Completely positive dynamical semigroups of *N*-level systems*

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We establish the general form of the generator of a completely positive dynamical semigroup of an N-level quantum system, and we apply the result to derive explicit inequalities among the physical parameters characterizing the Markovian evolution of a 2-level system.

I. INTRODUCTION

In this paper we establish the general form of the generator of a completely positive dynamical semigroup of an N-level quantum system (Sec. II) and we find the conditions, in the form of explicit inequalities, that complete positivity imposes on the physical parameters which characterize the Markovian evolution of a two-level system (Sec. III). The term *dynamical semigroup* was introduced by one of us to mean a continuous one parameter semigroup Λ: $t \to \Lambda_t$, $t \in \mathbb{R}^+$, of positive trace preserving linear maps Λ_t: $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$, where $\mathcal{T}(\mathcal{H})$ is the Banach space [under the trace norm $\|\sigma\|_1 = \mathrm{tr}(\sigma^*\sigma)^{1/2}$] of trace class operators on a complex separable Hilbert space $\mathcal{H}$.[1] Other terminologies which have been used in the literature are "quantum stochastic process"[2] and " (stationary) noncommutative Markov process."[3] Since Λ_t is a contraction,[4] it follows from the Hille-Yosida theorem[5] that there exists a linear operator L: $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ with dense domain of definition $D(L)$ such that

$$\lim_{t\downarrow 0} \|L\sigma - t^{-1}(\Lambda_t\sigma - \sigma)\|_1 = 0, \quad \sigma \in D(L).$$

L is called the *generator* of the semigroup. Therefore, if we regard $\mathcal{H}$ as the Hilbert space associated to some quantum system, we can interpret Λ_t as the integrated form of a Markovian master equation for the density operator representing the state of the system

$$\frac{d\rho}{dt} = L\rho, \quad \rho \in \mathcal{T}(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}(\rho) = 1. \tag{1.1}$$

Master equations of the form (1.1) are encountered in a wide variety of physical problems such as quantum optics, laser action, superradiance, oscillator damping, atomic and spin relaxation, decay of unstable systems, etc.[6–14] Generally speaking, an equation of the form (1.1) gives a correct description of the irreversible evolution of a quantum open system in contact with stationary surroundings, provided the decay time τ_R of the correlations of the "reservoir" is much shorter than the typical relaxation times τ_S of the system, so that memory effects can be neglected. If the latter condition is not met, one has in principle to solve for ρ a formally more complicated intergrodifferential equation with memory which is usually referred to as the generalized master equation (gme).[8,15–19] Recently, it has been shown by Davies that under suitable assumptions the gme does indeed go over into an equation of the form (1.1) with a rescaled time variable in the limit when the coupling of the system to its surroundings is made to tend to zero (weak-coupling limit, $\tau_S \to \infty$).[20] It is also possible to obtain (1.1) rigorously in the limit $\tau_R \to 0$. This has been called the limit of *singular reservoir*.[21] See our next paper for an explicit model thereof.[22]

In order to proceed further we need to recall the notion of completely positive map. Let $M(n)$ denote the C^* algebra of the $n \times n$ complex matrices and 1_n the identity map $M(n) \to M(n)$. A linear map α: $\mathcal{A} \to \mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ C^* algebras, is said to be *completely positive* if the tensor product map $\alpha^{(n)} = \alpha \otimes 1_n$: $\mathcal{A} \otimes M(n) \to \mathcal{B} \otimes M(n)$ is positive for all positive integers n (if $\alpha^{(p)}$ is positive for a given positive integer p, then α is called p positive).[23] For the theory of positive and completely positive maps of C^* algebras see Refs. 23–30. To show that complete positivity is actually a stronger condition than positivity, we give in Appendix A a general example of a positive map which is not two positive. Now let Λ be a dynamical semigroup and let $\mathcal{B}(\mathcal{H})$ denote the C^* algebra of bounded operators on $\mathcal{H}$. Let $\Lambda^*: t \to \Lambda_t^*$, $t \in \mathbb{R}^+$, be the positive, normal (i.e., ultraweakly continuous), and identity, preserving semigroup $\Lambda_t^*: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, dual to Λ, defined by

$$\mathrm{tr}[(\Lambda_t\sigma)A] = \mathrm{tr}[\sigma(\Lambda_t^*A)], \quad \sigma \in \mathcal{T}(\mathcal{H}), \quad A \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}^+ \tag{1.2}$$

