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Interaction between classical and quantum systems and the measurement of quantum observables

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Abstract. Quantum mechanics presumes classical measuring instruments with which they interact. The problem of measurement interaction between classical and quantum systems is posed and solved. The restriction to compatible measurements comes about naturally as the condition for the integrity of the classical system. A technical device is the perspective on classical mechanics as quantum mechanics with essentially hidden dynamical variables.

Keywords. Measurement theory; quantum observables; quantum theory; classical dynamics.

1. Introduction

The need for interaction between classical and quantum systems

Classical mechanics is the crystallization of our everyday experiences of matter and motion. During this century, we have found, however, that to deal with matter in the minute and matter in the subtle we must use quantum mechanics (Jammer 1966). Quantum mechanics has many points of similarity with classical mechanics and these aid us in developing quantum mechanics; but there are also many essential points of difference. The most important of these points of difference is that not all dynamical variables can be measured at the same time. The dynamical variables constitute a noncommuting algebra from which a commuting subalgebra is selected by any possible measurement. Such a state of affairs is beyond our everyday experience, though it may not be totally alien, in that, poetic experience, dream experience and extraordinary states of awareness share kinship with the structure of quantum mechanics.

Measurement in quantum mechanics is the physical process by which " pointer readings " are obtained which correspond to numerical values of a commuting subalgebra of dynamical variables. The remarkable feature of quantum-mechanical measurements is that not all dynamical variables can be measured simultaneously even in principle. Yet the measurement of a maximal commuting subalgebra of dynamical variables would yield, in the case of pure states, a complete specification of the state. Even a pure state can yield a dispersion in the measurement of one or more dynamical variables. So the measurement process should be such as to produce classical pointer readings on the one hand; and lead to unambiguous measurements for a " compatible " set of measurements, a measurement of a commuting set of dynamical variables (Bohr 1963; Dirac 1958) on the other hand.

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Quantum mechanics as a physical theory, then, must presuppose classical systems which can be influenced by the quantum system. It must therefore, require the *coupling* of classical and quantum systems. [The classical measuring instrument must be a *classical system* with a low dynamic inertia *which undergoes a catastrophe* so that the pointer readings can be *recorded.*] It is, however, known that the general structure of classical and quantum dynamics are different (Moyal 1949). It is customary to avoid the problem of coupling of classical and quantum systems and deal with models of the measurement process using quantum systems which are treated semiclassically (d'Espagnat 1971).

In this paper I shall proceed in a different manner. I introduce a direct method of dealing with the interaction of classical and quantum systems. It is made possible by the discovery that a classical system can be embedded in a quantum system with a continuum of superselection sectors. If the classical system is to preserve its integrity, the couplings to the quantum system must be suitably restricted. The notion of *compatible measurements* emerges as *consequence of* this *principle of integrity* of the classical system. As far as I am able to tell, the theory developed in this paper is consistent with the traditional ideas of measurement theory and provides the solution to the long-standing problem of providing a dynamical framework for quantum measurement theory.

2. Classical systems as quantum systems with superselection

Quantum mechanical states are vectors (or, rather, rays) in a linear vector space and can be *superposed* (Dirac 1958). The result of superposition is a pure state, a *coherent* weighted combination of the two states; it is to be contrasted with a mixture which is an *incoherent* weighted combination of the two states. In classical mechanics the pure states are those corresponding to precise values for all dynamical variables. As such, we cannot but have incoherent additions between states; there are no coherent combinations of two pure states which can be identified as a pure state.

There is one situation in which coherent combinations between two pure states of a quantum system are not identified: this arises in the case of a quantum system with "superselection rules" (Wick *et al.*, 1952; *see* also Jordan 1969). If we have subsets of states which are such that no dynamical variable which connects these two subsets can be measured, then the relative phase of any two states belonging to these two subsets becomes irrelevant. The two subsets of states are now labelled superselection sectors. The nonexistence of matrix elements between superselection sectors implies that any dynamical variable which has a constant value within a superselection sector, but different values in different superselection sectors, obeys an inviolable selection rule—a "superselection rule". It is believed that the electric charge, baryon number and oddness of fermions generate superselection rules.

