and that we should have the correspondence

$$W^\mu_3 \leftrightarrow \Lambda_3 \leftrightarrow D_3 \ , \qquad B^\mu \leftrightarrow \frac{g}{\frac{1}{2}g'}\Lambda_0 \leftrightarrow \Phi_0 \quad , \tag{43}$$

that is,

$$A_\mu \leftrightarrow \Lambda'_A = \sin\theta\ \Lambda_3 + \cos\theta\left(\frac{g}{\frac{1}{2}g'}\Lambda_0\right) \leftrightarrow \Phi_A = \sin\theta\ \Phi_3 + \cos\theta\ \Phi_0\ ,$$

$$Z_\mu \leftrightarrow \Lambda'_Z = \cos\theta\ \Lambda_3 - \sin\theta\left(\frac{g}{\frac{1}{2}g'}\Lambda_0\right) \leftrightarrow \Phi_Z = \cos\theta\ \Phi_3 - \sin\theta\ \Phi\ , \tag{44}$$

where Φ_A and Φ_Z are fictitious fields.

If the gauge conditions (27) and (28) are not to change under the gauge transformation, the gauge functions $\Lambda^\pm$, Λ_3 and Λ_0 should fulfil the relations:

$$\begin{aligned} &\Box\Lambda_A + Q_A/\cos\theta = 0\ , \qquad (\Box + M_Z^2/\eta)\,\Lambda_Z + Q_Z/\cos\theta = 0\ , \\ &(\Box + M^2/\xi)\,\Lambda^+ + Q^+ = 0\ , \end{aligned} \tag{45}$$

where Q_A, Q_Z, and $Q^\pm$ are given by (34)–(36). We may express these equations in terms of $\Lambda^\pm$, Λ'_A, and Λ'_Z, and then replace $\Lambda^\pm$, Λ'_A and Λ'_Z by fictitious fields $\Phi^\pm$, Φ_A and Φ_Z respectively according to (40) and (44). We find

$$\beta_A \equiv \Box\,\Phi_A + ie\,\partial^\mu(W_\mu^-\Phi^+ - W_\mu^+\Phi^-) = 0\ , \tag{46}$$

$$\begin{aligned} \beta_Z \equiv{}& (\Box + M_Z^2/\eta)\,\Phi_Z - iG\cos^2\theta\,\partial^\mu\,(W_\mu^-\Phi^+ - W_\mu^+\Phi^-) \\ &+ \frac{GM_Z}{2\eta}\,\psi\Phi_Z - \frac{GM}{2\eta}\,(S^+\Phi^- + S^-\Phi^+) = 0\ , \end{aligned} \tag{47}$$

$$\begin{aligned} \beta^+ \equiv{}& (\Box + M^2/\xi)\,\Phi^+ + iG\cos\theta\ \partial^\mu\,[(Z_\mu\cos\theta + A_\mu\sin\theta)\,\Phi^+] \\ &+ \frac{GM\cos\theta}{2\xi}(\psi + i\chi)\,\Phi^+ + ie\,\partial^\mu\,[W_\mu^+(\Phi_A + \cot\theta\Phi_Z)] \\ &- \frac{eM}{2\xi}\,S^+\left(2\Phi_A - \frac{G}{e}\cos 2\theta\Phi_Z\right) = 0\ . \end{aligned} \tag{48}$$

The fictitious Lagrangian is, therefore,

$$\mathcal{L}(\bar\Phi, \Phi) = -\bar\Phi_A\beta_A - \bar\Phi_Z\beta_Z - \bar\Phi^+\beta^+ - \bar\Phi^-\beta^-\ , \tag{49}$$

where β^- is the adjoint of β^+. The Lagrangian (49) will generate the necessary additional amplitude to cancel the non-gauge invariant amplitude in to the Weinberg Lagrangian [10].