(Λ^* provides the evolution in the Heisenberg picture). We say that Λ is a completely positive dynamical semigroup if the map Λ_t^*, $t \in \mathbb{R}^+$, is completely positive. One can argue that dynamical semigroups describing the evolution of physical systems should be completely positive. Indeed, assume we have a quantum system S coupled to a reservoir R. If we regard the total system $S+R$ as isolated, its dynamics will be given by a one-parameter group $U: t \to U_t$ of unitary transformations of $\mathcal{H}_S \otimes \mathcal{H}_R$, the tensor product of the Hilbert spaces associated to S and to R, respectively. Assume that $S+R$ has been initially prepared in a product state $\rho \otimes \sigma$, $\rho \in \mathcal{T}(\mathcal{H}_S)$, $\sigma \in \mathcal{T}(\mathcal{H}_R)$, in which S and R are uncorrelated. The Heisenberg reduced dynamics of S, $\Phi: t \to \Phi_t: \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$, $t \in \mathbb{R}^+$, is defined by

$$\mathrm{tr}\{(\rho \otimes \sigma)[U_t^*(A \otimes \mathbb{1})U_t]\} = \mathrm{tr}_S[\rho(\Phi_t A)], \tag{1.3}$$

where tr_S denotes the trace on $\mathcal{T}(\mathcal{H}_S)$. It is easy to see that Φ_t is completely positive. The proof can be found in a paper by Kraus,[25] who was the first, to our knowledge, to recognize the physical significance of complete positivity, in connection with state changes produced by quantum measurements. For the reader's convenience, we give in Appendix B a straightforward independent proof based on the definition. One can show by continuity that complete positivity will not be destroyed by any of the limiting procedures, such as weak-coupling or the singular-reservoir limit, which give rise from Φ_t to a dynamical semigroup. Other arguments justifying complete positivity have been given by Accardi[3] and by Lindblad,[31] which are based on the requirement of positivity, respectively, of quasiconditional expectations on the algebras of local (in time) observables and of the dynamics of the system $S+S'$, where S' is an auxiliary N-level system coupled trivially to the open system S. Both these arguments do not make reference to the dynamics of S being a subdynamics of a global unitary dynamics. We have received Lindblad's preprint after the completion of the first version of the present paper. In his work, using methods different from ours, the author gives the general form of the generator of a norm continuous completely positive dynamical semigroup. This result generalizes our theorem 2.2.

II. DYNAMICAL SEMIGROUPS OF N-LEVEL SYSTEMS

We now proceed to determine the structure of the generator of a completely positive dynamical semigroup of an N-level system. For such a system, we have the identifications $\mathcal{T}(\mathcal{H})=\mathcal{B}(\mathcal{H})=M(N)$ and if Λ is a completely positive dynamical semigroup thereof, it is clear that the map $\Lambda_t: M(N)\to M(N)$, $t\in\mathbb{R}^+$, is completely positive. We call Λ a completely positive dynamical semigroup of $M(N)$.

Let $\mathcal{P}_N$ denote the set of all complete families $\{P_1,P_2,\ldots,P_N\}$ of mutually orthogonal one-dimensional self-adjoint projections in $M(N)$: $P_iP_j=\delta_{ij}P_i$, $P_i=P_i^*$, $\sum_{i=1}^N P_i=1$. The following theorem is a special case of theorem 5 of Ref. 32:

Theorem 2.1. In order for a linear map $L: M(N)\to M(N)$ to be the generator of a dynamical semigroup of $M(N)$ it is necessary and sufficient that the conditions

$$\mathrm{tr}[P_r(LP_s)]\geq 0,\quad r\neq s=1,2,\ldots,N \tag{2.1}$$

and

$$\sum_{r=1}^{N}\mathrm{tr}[P_r(LP_s)]=0,\quad s=1,2,\ldots,N \tag{2.2}$$

hold for all $\{P_1,P_2,\ldots,P_N\}\in\mathcal{P}_N$. Condition (2.2) is necessary and sufficient for L to generate a trace preserving semigroup, whereas (2.1) expresses the positivity requirement.

Theorem 2.2. A linear operator $L: M(N)\to M(N)$ is the generator of a completely positive dynamical semigroup of $M(N)$ if it can be expressed in the form

$$L:\rho\to L\rho=-i[H,\rho]+\frac{1}{2}\sum_{i,j=1}^{N^2-1}c_{ij}\{[F_i,\rho F_j^*]+[F_i\rho,F_j^*]\},\quad \rho\in M(N), \tag{2.3}$$

where $H=H^*$, $\mathrm{tr}(H)=0$, $\mathrm{tr}(F_i)=0$, $\mathrm{tr}(F_i^*F_j)=\delta_{ij}$, $(i,j=1,2,\ldots,N^2-1)$, and $\{c_{ij}\}$ is a complex positive matrix. For a given L, H is uniquely determined by the condition $\mathrm{tr}(H)=0$ and $\{c_{ij}\}$ is uniquely determined by the choice of the F_i's.

Remark. We may call $-i[H,\cdot]$ the "Hamiltonian" part of the generator and $L+i[H,\cdot]$ its dissipative part. In general, H is not the same as the Hamiltonian H_0 of the free N-level system.[20,22] The proof of theorem 2.2 is based on some lemmas.

Lemma 2.1 $t\to\Lambda_t$ is a completely positive dynamical semigroup of $M(N)$ iff $t\to\Lambda_t\otimes 1_N$ is a dynamical semigroup of $M(N)\otimes M(N)$.