I find it more convenient to put the emphasis somewhat differently and view superselection as a "principle of impotence". In a quantum theory let us designate certain dynamical variables as being *unobservable in principle*. Consider all dynamical variables which commute with the set Z of non-observable dynamical variables. They form a subalgebra called the commutant Z' of the set Z. This is the subalgebra of *observables*. In general the algebra of observables is non-commutative like the algebra of dynamical variables.

The remarkable fact is that we could enlarge the set Z to the point where the commutant of observables, Z', is commutative. Then all the observables can be measured simultaneously. This could be a suitable model for a classical system, especially if each pure state is a superselection sector by itself. Then the absence of superpositions for classical systems would be understandable.

For Hamiltonian system with *one* degree of freedom and with commuting canonical variables χ, p the equations of motion take the form

$$\dot{\chi} = \frac{\partial H}{\partial p} = -i\,[\chi,\ \hat{H}]$$

$$\dot{p} = \frac{\partial H}{\partial \chi} = -i\,[p,\ \hat{H}] \tag{2.1}$$

where,

$$\hat{H} = -i\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial \chi} - \frac{\partial H}{\partial \chi}\frac{\partial}{\partial p}\right). \tag{2.2}$$

The operators $\hat{\chi}$ and $\hat{p}$ defined by

$$\hat{\chi} = +i\frac{\partial}{\partial p}\ ;\quad \hat{p} = -i\frac{\partial}{\partial \chi} \tag{2.3}$$

have the property

$$[\chi,\ \hat{\chi}] = 0\ ; \qquad [\chi,\ \hat{p}] = i\ :$$

$$[p,\ \hat{\chi}] = -i\ ;\quad [p,\ \hat{p}] = 0. \tag{2.4}$$

Thus the quantities

$$\omega = (\chi,\ p) \quad \text{and} \quad \pi = (-\hat{p},\ \hat{\chi}) = i\frac{\partial}{\partial \omega}$$

may be viewed as the canonical coordinate and momentum operators of a quantum system with *two* degrees of freedom. The equations of motion of the classical system can be viewed as the equation of motion of the quantum system with the Hamiltonian operator

$$H_{\text{op}} = i\,\frac{\partial H(\omega)}{\partial \omega^{\mu}}\,\epsilon^{\mu\nu}\,\frac{\partial}{\partial \omega^{\nu}}$$

$$\epsilon^{\mu\nu}(\omega) = [\omega^{\mu},\ \omega^{\nu}]_{\text{p.b.}} \tag{2.5}$$

in the form

$$\dot{\omega} = -i\,[\omega,\ H_{\text{op}}] \equiv i\,(\omega H_{\text{op}} - H_{\text{op}}\,\omega) \tag{2.6}$$

We note that the Hamiltonian operator is *linear* in the quantum momenta $i\,(\partial/\partial\omega)$ and hence any phase space density $\rho(\omega)$ is mapped into a new phase space density $\tilde{\rho}(\omega)$ such that $\tilde{\rho}(\tilde{\omega}) = \rho(\omega)$ where $\tilde{\omega}$ are the displaced values obtained by solving (2.6). If instead of this Schrödinger form of time development we were to view the time development in terms of the Heisenberg picture, we have the result

$$f(\omega) \to f(\tilde{\omega}) \tag{2.7}$$

where $\tilde{\omega}$ is the solutiou to (2.6). It is important to note that $\tilde{\omega}$ is a function of ω alone and not of ω and π by virtue of the linearity of H_{op} in the quantum canonical momenta π.

Let us now endow the quantum system with two degrees of freedom with the *superselection principle* that the quantum momenta $\pi = i\,(\partial/\partial\omega)$ are *unobservable at all times* and under all conditions. This implies and is implied by the identification of the observables with the commutative algebra of functions $f\,(\omega)$ of the coordinate operators. [This construction of a quantum theory embedding the classical theory is to be contrasted with the work of Coopman 1931; see also, Jordan and Sudarshan 1961].