The Faddeev-Popov Jacobian J_ϕ corresponding to (49) is

$$J_\Phi = \int \exp\{ i \int d^4x \, \mathcal{L}(\bar{\Phi}, \Phi) \} \, d[\bar{\Phi}, \Phi] \ , \qquad \Phi \equiv \{ \Phi^\pm, \Phi_A, \Phi_Z \} ,$$

$$= \det(1 + \mathcal{G} \cdot \gamma_\Phi) \ , \tag{50}$$

where γ_Φ is given by

$$[\gamma_\Phi] \begin{bmatrix} \Phi^+ \\ \Phi^- \\ \Phi_A \\ \Phi_Z \end{bmatrix} = \begin{bmatrix} \beta^+ - (\Box + M^2/\xi)\, \Phi^+ \\ \beta^- - (\Box + M^2/\xi)\, \Phi^- \\ \beta_A - \Box \Phi_A \\ \beta_Z - (\Box + M_Z^2/\eta)\, \Phi_Z \end{bmatrix} . \tag{51}$$

When one writes down (49) and (17) explicitly, one observes that the f-Lagrangian (49) is the "same" as the f-Lagrangian (17) if the fields $\Phi_A, \Phi_Z, \Phi^-, \Phi^+$ and their adjoints are respectively replaced by $\bar{D}_A, \bar{D}_Z, \bar{D}^+, \bar{D}^-$ and their adjoints. Such a replacement of fictitious fields, which do not appear in external states of observable processes, does not affect physics and, therefore, the f-Lagrangians (49) (obtained by gauge-condition considerations) and (17) (obtained by unitarity considerations) are equivalent. This indicates that the sources upsetting gauge invariance are the same as those upsetting unitarity. Once one eliminates one of these troubles the other trouble is automatically eliminated.

5. The precise meaning of the functional determinant

In the literature [8] the perturbation amplitudes of a functional determinant $J_\phi = \det(1 + \mathcal{G} \cdot \gamma_\phi)$, for example, are given by expanding $\ln \det(1 + \mathcal{G} \cdot \gamma_\phi) = \mathrm{Tr} \ln(1 + \mathcal{G} \cdot \gamma_\phi)$ in a power series of the coupling constants in the form

$$\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \mathrm{Tr}\, (\mathcal{G} \cdot \gamma_\phi)^n \ .$$

One observes that the path integral (50) reduces to (38) by a change of variables

$$\Phi^\pm \to \phi^\pm, \qquad \Phi_A \to \phi_A = \Phi_A/\cos\theta \ , \qquad \Phi_Z \to \phi_Z = \Phi_Z/\cos\theta \ . \tag{52}$$

Thus, $(1 + \mathcal{G} \cdot \gamma_\Phi)$ in (50) and $(1 + \mathcal{G} \cdot \gamma_\phi)$ in (38) are related by a formal similarity transformation, i.e.

$$[1 + \mathcal{G} \cdot \gamma_\Phi] = C^{-1} [1 + \mathcal{G} \cdot \gamma_\phi] \, C \ , \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix} . \tag{53}$$

With usual definition of a determinant, one obtains

$$\det(1 + \mathcal{G} \cdot \gamma_\Phi) = \det(1 + \mathcal{G} \cdot \gamma_\phi) , \qquad \det C^{-1} \det C = 1 . \tag{54}$$

The fourth order term in ln det $(1 + \mathcal{G} \cdot \gamma_\Phi)$, for example, involves only the term of the form $T_1 \equiv G_\xi(x_1 - x_2)\, G_0(x_2 - x_3)\, G_\xi(x_3 - x_4)\, G_0(x_4 - x_1)$ where G_ξ corresponds to $(\Box + M^2/\xi)^{-1}$ and G_0 corresponds to $\Box^{-1}$. However, the fictitious Lagrangian corresponding to det $(1 + \mathcal{G} \cdot \gamma_\phi)$ gives, in addition to the term T_1, a term of the form $G_\xi(x_1 - x_2)\, G_0(x_2 - x_1)\, G_\xi(x_3 - x_4)\, G_0(x_4 - x_3)$ representing the diagram with two separate ficititious loops. This additional term is necessary for unitarity and gauge independence of the theory. A similar situation also occurs in the self-energy diagrams with two separate single loops in the massless Yang-Mills theory [8, 5]. Therefore, the functional determinant in gauge theories should be defined and understood in terms of the f-Lagrangian (cf. appendix).