Proof. From theorem 5 of Ref. 28, a linear map $\Gamma: M(N)\to M(N)$ is completely positive iff $\Gamma\otimes 1_N$ is positive. Expressing an element

$$\hat{A}\in M(N)\otimes M(N)\ \text{ as }\ \hat{A}=\sum_{i,j=1}^{N}A_{ij}\otimes E_{ij},\ A_{ij}\in M(N),$$

$(E_{ij})_{rs}=\delta_{ir}\delta_{js}$ and denoting by Tr the trace on $M(N)\otimes M(N)$ we have

$$\mathrm{Tr}[(\Gamma\otimes 1_N)\hat{A}]=\sum_{i,j=1}^{N}\mathrm{Tr}(\Gamma A_{ij}\otimes E_{ij})=\sum_{i,j=1}^{N}\mathrm{tr}(\Gamma A_{ij})\,\mathrm{tr}(E_{ij})=\sum_{i=1}^{N}\mathrm{tr}(\Gamma A_{ii}).$$

Hence Γ is trace preserving iff $\Gamma\otimes 1_N$ is trace preserving. QED

Lemma 2.2. Let Γ be a linear operator $M(N)\to M(N)$ and let $\{F_\alpha\}_{\alpha=1,2,\ldots,N^2}$ be a complete orthonormal set (c.o.s) in $M(N)$, viz., $(F_\alpha,F_\beta)=\mathrm{tr}(F_\alpha^*F_\beta)=\delta_{\alpha\beta}$. Then Γ can be uniquely written in the form

$$\Gamma: A\to\Gamma A=\sum_{\alpha,\beta=1}^{N^2}c_{\alpha\beta}F_\alpha AF_\beta^*,\quad A\in M(N). \tag{2.4}$$

Moreover, if $\Gamma A^*=(\Gamma A)^*$, then $c_{\alpha\beta}=\langle c_{\beta\alpha}\rangle$.

Proof. First note that

$$\sum_{\alpha=1}^{N^2}F_\alpha^*AF_\alpha=\mathbb{1}\,\mathrm{tr}(A),\ \forall A\in M(N). \tag{2.5}$$

Indeed, the left-hand side of (2.5) is invariant under a change of c.o.s. $F_\alpha\to E_\alpha$ and choosing $\{E_\alpha\}=\{E_{ij}\}$, we have

$$\sum_{i,j=1}^{N}E_{ij}^*AE_{ij}=\sum_{i,j=1}^{N}E_{ji}AE_{ij}=\left(\sum_{j=1}^{N}E_{jj}\right)\left(\sum_{i=1}^{N}A_{ii}\right)=\mathbb{1}\,\mathrm{tr}(A).$$

Now let $\mathcal{L}(M(N))$ denote the vector space of linear operators $M(N)\to M(N)$ and let $\{G_\alpha\}$ be a c.o.s. in $M(N)$. $\mathcal{L}(M(N))$ becomes a unitary space with the inner product

$$\langle\Gamma,\Phi\rangle=\sum_{\alpha=1}^{N^2}(\Gamma G_\alpha,\Phi G_\alpha)=\sum_{\alpha=1}^{N^2}\mathrm{tr}[(\Gamma G_\alpha)^*(\Phi G_\alpha)].$$

Define

$$\Gamma_{\alpha\beta}: A\to\Gamma_{\alpha\beta}A=F_\alpha AF_\beta^*\quad(\alpha,\beta=1,2,\ldots,N^2). \tag{2.6}$$

Then $\{\Gamma_{\alpha\beta}\}$ is a c.o.s. in $\mathcal{L}(M(N))$. Indeed, using (2.5) we have

$$\langle\Gamma_{\alpha\beta},\Gamma_{\mu\nu}\rangle=\sum_{\lambda=1}^{N^2}\operatorname{tr}[(\Gamma_{\alpha\beta}G_\lambda)^*(\Gamma_{\mu\nu}G_\lambda)]$$
$$=\sum_{\lambda=1}^{N^2}\operatorname{tr}[(F_\alpha G_\lambda F_\beta^*)^*(F_\mu G_\lambda F_\nu^*)]$$
$$=\operatorname{tr}\left[F_\beta\left(\sum_{\lambda=1}^{N^2}G_\lambda^*F_\alpha^*F_\mu G_\lambda\right)F_\nu^*\right]$$
$$=\operatorname{tr}(F_\alpha^*F_\mu)\operatorname{tr}(F_\nu^*F_\beta)=\delta_{\alpha\beta}\delta_{\beta\nu}.$$

The last assertion of the lemma is now easily verified.

QED

Lemma 2.3. Let $\{F_\alpha\}_{\alpha=1,2,\ldots,N^2}$ be a c.o.s. in $M(N)$ such that $F_{N^2}=(1/N)^{1/2}\,\mathbb{1}$ and let L be a linear operator $M(N)\to M(N)$ such that $(LA)^*=LA^*$ and $\operatorname{tr}(LA)=0$ for all $A\in M(N)$. Then L can be uniquely written in the form

$$L:A\to LA=-i[H,A]+\frac{1}{2}\sum_{i,j=1}^{N^2-1}c_{ij}\{[F_i,AF_j^*]+[F_iA,F_j^*]\},\quad(2.7)$$

where $H=H^*$, $\operatorname{tr}(H)=0$, and $c_{ij}=\langle c_{ji}\rangle_{\text{av}}$.