State vectors for the quantum system are given, in the Schrödinger representation, by their wave functions $\psi\,(\omega)$. But because of the superselection principle, the relative phase of the distinct ideal eigenstates of coordinate operators is unmeasurable and, therefore, irrelevant. Hence, we are led to the equivalence

$$\psi\,(\omega) \sim \psi\,(\omega)\,\exp\,\{i\phi\,(\omega)\} \tag{2.8}$$

Therefore, only the absolute value of $\psi\,(\omega)$ is relevant and may be taken as the positive square root of the phase space density

$$\psi\,(\omega) = \sqrt{\rho\,(\omega)}. \tag{2.9}$$

The ideal eigenstates of the coordinate operators is identified with the classical state corresponding to a point in phase space. The time development is given by (2.6) and (2.7) and leads to a *trajectory in phase space* which is entirely observable. The possibility of being able to observe the entire trajectory is to be directly traced to the linearity of the Hamiltonian operator (2.5) in the quantum momentum operators.

It is to be noted that the Hamiltonian operator (2.5) is *not observable*: What is *observable* is the associated *energy function* $H\,(\omega)$ which is a function of the quantum coordinate operators only.

The restriction to the study of a classical system with one degree of freedom and its mapping on to a quantum system with two degrees of freedom with a superselection principle can be generalized to a system with f degrees of freedom. In this case

$$\omega = (q_1, \ldots, q_f\,;\; p_1, \ldots, p_f)$$
$$\pi = i\left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_f}\,;\; \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_f}\right). \tag{2.10}$$

The observables are functions of the ω only, and the Hamiltonian operator being linear in the quantum momenta π the trajectories continue to be observable.

We can generalize the system even further. Let ω denote the entire set of classical dynamical variables. We can then map it onto the superselection sectors of a quantum theory with ω and $i(\partial/\partial\omega)$ as coordinate and momentum operators and a Hamiltonian operator (2.5) linear in the momentum operators. The generalized "trajectory" is now the specification of all ω as functions of time and this is entirely observable.

Not only the Hamiltonian, but all the generators of canonical transformations including displacements, rotations and transformations to moving frames are linear in the momenta (Sudarshan and Mukunda 1974). None of them are observable, but there are associated dynamical quantities of momentum, angular momentum and moment of mass which are either constant or have simple time dependence. I shall not elaborate on these generalizations in this paper.

What I have presented here is the complete equivalence of a classical system with a suitable quantum system endowed with a superselection principle: Classical mechanics as a hidden variable theory!

3. Coupling of classical and quantum systems

Since a classical system is a special kind of quantum system, we may couple a classical system with a quantum system provided we pay attention to the superselection principle: the momentum operators $\pi = i\,(\partial/\partial\omega)$ shall continue to remain unobservable. The dynamical variables are elements of the noncommutative algebra generated by ω, π and the quantum system variable which I collectively denote by ξ. These variables ξ may involve canonical pairs Q, P or spin variables $\underset{\sim}{S}$, or more general quantities. (In this paper I deal with systems with a finite number of degrees of freedom and specifically exclude dynamical fields, for technical reasons). Given such a system, the Hamiltonian operator may be written

$$H_{\text{op}} = H^{0}_{\text{op}} + H^{\text{int}}_{\text{op}} \tag{2.11}$$

with

$$H^{0}_{\text{op}} = i\frac{\partial H}{\partial\omega^{\mu}}\,\epsilon^{\mu\nu}(\omega)\,\frac{\partial}{\partial\omega^{\nu}} + X(\xi)$$

$$= \frac{\partial H}{\partial\omega^{\mu}}\,\epsilon^{\mu\nu}(\omega)\,\pi_{\nu} + X(\xi).$$

$$H^{\text{int}}_{\text{op}} = i\phi^{\mu}(\omega, \xi)\,\frac{\partial}{\partial\omega^{\mu}} + h(\omega, \xi) \tag{2.12}$$

$$= \phi^{\mu}(\omega, \xi)\,\pi_{\mu} + h(\omega, \xi).$$

Here H is a function of ω only and X is a function of ξ only ; ϕ^{μ} and h depend on both set of variables to describe interaction. We can absorb the " free Hamiltonian " terms into the interaction part and rewrite

$$H_{\text{op}} = \phi^{\mu}(\omega, \xi)\,\pi_{\mu} + X(\omega, \xi). \tag{3.1}$$