6. A scale transformation of the fictitious fields

Let us consider the following scale transformation for the fictition fields:

$$\begin{aligned} &\Phi_A \to a\Phi_A , && \bar{\Phi}_A \to \bar{\Phi}_A/a , \\ &\Phi_Z \to b\Phi_Z , && \bar{\Phi}_Z \to \bar{\Phi}_Z/b , \\ &\Phi^+ \to c\Phi^+ , && \bar{\Phi}^+ \to \bar{\Phi}^+/c , \\ &\Phi^- \to d\Phi^- , && \bar{\Phi}^- \to \bar{\Phi}^-/d , \end{aligned} \tag{54}$$

where a, b, c, and d are real constants. The structure and the free Lagrangian terms of the Lagrangian (49), for example, are unchanged under the transformation (54). But the couplings of the interaction terms are changed. However, they change in such a way that the amplitudes due to the Lagrangian (49) are not changed. This is due to the fact that the fictitious particles are not allowed to appear in the external states of a physical process. Thus, whenever a fictitious particle is created at some vertex, it must be annihilated at another vertex in each Feynman diagram. In other words, the interaction term involving $\bar{\Phi}_i \Phi_j$ $(i, j = +, -. A, Z)$ must be always accompanied by the term involving $\bar{\Phi}_j \Phi_i$. We see that the product of the coupling strength of any such pair of terms is not changed by the transformation (54). Therefore, we could have a class of the f-Lagrangian which are related by the transformation (54) and gives rise to the same amplitudes.

7. Remarks and conclusions

By explicit calculations up to the 2-loop level, we show that the f-Lagrangians (17) and (20) lead to the same net amplitudes which restore unitarity and gauge in-

dependence for $\xi = 1, \eta > 0$ and arbitrary α. The f-Lagrangian (25) turns out to be identical to (49), while (49) is obtained without the condition $\alpha = \xi^{-1} = \eta^{-1}$. Thus it is not surprising that (25) does restore unitarity for arbitrary α, ξ and η, although it is obtained under the condition $\alpha = \xi^{-1} = \eta^{-1}$. This shows that one must be careful in using the constraints in the Lagrange multiplier formalism. Dirac has pointed out that constraint equations are "weak" as contrasted with the "strong" dynamical equations because with a dynamical equation one can take the Poisson bracket of both side with any dynamical variable and get other valid equations of motion, while with a constraint equation one cannot do this in general [11].

Using the method based on the Lagrange multiplier formalism we can construct a new class of renormalizable theories for massive charged vector bosons. Take the Lagrangian (2), for example. It is usually obtained from a symmetric Lagrangian with a "quartic potential" of scalar fields, in which the parameters are so arranged that symmetry is spontaneously broken and one of the scalar fields develops a vacuum expectation value. Now, we consider the Lagrangian $\mathcal{L}'_{\mathrm{W1}} = \mathcal{L}_{\mathrm{W1}}|_{\lambda=0}$. We see that $\mathcal{L}'_{\mathrm{W1}}$ involves massive charged vector bosons and that it is not invariant under the usual local gauge transformation. It has no "quartic potential" and apparently has nothing to do with spontaneously broken symmetry. However, we can construct a new theory by starting with the Lagrangian (1) with $\mathcal{L}_{\mathrm{W1}}$ replaced by $\mathcal{L}'_{\mathrm{W1}}$ and repeating the steps in eqs. (1)–(17). Such a theory is also unitary and renormalizable by standard power counting. (This will be considered in detail in a separate paper.) Thus, the new method discussed above has more general validity for it could be applied to theories with or without spontaneously broken gauge symmetry [6].

Appendix

In this appendix, we check unitarity by explicit calculations at both 1-loop and 2-loop levels and illustrate the discussion about functional determinant in sect. 5.