Proof. From (2.4) we have

$$LA=\frac{1}{N}c_{N^2N^2}A+\left(\frac{1}{N}\right)^{1/2}\sum_{i=1}^{N^2-1}(c_{iN^2}F_iA+c_{N^2i}AF_i^*)$$
$$+\sum_{i,j=1}^{N^2-1}c_{ij}F_iAF_j^*=-i[H,A]+\{G,A\}$$
$$+\sum_{i,j=1}^{N^2-1}c_{ij}F_iAF_j^*,\quad(2.8)$$

where $H=(1/2i)(F^*-F)$ and $G=(1/2N)C_{N^2N^2}\mathbb{1}+(1/2)(F^*+F)$, with $F=(1/N)^{1/2}\sum_{i=1}^{N^2-1}c_{iN^2}F_i$. Now

$$0=\operatorname{tr}(LA)=\operatorname{tr}\left[\left(2G+\sum_{i,j=1}^{N^2-1}c_{ij}F_j^*F_i\right)A\right],\ \forall\,A\in M(N)$$

implies $G=-\frac{1}{2}\sum_{i,j=1}^{N^2-1}c_{ij}F_j^*F_i$, whence (2.7) follows. The uniqueness follows from dimensionality considerations, since $\operatorname{tr}(LA)=0$, $\forall\,A\in M(N)$ implies N^2-independent conditions on L. QED

Lemma 2.4. Let $\{F_\alpha\}_{\alpha=1,2,\ldots,N^2}$ be a c.o.s. in $M(N)$. Then

$$\left\{\underset{(\alpha)}{\hat P}\ \Big|\ \underset{(\alpha)}{\hat P}=\sum_{i,j=1}^{N}\underset{(\alpha)}{P}_{ij}\otimes E_{ij};\ \underset{(\alpha)}{P}_{ij}=F_\alpha E_{ij}F_\alpha^*;\ \ \alpha=1,2,\ldots,N^2\right\}$$

is a complete family of mutually orthogonal self-adjoint projections in $M(N)\otimes M(N)$.

Proof. An element $\hat P=\sum_{i,j=1}^{N^2}P_{ij}\otimes E_{ij}$ of $M(N)\otimes M(N)$ is a self-adjoint projection iff

$$P_{ij}^*=P_{ji}\ \text{ and }\ \sum_{l=1}^{N}P_{il}P_{lj}=P_{ij}\quad(i,j=1,2,\ldots,N).\ (2.9)$$

Two such projections $\hat P$ and $\hat Q$ are orthogonal iff

$$\sum_{l=1}^{N}P_{il}Q_{lj}=0\quad(i,j=1,2,\ldots,N).\quad(2.10)$$

We have

$$\underset{(\alpha)}{P}_{ij}^*=(F_\alpha E_{ij}F_\alpha^*)^*=F_\alpha E_{ij}^*F_\alpha^*=\underset{(\alpha)}{P}_{ji}$$

and

$$\sum_{l=1}^{N}\underset{(\alpha)}{P}_{il}\underset{(\beta)}{P}_{lj}=\sum_{l=1}^{N}F_\alpha E_{il}F_\alpha^*F_\beta E_{lj}F_\beta^*$$
$$=F_\alpha E_{ij}F_\beta^*\operatorname{tr}(F_\alpha^*F_\beta)=\delta_{\alpha\beta}\underset{(\alpha)}{P}_{ij}.\qquad\text{QED}$$

Proof of theorem 2.2. The "if" part: If $t\to\Lambda_t$ is the semigroup generated by (2.3), the generator of the semigroup $t\to\Lambda_t\otimes 1_N$ is $L\otimes 1_N$. By Lemma 2.1 we must show that $\{c_{pq}\}\geq 1$ implies $L\otimes 1_N$ to satisfy the conditions of Theorem 2.1. Since $\operatorname{tr}(L\rho)=0$ for all $\rho\in M(N)$, we need only check that

$$\operatorname{Tr}\left\{\underset{(1)}{\hat P}\left[(L\otimes 1_N)\,\underset{(2)}{\hat P}\right]\right\}\geq 0$$

for all pairs $\underset{(1)}{\hat P}$, $\underset{(2)}{\hat P}$ of mutually orthogonal self-adjoint projections in $M(N)\otimes M(N)$. And indeed, using (2.9) and (2.10), we get