The equations of motion for ω^{μ} and ξ are given by

$$\dot{\omega}^{\mu} = -\,\phi^{\mu} \tag{3.2}$$

$$\dot{\xi} = -i\,[\xi, \phi^{\mu}]\,\pi_{\mu} - i\,[\xi, X]. \tag{3.3}$$

By virtue of these equations $\omega^{\mu}(t)$ become functionally dependent on the noncommutative quantum variables ξ, but these same equations guarantee that they will continue to remain mutually commutative. We can also write down the equations of motion for π^{μ}. The superselection principle requires them to remain unobservable and also demands that $\omega^{\mu}(t)$ should not depend on π.

We can use (3.1) to calculate the higher time derivatives of ω^μ. We get

$$\ddot{\omega}^\mu = + i\,[\phi^\mu, H_{\text{op}}]$$
$$= \phi^\nu \left(\frac{\partial}{\partial \omega^\nu} \phi^\mu\right) + i\,[\phi^\mu, \phi^\nu]\,\pi_\nu + i\,[\phi^\mu, X]. \tag{3.4}$$

If $\omega\,(t)$ is to be independent of π for all t, we must have the coefficient of the π_ν term vanish. We get, therefore, the requirement

$$[\phi^\mu, \phi^\nu] = 0.$$

We could derive a stronger condition by observing that according to (3.2) the velocities are given by $-\phi^\mu\,(\omega)$ and these are all simultaneously measurable. Hence

$$[\Phi^\mu\,(\omega),\ \Phi^\nu\,(\omega')] = 0. \tag{3.5}$$

where ω' is a suitable point in the classical phase space which may or may not coincide with ω. [We may therefore differentiate with respect to ω' any number of times !] Barring singular "impulsive" interactions velocities and accelerations should also commute, by virtue of (3.4) and (3.5) we obtain

$$\big[[\phi^\mu\,(\omega), X], \phi^\nu\,(\omega')\big] = 0.$$

If we were to deal with higher derivatives of ω, we could deduce additional relations of the form

$$\big[[[\phi^\mu\,(\omega), X], X], \phi^\nu\,(\omega')\big] = 0$$
$$\Big[\big[[[\phi^\mu\,(\omega), X], X], X\big], \phi^\nu\,(\omega')\Big] = 0$$

and so on. All these relations are satisfied if :

$$[\phi^\mu, X] = f^\mu\,(\phi, \omega). \tag{3.6}$$

Consistency of the superselection principle for the interacting classical system and the observability of the "trajectory" can be translated into the requirement that the coupling functions $\Phi^\mu\,(\omega, \xi)$ are dependent only on a commutative subset of the quantum variables; the function may depend on other quantum dynamical variables also but in such a special manner that $[\Phi^\mu, X]$ depends only on these commuting sets of quantum variables. If, for example, we were to have Φ^μ dependent only on canonical coordinate operators, then X can depend linearly on the quantum momentum operators, unless the interactions are impulsive. [I am grateful to Narasimhaiengar Mukunda for a patient and critical discussion of these considerations.]

It is gratifying that we are naturally led to a measurement, *via* the classical trajectory, of only a commuting set of quantum dynamical variables. I discuss measurement in the next section.

4. Measurement

Let us now turn to measurement of quantum dynamical variables. We have seen in the last section that if we need to measure a subset of the maximal commuting set of quantities ζ, which themselves form a subset of the set ξ of dynamical variables of the quantum system, then we couple suitable functions $\Phi^{\mu}(\zeta, \omega)$ to the classical system through the nonobservable dynamical variables $i(\partial/\partial\omega^{\mu})$. Then the classical trajectory $\omega^{\mu}(t)$ now depends on the ζ. A consequence of this is the possibility of " branching " of the classical trajectory if the quantum variables Φ^{μ} are many-valued. The most familiar example of this is the splitting of a molecular beam in the Stern-Gerlach experiment. We now study the measurement problem more systematically.

Two kinds of measurement interaction may be distinguished: continuous measurements in which quantum dynamical variables are monitored continuously and discrete measurements in which instantaneous values are measured by one or more impulsive interactions.