We first verify that the theory defined by the physical amplitude A_1 given by (15) is unitary and gauge independent. The effective Lagrangian for the theory is $\mathcal{L}_{\mathrm{W}} + \mathcal{L}_1(D, \bar{D})$ given by (10) and (17). At the 1-loop level, we calculate the imaginary part of the 1-loop self-energy diagram of the physical vector boson $W^+(p)$ with the 4-momentum p_μ and the polarization vector $\epsilon_\mu(p)$ satisfying $\epsilon_\mu p^\mu = 0$. Based on unitarity and phase-space considerations, the contributions due to the unphysical intermediate states $(W_s^+\gamma)$, $(S^+\gamma)$, $(D^+\bar{D}_{\mathrm{A}})$ and $(\bar{D}^-D_{\mathrm{A}})$ must cancel among themselves, where W_s^+ is the negative spin-0 particle associated with the 4-vector field $W_\mu^+(x)$. We find that

$$\mathrm{W}^+(p) \to \mathrm{W}_{\mathrm{S}}^+(k)\,\gamma\,(q) \to \mathrm{W}^+(p): \qquad -e^2\epsilon_\mu\epsilon_\nu[-M^2g^{\mu\nu} - 2q^\mu q^\nu + M^2 \times (1-\alpha)\,q^\mu q^\nu/q^2]\ , \tag{A.1}$$

$$\mathrm{W}^+(p) \to \mathrm{S}^+(k)\,\gamma\,(q) \to \mathrm{W}^+(p): \qquad +e^2\epsilon_\mu\epsilon_\nu\,[-M^2 g^{\mu\nu} + M^2(1-\alpha)\,q^\mu q^\nu/q^2]\ , \tag{A.2}$$

$$\mathrm{W}^+(p) \to \mathrm{D}^+(k)\,\overline{\mathrm{D}}_\mathrm{A}(q) \to \mathrm{W}^+(p): \quad -e^2\epsilon_\mu\epsilon_\nu q^\mu q^\nu\ , \tag{A.3}$$

$$\mathrm{W}^+(p) \to \overline{\mathrm{D}}^-(k)\,\mathrm{D}_\mathrm{A}(q) \to \mathrm{W}^+(p): \quad -e^2\epsilon_\mu\epsilon_\nu q^\mu q^\nu\ , \tag{A.4}$$

where the overall signs are related to the facts that W^+_S has a negative metric, while S^+ has a positive metric, and D's are scalar-fermions. The sum of (A.1)–(A.4) indeed vanish for any α, as required by unitarity and gauge invariance.

To verify unitarity and gauge independence at the 2-loop level, we show that there is no net contribution to the imaginary amplitudes due to two unphysical particles (with masses $M/\xi^{\frac{1}{2}}$ and $M_z/\eta^{\frac{1}{2}}$) and one physical particle (with mass M) in the intermediate states of the 2-loop self-energy diagram of the neutral vector boson $\mathrm{Z}(p_1)$ with momentum p_1^μ and polarization ϵ_1^μ. The imaginary amplitude of the process $\mathrm{Z} \to \mathrm{W}^+\mathrm{S}^-\chi \to \mathrm{Z}$, for example, can be obtained by calculating the decay amplitude of the process $\mathrm{Z} \to \mathrm{W}^+\mathrm{S}^-\chi$. The decay amplitude for the direct transition from $\mathrm{Z}(p_1)$ to the state I_i, $i = 1, 2, 3, \ldots$, e.g. $\mathrm{Z}(p_1) \to \mathrm{I}_1$, $\mathrm{I}_1 = \mathrm{W}^+(p_2)\,\mathrm{S}^-(p_3)\,\chi(p_4)$, is denoted by $S(\mathrm{I}_i)$; while that for the two-step transition from $\mathrm{Z}(p_1)$ to the state I_i, $i = 1, 2, 3, \ldots$, e.g. $\mathrm{Z}(p_1) \to \mathrm{W}^+(p_2)\,\psi \to \mathrm{I}_1$, is denoted by $T(\mathrm{W}^+(p_2)\,\psi)$. In the calculations, the gauge parameter ξ is arbitrary and we set $\eta = \xi$ for simplicity. The results are as follows:

(a) $A_1(\mathrm{Z}(p_1) \to \mathrm{I}_1 \to \mathrm{Z}(p_1))\ , \qquad \mathrm{I}_1 = \mathrm{W}^+(p_2)\,\mathrm{S}^-(p_3)\,\chi(p_4)\ ,$

$$|a_1|^2 = |T(\mathrm{W}^+(p_2)\,\mathrm{W}^-_\mu) + T(\mathrm{S}^+\mathrm{S}^-(p_3)) + T(\psi\chi(p_4)) + \mathrm{S}(\mathrm{I}_1)|^2 \tag{A.5}$$

$$= +\,[a + b\cos^2\theta - f\cos^2\theta + d\cos(2\theta) - g\cos\theta\,(p_3\cdot p_4 + p_4\cdot p_4)]^2\ .$$

(b) $A_2(\mathrm{Z}(p_1) \to \mathrm{I}_2 \to \mathrm{Z}(p_1))\ , \qquad \mathrm{I}_2 = \mathrm{W}^+(p_2)\,\mathrm{S}^-(p_3)\,\mathrm{Z}_\mathrm{S}(p_4)\ ,$

$$|a_2|^2 = -\,|T(\mathrm{W}^+(p_2)\,\mathrm{W}^-_\mu) + T(\mathrm{W}^+(p_2)\,\mathrm{S}^-) + T(\mathrm{S}^+\mathrm{S}^-(p_3)) + \mathrm{T}(\psi\mathrm{Z}_\mathrm{S}(p_4))|^2$$

$$= -\,[a + b\cos^2\theta\,\sin^2\theta - d\sin^4\theta - g\cos\theta\,\sin^2\theta\,(p_3\cdot p_4 + p_4\cdot p_4)]^2\ . \tag{A.6}$$

(c) $A_3(\mathrm{Z}(p_1) \to \mathrm{I}_3 \to \mathrm{Z}(p_1))\ , \qquad \mathrm{I}_3 = \mathrm{W}^+(p_2)\,\mathrm{W}^-_\mathrm{S}(p_3)\,\mathrm{Z}_\mathrm{S}(p_4)\ ,$

$$|a_3|^2 = |\,T(\mathrm{W}^+(p_2)\,\mathrm{W}^-_\mu) + T(\mathrm{W}^+_\mu\mathrm{W}^-_\mathrm{S}(p_3)) + T(\mathrm{W}^+(p_2)\,\mathrm{S})$$

$$+\,T(\mathrm{S}^+\mathrm{W}^-_\mathrm{S}(p_3)) + T(\psi\mathrm{Z}_\mathrm{S}(p_4) + \mathrm{S}(\mathrm{I}_3)|^2$$

$$= +\,[a - b\cos^4\theta + f - d\,(\sin^4\theta + \cos^2\theta) + g\cos^3\theta\,(p_3\cdot p_4 + p_4\cdot p_4)]^2\ . \tag{A.7}$$

(d) $A_4(\mathrm{Z}(p_1) \to \mathrm{I}_4 \to \mathrm{Z}(p_1))$, $\qquad \mathrm{I}_4 = \mathrm{W}^+(p_2)\, \mathrm{W}_\mathrm{S}^-(p_3)\, \chi(p_4)$,

$$|a_4|^2 = -\,|T(\psi\chi(p_4)) + T(\mathrm{W}^+(p_2)\,\mathrm{S}^-) + T(\mathrm{S}^+\mathrm{W}_\mathrm{S}^-(p_3))|^2$$

$$= -\,[a + f\sin^2\theta - d\sin^2\theta]^2 \ . \tag{A.8}$$

(e) $A_5(\mathrm{Z}(p_1) \to \mathrm{I}_5 \to \mathrm{Z}(p_1))$, $\qquad \mathrm{I}_5 = \mathrm{W}^+(p_2)\,\mathrm{D}^-(p_3)\,\overline{\mathrm{D}}_\mathrm{Z}(p_4)$,