$$\operatorname{Tr}\left\{\underset{(1)}{\hat P}\left[(L\otimes 1_N)\,\underset{(2)}{\hat P}\right]\right\}=\sum_{i,j=1}^{N}\operatorname{tr}\left[\underset{(1)}{P}_{ij}\left(L\,\underset{(2)}{P}_{ji}\right)\right]$$
$$=-i\sum_{i,j=1}^{N}\operatorname{tr}\left(\underset{(1)}{P}_{ij}\left[H,\underset{(2)}{P}_{ji}\right]\right)$$
$$+\sum_{p,q=1}^{N^2-1}c_{pq}\sum_{i,j=1}^{N}\left[\operatorname{tr}\left(\underset{(1)}{P}_{ij}F_p\underset{(2)}{P}_{ji}F_q^*\right)\right.$$
$$\left.-\tfrac{1}{2}\operatorname{tr}\left(\underset{(1)}{P}_{ij}F_q^*F_p\underset{(2)}{P}_{ji}+\underset{(1)}{P}_{ij}\underset{(2)}{P}_{ji}F_q^*F_p\right)\right]$$
$$=\sum_{p,q=1}^{N^2-1}c_{pq}\sum_{i,j=1}^{N}\operatorname{tr}\left(\underset{(1)}{P}_{ij}F_p\underset{(2)}{P}_{ji}F_q^*\right)$$
$$=\sum_{p,q=1}^{N^2-1}c_{pq}\sum_{i,j,k,l=1}^{N}\operatorname{tr}\left(\underset{(1)}{P}_{ik}\underset{(1)}{P}_{kj}F_p\underset{(2)}{P}_{jl}\underset{(2)}{P}_{li}F_q^*\right)$$
$$=\sum_{k,l=1}^{N}\sum_{p,q=1}^{N^2-1}c_{pq}\operatorname{tr}\left[\left(\sum_{j=1}^{N}\underset{(1)}{P}_{kj}F_p\underset{(2)}{P}_{jl}\right)\times\left(\sum_{j=1}^{N}\underset{(1)}{P}_{kj}F_q\underset{(2)}{P}_{jl}\right)^*\right]\geq 0,$$

since $\{c_{pq}\}\geq 0$.

The "only if" part: If a linear operator $L:M(N)\to M(N)$ generates a completely positive dynamical semigroup of $M(N)$ we have $\operatorname{tr}(LA)=0$ and $(LA)^*=LA^*$ for all $A\in M(N)$. Hence, by Lemma 2.3, L can be written in the form (2.3) with $H=H^*$, $\operatorname{tr}(H)=0$, and $c_{ij}=\langle c_{ji}\rangle_{\text{av}}$. Since the matrix $\{c_{ij}\}$ is self-adjoint, we can choose another orthonormal set of traceless matrices $\{G_1,G_2,\ldots,G_{N^2-1}\}$ such that

$$L\rho=-i[H,\rho]+\frac{1}{2}\sum_{p=1}^{N^2-1}\lambda_p\{[G_p,\rho G_p^*]+[G_p\rho,G_p^*]\},\quad\rho\in M(N).$$

Define

$$\underset{(q)}{\hat P}=\sum_{i,j=1}^{N}(G_qE_{ij}G_q^*)\otimes E_{ij},\ q=1,2,\ldots,N^2-1\ \text{ and}$$
$$\hat P=\frac{1}{N}\sum_{i,j=1}^{N}E_{ij}\otimes E_{ij}.$$

Then, by Lemma 2.4, Theorem 2.1, and Lemma 2.1 we have

$$0\leq N\operatorname{Tr}\left\{\underset{(q)}{\hat P}[(L\otimes 1_N)\hat P]\right\}$$

$$= \sum_{p=1}^{N^2-1} \lambda_p \sum_{i,j=1}^{N} \operatorname{tr}(G_q E_{ij} G_q^* G_p E_{ji} G_p^*)$$

$$= \sum_{p=1}^{N^2-1} \lambda_p \operatorname{tr}(G_q^* G_p) \operatorname{tr}(G_q G_p^*) = \lambda_q, \quad q = 1, 2, \ldots, N^2 - 1.$$

The uniqueness of H and of $\{c_{ij}\}$ follows from Lemma 2.3. QED

III. TWO-LEVEL SYSTEM

In Ref. 33, Theorem 2.1 was applied to give the following characterization of the generator of a dynamical semigroup of $M(2)$.

Theorem 3.1. A linear operator $L: M(2) \to M(2)$ is the generator of a dynamical semigroup $t \to \Lambda_t$ of $M(2)$ iff it can be written in the form

$$L: \rho \to L\rho = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{3} c_{ij} \{[F_i, \rho F_j] + [F_i \rho, F_j]\}, \quad \rho \in M(2), \tag{3.1}$$

where (i) $H = \sum_{i=1}^{3} h_i F_i, \; h_i \in \mathbb{R}$;

(ii) $F_i = F_i^*$ and

$$F_i F_j = \tfrac{1}{4} \delta_{ij} \mathbb{1} + \frac{i}{2} \sum_{k=1}^{3} \epsilon_{ijk} F_k (\Rightarrow \operatorname{tr}(F_i F_j) = \tfrac{1}{2} \delta_{ij}, \; \operatorname{tr}(F_i) = 0);$$