For impulsive interactions we consider a singular perturbation of the un coupled quantum and classical systems idealized in the form

$$\begin{aligned} H_{\text{int}} &= V(\omega, \xi)\,\delta(t-t_0) \\ &= \left\{ i\Phi^{\mu}(\omega, \xi)\frac{\partial}{\partial\omega^{\mu}} + X(\omega, \xi)\right\}\delta(t-t_0). \end{aligned} \tag{4.1}$$

The effect of this impulsive interaction is obtained by going to the interaction picture. (I am grateful to Baidyanath Misra for a discussion of this question). The generator of interactions is the time-ordered unitary operator

$$\begin{aligned} U &= (\exp\{-i\textstyle\int V(\omega(t), \xi(t)\,\delta(t-t_0)\,dt\})_{+} \\ &= \exp\left(\Phi^{\mu}\frac{\partial}{\partial\omega} - iX\right). \end{aligned} \tag{4.2}$$

The classical system is a quantum system with the state vector $\psi(\omega)$ with the distinct values of ω corresponding to distinct superselection sectors. So essentially only $|\psi(\omega)|$ is relevant. The integrity of the classical system demands that this feature be preserved by the transformation (4.2). If the state vector of the coupled system is denoted by $\Psi(\omega, \zeta)$

$$\Psi(\omega, \zeta) \to \Psi_I(\omega, \zeta)$$

where

$$\Psi_I(\omega_I, \zeta_I) = \Psi(\omega, \zeta) \tag{4.3}$$

$$\begin{aligned} \omega_I &= U\omega U^{-1} \\ \zeta_I &= U\zeta U^{-1} \\ \Psi_I &= U^{-1}\psi. \end{aligned} \tag{4.4}$$

We require that the $\partial/\partial\omega$ do not enter into the expression of ω_I. Since

$$\begin{aligned} \omega_I^{\mu} &= \omega^{\mu} - \Phi^{\mu} + \frac{1}{2!}\left[\Phi^{\nu}\frac{\partial}{\partial\omega^{\nu}} - iX,\ \Phi^{\mu}\right] \\ &\quad - \frac{1}{3!}\left[\Phi^{\lambda}\frac{\partial}{\partial\omega^{\lambda}} - iX,\ \left[\Phi^{\nu}\frac{\partial}{\partial\omega^{\nu}} - iX,\ \Phi^{\mu}\right]\right] + \dots \end{aligned} \tag{4.5}$$

these requirements are met if

$$\Phi^{\mu} = \Phi^{\mu}(\omega\ \zeta) \qquad X = 0. \tag{4.6}$$

The condition (4.6) together with (4.5) imply the possibility of measuring all members of a complete set ζ of commuting observables. We may write the expression for $\omega_I{}^{\mu}$ given by (4.5) as the solution of the differential equations

$$\frac{\partial \omega^{\mu}(\tau)}{\partial \tau} + \Phi^{\mu}\left(\omega(\tau),\ \zeta\right) = 0 \tag{4.7}$$

with the boundary conditions

$$\omega^{\mu}(0) = \omega^{\mu}(t_0) = \omega^{\mu}(t_0 -)$$
$$\omega^{\mu}(1) = \omega_I{}^{\mu}(t_0) = \omega^{\mu}(t_0 +). \tag{4.8}$$

Since all the ζ can be simultaneously diagonalized (4.7) may be viewed as a set of differential equations labelled by a set of parameters; (4.8) then yields a "branching of the trajectories" according to the quantization of the set ζ.

If we make repeated observations, we must guarantee the integrity of the classical system; this entails the "compatibility" of the different measurement interactions. If we denote by them by

$$i\Phi_1{}^{\mu}\frac{\partial}{\partial \omega^{\mu}}\cdot\delta(t-t_1) \quad \text{and} \quad i\Phi_2{}^{\mu}\frac{\partial}{\partial \omega^{\mu}}\,\delta(t-t_2),$$

then in the interaction picture

$$[\Phi_1{}^{\mu}(t_1),\ \Phi_2{}^{\nu}(t_2)] = 0. \tag{4.9}$$

A special limiting case of repeated observations is the situation of continuous observation. The discussion in the last section shows that in this case we should demand

$$\Phi^{\mu} = \Phi^{\mu}(\omega,\ \zeta)$$
$$[X,\ \Phi^{\mu}] = f^{\mu}(\omega,\ \zeta) \tag{4.10}$$

with the interaction in the form (3.1). We cannot, in this case, choose $X \equiv 0$, since it involves the "free" Hamiltonian of the quantum system. It is interesting and important to note that the quantities that can be continuously observed need not be constants of motion.