$$|a_5|^2 = -\,|T(\mathrm{W}^+(p_2)\,\mathrm{W}_\mu^-) + T(\mathrm{W}^+(p_2)\,\mathrm{S}^-) + T(\overline{\mathrm{D}}^-\mathrm{D}^-(p_3))|^2 \tag{A.9}$$

$$= -\,[-b\cos^3\theta + f\cos\theta - d\cos^3\theta + g\cos^2\theta\,(p_3\cdot p_4 + p_4\cdot p_4)]^2 \ .$$

(f) $A_6(\mathrm{Z}(p_1) \to \mathrm{I}_6 \to \mathrm{Z}(p_1))$, $\qquad \mathrm{I}_6 = \mathrm{W}^+(p_2)\,\overline{\mathrm{D}}^+(p_3)\,\mathrm{D}_\mathrm{Z}(p_4)$

$$|a_6|^2 = -\,|T(\mathrm{W}^+(p_2)\,\mathrm{W}_\mu^-) + T(\mathrm{W}^+(p_2)\,\mathrm{S}^-) + T(\mathrm{D}^+\overline{\mathrm{D}}^+(p_3))|^2 \tag{A.10}$$

$$= -\,[-b\cos^3\theta + f\cos\theta - d\cos^2\theta + g\cos^2\theta\,(p_3\cdot p_4 + p_4\cdot p_4)]^2 \ .$$

$$a = -\frac{G^2\cos\theta\ Y_4 Z_3}{(p_1 - p_4)^2 - m_\psi^2} - \frac{G^2\cos\theta\,\sin^2\theta\ \epsilon_1\cdot\epsilon_2}{2} \ ,$$

$$b = \frac{G\cos\theta}{(-2p_1\cdot p_2 + M_\mathrm{Z}^2)}\,[-2Y_2Z_4 - 2Z_1Y_4 + \epsilon_1\cdot\epsilon_2(p_1\cdot p_4 + p_2\cdot p_4)] \ ,$$

$$\mathrm{d} = \frac{G^2\cos\theta\ Y_3Z_4}{(-2p_1\cdot p_3 + M_\mathrm{Z}^2)} \ , \qquad f = \frac{G^2\cos\theta\,\sin^2\theta M_\mathrm{Z}^2\epsilon_1\cdot\epsilon_2}{2\xi\,(2p_3\cdot p_4 + M_\mathrm{Z}^2/\eta)} \ ,$$

$$\mathrm{g} = \frac{G^2\cos^2\theta\,\sin^2\theta\,(1-\xi^{-1})\,M_\mathrm{Z}^2\epsilon_1\cdot\epsilon_2}{(-2p_1\cdot p_2 + M_\mathrm{Z}^2)\,(2p_4\cdot p_3 + M_\mathrm{Z}^2/\eta)} \ , \qquad Y_i = \epsilon_1\cdot p_i \ , \qquad Z_i = \epsilon_2\cdot p_i \ ,$$

where

$$\mathrm{Im}\,A_i = \int |a_i|^2\,\delta(p_1^2 - M^2)\,\delta(p_2^2 - M^2/\xi)\,\delta(p_3^2 - M_\mathrm{Z}^2/\eta)\,\delta^4(p_1 - p_2 - p_3 - p_4)$$

$$\times\,\theta(p_{20})\,\theta(p_{30})\,\theta(p_{40})\,\mathrm{d}^4p_2\mathrm{d}^4p_3\mathrm{d}^4p_4\,(2\pi)^{-5} \ , \qquad i = 1, 2, 3, \ldots \ ;$$

$\epsilon_1^\mu(p_1)$ and $\epsilon_2^\mu(p_2)$ are the polarization vectors for vector bosons $\mathrm{Z}(p_1)$ and $\mathrm{W}(p_2)$ respectively. The sum of the unphysical amplitudes (A.5)–(A.10) is zero for arbitrary ξ and $\eta = \xi$, as required by unitarity.