(iii) $\{c_{ij}\} = \begin{pmatrix} \gamma - 2\gamma_1 & -ia_3 & i\,a_2 \\ ia_3 & \gamma - 2\gamma_2 & -ia_1 \\ -ia_2 & ia_1 & \gamma - 2\gamma_3 \end{pmatrix}, \quad \gamma = \gamma_1 + \gamma_2 + \gamma_3$;

(iv) $\gamma_1, \gamma_2, \gamma_3 \geq 0$;

(v) $a_i = \gamma_i m_i^0 + \sum_{j,k=1}^{3} \epsilon_{ijk} m_j^0 h_k$;

(vi) $m_i^0 = 0$ if $\gamma_1\gamma_2\gamma_3 = 0$;

(vii) $(m_1^0, m_2^0, m_3^0) \in S = \left\{ (z_1, z_2, z_3) \,\middle|\, z_1, z_2, z_3 \in \mathbb{R}; \inf_{x_1^2+x_2^2+x_3^2=1} \left[\sum_{i=1}^{3} \left(\gamma_i x_i (x_i - z_i) + \sum_{j,k=1}^{3} \epsilon_{ijk} x_i h_j z_k \right) \right] \geq 0; \; x_1, x_2, x_3 \in \mathbb{R} \right\}$ if $\gamma_1\gamma_2\gamma_3 > 0$.

Let $\rho_t = \Lambda_t \rho_0$ be the density matrix describing the system at time $t \geq 0$ and define the polarization components $M_i(t) = \operatorname{tr}(\rho_t F_i)$, $i = 1, 2, 3$. One easily verifies that the latter satisfy the following equations of motion (Bloch equations[34]):

$$\frac{dM_i(t)}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} h_j (M_k(t) - M_k^0) - \gamma_i (M_i(t) - M_i^0), \quad i = 1, 2, 3, \tag{3.2}$$

where $M_i^0 = \frac{1}{2} m_i^0$ $(i = 1, 2, 3)$. $\mathbf{M}^0$ is a stationary state and it is the only stationary state iff $\gamma_1\gamma_2\gamma_3 > 0$ (in the latter case every state approaches $\mathbf{M}^0$ as $t \to \infty$).

If, for instance, we think of $M(2)$ as the algebra of observables of a spin-$\frac{1}{2}$ magnetic moment, we can interpret Eqs. (3.2) as describing spin relaxation in a molecular surrounding under the action of an external magnetic field $\mathbf{H} = (1/\hbar g)\mathbf{h}$, g being the gyromagnetic ratio. γ_1, γ_2, and γ_3 are damping factors which are directly related to the relaxation times of the polarization components towards their equilibrium values; they are in fact inverse relaxation times $\gamma_i = 1/T_i$, if L commutes with its Hamiltonian part $-i[H, \cdot]$.

If $\gamma = 0$, we have $L = -i[H, \cdot]$. This corresponds to a purely Hamiltonian evolution which is of course completely positive. Let $\gamma > 0$ and define $\kappa_i = \gamma - 2\gamma_i$. Then, it follows from Theorem 2.2 that in order for the evolution to be completely positive it is necessary and sufficient that

$$\begin{aligned} &\text{(a) } \kappa_1 + \kappa_2 + \kappa_3 \geq 0, \\ &\text{(b) } \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 \geq a_1^2 + a_2^2 + a_3^2, \\ &\text{(c) } \kappa_1\kappa_2\kappa_3 \geq \sum_{i=1}^{3} \kappa_i a_i^2. \end{aligned} \tag{3.3}$$

Conditions (3.3) are equivalent to the following:

$$\begin{aligned} &\text{(a) } \kappa_1, \kappa_2, \kappa_3 \geq 0, \\ &\text{(b) } a_1 = (\kappa_2\kappa_3)^{1/2} y_1, \\ &\text{(c) } a_2 = (\kappa_3\kappa_1)^{1/2} y_2, \\ &\text{(d) } a_3 = (\kappa_1\kappa_2)^{1/2} y_3, \\ &\text{(e) } y_1^2 + y_2^2 + y_3^2 \leq 1. \end{aligned} \tag{3.4}$$

In terms of the γ_i's, (3.4) (a) can be written

$$\gamma_1 + \gamma_2 \geq \gamma_3, \quad \gamma_2 + \gamma_3 \geq \gamma_1, \quad \gamma_3 + \gamma_1 \geq \gamma_2, \tag{3.5}$$

showing that no two relaxation times can be much longer than the third.

In particular, we see that no two γ_i's can be zero without the third being zero too. Hence a non-Hamiltonian completely positive evolution admits for at most a one dimensional manifold of equilibrium states. This is the case when one of the γ_i's, say γ_1, is zero. Then $\gamma_2 = \gamma_3$ and there is essentially only one relaxatiom time.