As simple examples of continuous measurements we may consider the Stern-Gerlach experiment with the Hamiltonian operator

$$H = \frac{-i}{m}\,\mathrm{p}\cdot\frac{\partial}{\partial \mathrm{q}} - i\Gamma\, S_3\,\frac{\partial}{\partial p_3} - \gamma\, B_3.\mathrm{S}_3$$

where Γ is proportional to the magnetic field gradient. The equations of motion can be solved to yield, without any essential loss of generality,

$$q_1(t) = \frac{1}{m}\,p_1\,t\ ;\quad q_2(t) = 0\ ;$$

$$q_3(t) = \pm\tfrac{1}{2}\,\Gamma\, t^2.$$

where the spin is taken to have values $\pm\frac{1}{2}$. We get *two* parabolas for the trajectory.

Another simple example is the measurement of the quantum coordinate Q of harmonic oscillator by an impulsive interaction:

$$H = -\frac{i}{m} p \cdot \frac{\partial}{\partial q}$$

$$+ \frac{1}{2m} (P^2 + \omega^2 M^2 Q^2)$$

$$+ ig\, Q \cdot \frac{\partial}{\partial p} \cdot \delta\,(t - t_0)$$

The classical trajectory would be a straight line at uniform velocity excepting for the sudden jump in the momentum by the amount $\triangle p = \lambda Q$.

If we wish to measure the coordinate at a later instant, it could be compatible only if it is an integral number of periods later when we recover the same value.

5. Concluding remarks

The superselection principle applicable to the classical system viewed as a quantum system makes different phase space configurations belong to different superselection sectors and hence their relative phase is nonmeasurable. The loss of this phase information is compensated by the continuous observability of the phase space trajectory. It is of considerable interest to note that if the classical system was coupled to a quantum system in an eigenstate of a set of variables ζ and the subsequent interaction is in terms of a set of variables η which commute among themselves but not with ζ the different components of the split classical trajectory have phase relations, but these phase relations cannot be measured in any fashion except by giving up the information on the variables η and then proceeding to measure ζ. Any definitive measurement of η destroys any phase relations which exist.

We may view measurement as being destruction of any phase information so that the component beam becomes a genuine physical system in itself. *Measurement* may thus be viewed as the *process of one-becoming-two*.

We point out that classical mechanics is viewed in this paper as *quantum mechanics with hidden variables*. The hiddenness of the quantities $i(\partial/\partial\omega)$ and functions of them is an essential property which must be maintained to preserve the integrity of the classical system. Unlike the orthodox quantum theories with superselection rules, here the Hamiltonian does not preserve the superselection sectors, but causes continuous and lawful evolution of the system from one sector to another so that we have a nontrivial classical trajectory. This leads to no inconsistencies, since the Hamiltonian is not an observable but it is associated in a (projective, up to—neutral element) correspondence with the energy operator which is an observable.

The position taken in this paper is that all classical dynamical variables can be measured and that quantum dynamical variables are to be measured by coupling a classical system to quantum system. Thus, we have seen that the interaction (3.1) converts the dynamical variables $-\Phi^{\mu}$ into the velocities ω^{μ} along the classical trajectory. The question of measuring classical dynamical variables, the catastrophic configuration of pointers that give pointer readings and the irreversibility that is implicit in a recordable measurement and, finally, the role of the observer or rather, the presiding intelligence in the measurement protocol and its authentication (Wigner 1952) are questions too profound to be discussed in this paper. My understanding is well summarized by the smrti:

Sarvāgamānamācāram
pratyāpi parikalpayet
ācāra prabhavo dharmo
dharmasya praburacyutaḥ

[All authoritative formulations stress the proper procedure; proper procedure is the prerequisite to natural law. The immutable awareness (the Self) is the presiding intelligence.]

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