The physical amplitude A_2, which can be regarded as given by the effective Lagrangian $\mathcal{L}_W + \mathcal{L}_2(D, \bar{D})$ giving by (10) and (20), has also been verified to be unitary and gauge independent. Explicit calculations at both the 1-loop and the 2-loop (with arbitrary $\eta > 0$ and $\xi = 1$) levels are given in ref. [1]. We may remark that when η is arbitrary and $\xi = 1$ the net results (A.5)–(A.10) for the amplitude A_1 given by (15) become identical to the results given in ref. [1] for the amplitude A_2, as discussed in sect. 7. (There is a misprint in ref. [1], that is, the quantity A_f in eq. (16) should read

$$-[-b\cos^3\theta + (c/\eta)\cos\theta - d\cos^3\theta]\,[-b\cos^3\theta + c\cos\theta - d\cos^3\theta] \quad \text{(A.11)}$$

instead of its square with a minus sign.)

To illustrate the discussions in sect. 5 about functional determinant in gauge theories, let us consider the 4th order self-energy diagram of $W^+(p)$ with two separate loops. Unitarity requires that the contributions due to the unphysical intermediate states $(W_S^+\gamma)$, $(S^+\gamma)$, $(D^+\bar{D}_A)$ and $(\bar{D}^-D_A)$ must cancel among themselves. According to (15), with the usual definition of determinant

$$\det(1+\bar{M}) = \exp\left\{\sum_{n=1}^{\infty}(-1)^{n+1}\,\mathrm{Tr}\,(\bar{M}^n)/n\right\}, \quad \text{(A.12)}$$

we obtain the following amplitudes B_i due to the unphysical intermediate states:

$$B_1(W^+(p) \to (W_S^+\gamma)/(S^+\gamma) \to W_S^+\gamma)/(S^+\gamma) \to W^+), \quad \text{(A.13)}$$

$$B_2(W^+(p) \to (W_S^+\gamma)/(S^+\gamma) \to W^+ \to (D^+\bar{D}_A)/(\bar{D}^-D_A) \to W^+), \quad \text{(A.14)}$$

$$B_3(W^+(p) \to (D^+\bar{D}_A)/(\bar{D}^-D_A) \to W^+ \to (W_S^+\gamma)/(S^+\gamma) \to W^+), \quad \text{(A.15)}$$

where the notation $W^+ \to (W_S^+\gamma)/(S^+\gamma) \to W^+$, for example, denotes the sum of the two amplitudes $W^+ \to W_S^+\gamma \to W^+$ and W^+, given by (A.1) and (A.2), etc. With the help of (A.1)–(A.4), we see that (A.13) + (A.14) + (A.15) $\neq 0$ because the amplitudes (A.13) and (A.14) cancel each other while (A.15) remains. As discussed in sect. 5, the fictitious Lagrangian corresponding to $\det(1 + \mathcal{G}\gamma_\phi)$ leads to terms of the form $[G_\xi(x_1 - x_2)\,G_0(x_2 - x_1)]\,[G_\xi(x_3 - x_4)\,G_0(x_4 - x_3)]$ representing the 4th order amplitudes with two separate fictitious loops aside from those given by (A.14) and (A.15). This particular type of terms involves two separate traces which does not appear in the usual definition (A.12) for a functional determinant. They arise from the non-commutativity of the elements of the determinant. There are four such amplitudes and their sum is

$$B_4(W^+(p) \to (D^+\bar{D}_A)/(\bar{D}^-D_A) \to W^+ \to (D^+\bar{D}_A)/(\bar{D}^-D_A) \to W^+). \quad \text{(A.16)}$$

The amplitude (A.16) cancels (A.15), so that the sum (A.13) + (A.14) + (A.15) + (A.16) vanishes as required by unitarity. Therefore, the formal functional determinant in gauge theories should be defined and understood in terms of a fictitious Lagrangian, instead of the incorrect expansion (A.12).

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