As a special example, we consider the case $\gamma_1 = \gamma_2 = \gamma_\perp > 0$, $\gamma_3 = \gamma_\parallel > 0$, and $h_1 = h_2 = 0$. Then we have $\kappa_1 = \kappa_2 = \gamma_\parallel$, $\kappa_3 = 2\gamma_\perp - \gamma_\parallel$ and conditions (3.4) become

$$\text{(a) } 2\gamma_\perp \geq \gamma_\parallel, \tag{3.6}$$

(b) $a_i = [\gamma_\parallel(2\gamma_\perp - \gamma_\parallel)]^{1/2} y_i, \quad i = 1, 2, \quad a_3 = \gamma_\parallel y_3,$

$y_1^2 + y_2^2 + y_3^2 \leq 1.$

For the equilibrium state we get

$$\begin{aligned} M_1^0 &= [\gamma_\parallel(2\gamma_\perp - \gamma_\parallel)]^{1/2} \left(\frac{\gamma_\perp y_1 - h_3 y_2}{2(\gamma_\perp^2 + h_3^2)} \right), \\ M_2^0 &= [\gamma_\parallel(2\gamma_\perp - \gamma_\parallel)]^{1/2} \left(\frac{\gamma_\perp y_2 + h_3 y_1}{2(\gamma_\perp^2 + h_3^2)} \right), \quad M_3^0 = y_3. \end{aligned} \tag{3.7}$$

If the system is rotationally symmetric about the direction of the magnetic field, we have $M_1^0 = M_2^0 = 0$. In this case $\gamma_\perp$ and $\gamma_\parallel$ are, respectively, the inverse transverse and the inverse longitudinal relaxation times and (3.6) (a) is written

$$T_\parallel \geq \tfrac{1}{2} T_\perp, \tag{3.8}$$

a relation which had been previously derived by Favre and Martin for a spin system weakly coupled to a high-temperature bath.[18] To our knowledge, relation (3.8) is experimentally satisfied in all known cases.

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APPENDIX A

The following proposition provides a fairly general example of a positive map which is not two positive.

Proposition. Let $\mathcal{A}$ be a non commutative C^* algebra which identity and let β be a $*$ antiautomorphism of $\mathcal{A}$. Then β is not two positive.

Proof. Let $\hat{A}$ be a self-adjoint element of $\mathcal{A}\otimes M(2)$. It has the form $\hat{A}=\sum_{i,j=1}^{2}A_{ij}\otimes E_{ij}$, where $(E_{ij})_{rl}=\delta_{ir}\delta_{jl}$ and $A_{ij}^*=A_{ji}$, and we have

$$\beta^{(2)}(\hat{A}^2)-[\beta^{(2)}(\hat{A})]^2=\beta[A_{12},A_{12}^*]\otimes E_{11}+\beta[A_{11}-A_{22},A_{12}]$$
$$\otimes E_{12}-\beta[A_{11}-A_{22},A_{12}^*]\otimes E_{21}-\beta[A_{12},A_{12}^*]\otimes E_{22}.$$

Assume $\beta^{(2)}$ is positive. Then, since $\beta^{(2)}$ is self-adjoint and identity preserving, we have $\|\beta^{(1)}\|=1$.[35] Therefore, $\beta^{(2)}$ satisfies Kadison's inequality[36] $\beta^{(2)}(\hat{A}^2)-[\beta^{(2)}(\hat{A})]^2 \geqslant 0$. By the above, this implies $\beta[A_{12},A_{12}^*]=0$, which, by the arbitrariness of A_{12}, contradicts the noncommutativeness of $\mathcal{A}$. QED

APPENDIX B

To simplify notations we drop the subscript t from Φ_t and U_t and write $\rho(A)$ in place of $\mathrm{tr}(\rho A)$. An element $\hat{A}\in\mathcal{B}(\mathcal{H}_S)\otimes M(n)$ admits of a unique decomposition $\hat{A}=\sum_{i,l=1}^{n}A_{il}\otimes E_{il}$, $A_{il}\in\mathcal{B}(\mathcal{H}_S)$, $(E_{il})_{rs}=\delta_{ir}\delta_{ls}$, and if $\hat{A}$ is positive we have

$$\hat{A}=\hat{B}^*\hat{B}=\sum_{i,j=1}^{n}\left(\sum_{l=1}^{n}B_{li}^*B_{lj}\right)\otimes E_{ij}.$$

A density operator on $\mathcal{B}(\mathcal{H}_S)\otimes M(n)$ can be written as a finite convex combination of states of the form $\rho\otimes\omega$, where ρ is a density operator on $\mathcal{B}(\mathcal{H}_S)$ and ω is a pure state on $M(n)$, viz., $\omega(E_{ij})=\bar{x}_i x_j$. Set $Q_{ki}=U^*(B_{ki}\otimes 1)U$ and $Q_s=\sum_{r=1}^{n}x_rQ_{sr}$. Then

$$(\rho\otimes\omega)[\Phi^{(n)}(\hat{B}^*\hat{B})]=(\rho\otimes\omega)\left[\sum_{i,j}\left(\sum_k\Phi(B_{ki}^*B_{kj})\otimes E_{ij}\right)\right]$$
$$=\sum_{i,j,k}\rho[\Phi(B_{ki}^*B_{kj})]\bar{x}_i x_j$$
$$=\sum_{i,j,k}(\rho\otimes\sigma)[U^*(B_{ki}^*B_{kj}\otimes \mathbb{1})U]\bar{x}_i x_j$$
$$=\sum_{i,j,k}\bar{x}_i(\rho\otimes\sigma)(Q_{ki}^*Q_{kj})x_j$$
$$=(\rho\otimes\sigma)\left(\sum_s Q_s^*Q_s\right)\geqslant 0. \qquad \text{QED}$$

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[1]A. Kossakowski, Rep. Math. Phys. **3**, 247 (1972).
[2]E.B. Davies, Commun. Math. Phys. **15**, 277 (1969); **19**, 83 (1970); **22**, 51 (1971).
[3]L. Accardi, "Non-Relativistic Quantum Mechanics as a Non-Commutative Markov Process," preprint Lab. di Cibernetica del C.N.R., Arco Felice (Naples), 1975.
[4]This property follows from Theorem 3.3 of Ref. 26 applied to the dual map Λ_t^* defined by formula (1.2).
[5]K. Yosida, *Functional Analysis* (Springer-Verlag, Berlin, 1972), 3rd ed.
[6]G.S. Agarwal, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1973), Vol. XI, pp. 1–75.
[7]H. Haken, *Laser Theory, Handbook of Physis*, 25/2c (Springer-Verlag, Berlin, 1970).
[8]F. Haake, in *Springer Tracts in Modern Physics*, Vol. 66, (Springer-Verlag, Berlin, 1973), pp. 98–168.
[9]W. Happer, Rev. Mod. Phys. **44**, 169 (1972).
[10]F. Bloch and R.K. Wangsness, Phys. Rev. **89**, 728 (1953).
[11]A. Abragam, *The Principles of Nuclear Magnetism* (Oxford U.P., London, 1961).
[12]N.M. Atherton, *Electron Spin Resonance* (Wiley, New York, 1973).
[13]S. Twareque Ali, L. Fonda, and G.C. Ghirardi, Nuovo Cimento A **25**, 134 (1974).
[14]A. Barchielli and L. Lanz, "Banach Space Representations of the Galilei Semigroup: Decay of an Unstable Particle," preprint, University of Milan, 1975.
[15]R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960); and in *Lectures in Theoretical Physics* (*Boulder*), edited by W.E. Brittin, B.W. Downs, and J. Downs (Interscience, New York, 1961), Vol. 3, p. 106.
[16]P.N. Argyres and P.L. Kelley, Phys. Rev. **134**, A98 (1964).
[17]G.G. Emch and G.L. Sewell, J. Math. Phys. **9**, 946 (1968).
[18]C. Favre and Ph. Martin, Helv. Phys. Acta **41**, 333 (1968).
[19]E. Presutti, E. Scacciatelli, G.L. Sewell, and F. Wanderlingh, J. Math. Phys. **13**, 1085 (1972).
[20]E.B. Davies, Commun. Math. Phys. **39**, 91 (1974). See also J.V. Pulé, Commun. Math. Phys. 38, 241 (1974), for a model which does not start directly from a gme.
[21]K. Hepp and E.H. Lieb, Helv. Phys. Acta **46**, 573 (1973).
[22]V. Gorini and A. Kossakowski, J. Math. Phys. (to be published). See also A. Frigerio and V. Gorini, J. Math. Phys. (to be published), and E.B. Davies, Z. Wahrscheinlichkeitstheorie Verw. Geb. 23, 261 (1972).
[23]W.F. Stingespring, Proc. Am. Math. Soc. **6**, 211 (1958).
[24]E. Størmer, Acta Math. **110**, 223 (1963).
[25]K. Kraus, Ann. Phys. (N.Y.) **64**, 311 (1971).
[26]E. Størmer, in *Lecture Notes in Physics*, edited by A. Hartkämper and H. Neumann (Springer-Verlag, Berlin, 1974), Vol. 29, pp. 85–106.
[27]W. Arveson, Acta Math. **123**, 141 (1969).
[28]M.D. Choi, Can. J. Math. **24**, 520 (1972).
[29]M.D. Choi, Linear Algebra Appl. (to be published).
[30]M.D. Choi, Ill. J. Math. 18, 565 (1974).
[31]G. Lindblad, "On the Generators of Quantum Dynamical Semigroups," preprint Royal Institute of Technology, Stockholm, TRITA-TFY-75-1, 1975.
[32]A. Kossakowski, Bull. Acad. Pol. Sci. Sér. Math. Astr. Phys. **20**, 1021 (1972).
[33]A. Kossakowski, Bull. Acad. Pol. Sci. Sér. Math. Astr. Phys. **21**, 649 (1973).
[34]F. Bloch, Phys. Rev. **70**, 460 (1946).
[35]B. Russo and H.A. Dye, Duke Math. J. **33**, 413 (1966).
[36]R.V. Kadison, Ann. Math. **56**, 494 (1952).