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Relation between Nambu and Hamiltonian mechanics

N. Mukunda and E. C. G. Sudarshan
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012, India
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The connection between Nambu's generalization of classical dynamics and conventional Hamiltonian ideas is explored. In particular, the possibility of embedding the dynamics of a Nambu triplet in a four-dimensional canonical phase-space formalism is proved.

INTRODUCTION

Motivated by Liouville's theorem and by the form of Euler's equations for a classical rigid rotator, Nambu had some time ago suggested very interesting generalizations of classical Hamiltonian dynamics.[1] There are two basic elements in Hamiltonian dynamics. Firstly, one has an even-dimensional phase space on which the Poisson bracket (PB) structure obeying the Jacobi identity is defined. Secondly, one has the Hamiltonian form for the equations of motion, according to which the evolution in time of a dynamical system is determined by a single function, the Hamiltonian, and consists of a time-dependent canonical transformation. The basic canonical structure is carried by a single canonical pair of variables, a Hamiltonian doublet, and the PB can already be set up for functions of a single doublet; generalization to several pairs is straightforward. In the simplest generalization suggested by Nambu, the primitive Hamiltonian doublet is replaced by a set of three variables, a Nambu triplet. A new kind of PB (not obeying the Jacobi identity) is defined for three functions at a time, and time evolution is determined by two Hamiltonians, not one. A new definition of canonical transformations can be given, and Liouville's theorem is maintained.

It is of interest to see to what extent Nambu's mechanics goes beyond the general framework of Hamiltonian ideas, and to determine whether a description using Nambu triplets can be embedded in a description using Hamiltonian variables. We shall examine this question in this paper for the case of a single Nambu triplet, and shall show that the dynamics of a system described by one such triplet can always be embedded in a conventional Hamiltonian scheme using two canonical pairs (four-dimensional phase space). However, extension of our result to several Nambu triplets is quite nontrivial and will not be attempted here. A solution along these lines was proposed by Bayen and Flato[2] but their solution was only partial. In particular they conjecture that one needs at least three pairs of canonical variables to reproduce the equations of a Nambu triplet: we disprove this conjecture. We shall give first a direct proof of our result, and then analyze the treatment of Ref. 2, which looks upon a Nambu system as a constrained three-dimensional Lagrangian system, to show that this too is in conformity with our general result. (See also Ref. 3 for further work on Nambu's mechanics.)

In the reverse direction one can ask whether a general Hamiltonian system can be described in such a way that its equations of motion appear in Nambu form. Here one has in mind the generalization in which the Hamiltonian doublet is replaced by an n-tuple of variables, and one has $(n-1)$ Hamiltonians to give the time evolution. We shall show that as a result of the structure of the canonical phase space of any dimension, the equations of motion of any Hamiltonian system can be put into the Nambu form.

Section I describes the way one can embed the Nambu scheme in a Hamiltonian framework, while in Sec. II we discuss the Lagrangian suggested in Ref. 2 for deriving Nambu's equations. Use is made here of Dirac's theory of constraints and of Dirac brackets.[4] Section III proves the possibility of writing Hamilton's equations for a system with any number of degrees of freedom in the Nambu form.

I. THE NAMBU SCHEME IN A HAMILTONIAN FRAMEWORK

Let x_1, x_2, x_3 be the independent members of a Nambu triplet. The equations of motion for x_j involve two algebraically independent functions $F(x_1, x_2, x_3)$ and $G(x_1, x_2, x_3)$ and are postulated by Nambu to be

$$\dot{x}_j = \frac{\partial(F, G)}{\partial(x_k, x_l)}, \quad j, k, l = \text{cyclic permutations of } 1, 2, 3. \tag{1.1}$$

The problem of embedding this system in the Hamiltonian framework can be stated as follows: Construct an even-dimensional phase space with variables $q_1, \ldots, q_n, p_1, \ldots, p_n$; choose a Hamiltonian $H(q, p)$ and three independent phase-space functions $\zeta_j(q,p)$ such that the right-hand sides of Hamilton's equations of motion for ζ_j are equal to the right-hand sides of Eq. (1.1) with ζ written in place of x:

$$\{\zeta_j(q,p), H(q,p)\} \equiv \frac{\partial(F(\zeta_1,\zeta_2,\zeta_3), G(\zeta_1,\zeta_2,\zeta_3))}{\partial(\zeta_k,\zeta_l)}. \tag{1.2}$$

(The curly brackets denote the conventional PB.) In particular, discover the smallest value of n for which this can be done.

We shall solve this problem by expressing the content of Eq. (1.1) in a different form. The essential point to realize is that Nambu's equations of motion are so constructed that both $F(x_1,x_2,x_3)$ and $G(x_1,x_2,x_3)$ are constants of motion. As a result, for a specific state of motion, it is enough to know how x_3, say, varies with time; the dependence of x_1 and x_2 on t can then be obtained by setting $F(x_1,x_2,x_3)$ and $G(x_1,x_2,x_3)$ equal to the constant values characteristic of that state of motion. In order that one have a nontrivial system of equations of motion, one must assume that the right-hand side of Eq. (1.1) is nonzero for at least one value of j. Without loss of generality, it can be assumed that

$$a(x_1,x_2,x_3) \equiv \frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_2} - \frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_1} \neq 0. \tag{1.3}$$

But this can be interpreted to mean that x_3, F, and G are three independent functions of x_1, x_2, x_3 and so could be used as an independent set in place of the latter. In particular, x_1 and x_2 could be expressed as certain functions, φ_1 and φ_2, of x_3, F, and G, and $a(x_1,x_2,x_3)$ also becomes a function ξ of these variables:

$$\begin{aligned} &x_1 = \varphi_1(x_3,F,G), \quad x_2 = \varphi_2(x_3,F,G), \\ &a(x_1,x_2,x_3) = \xi(x_3,F,G). \end{aligned} \tag{1.4}$$

The Nambu equations of motion (1.1) are then completely equivalent to the set

$$\dot{x}_3 = \xi(x_3,F,G), \quad \dot{F}=0, \quad \dot{G}=0, \tag{1.5}$$

supplemented, of course, by the expressions for x_1, x_2 given in Eq. (1.4).

We now wish to embed Eq. (1.5) in a Hamiltonian system. We therefore need a Hamiltonian $H(q,p)$, and three independent phase-space variables $\alpha(q,p)$, $\beta(q,p)$, and $\gamma(q,p)$ to serve, respectively, as x_3, F, and G, obeying the system of PB relations

$$\begin{aligned} &\{\alpha(q,p), H(q,p)\} = \xi(\alpha,\beta,\gamma), \\ &\{\beta, H\} = \{\gamma, H\} = 0. \end{aligned} \tag{1.6}$$

It is a known property of canonical coordinate systems in phase space that any function on phase space, which is not a mere constant, can always be made one of the members of a system of canonical coordinates.[5] Without loss of generality, we therefore assume a particularly simple form for the Hamiltonian H:

$$H(q,p) = p_1. \tag{1.7}$$

Equations (1.6) then simplify to

$$\begin{aligned} &\frac{\partial\alpha(q_1,p_1,q_2,p_2,\ldots)}{\partial q_1} = \xi(\alpha,\beta,\gamma), \\ &\frac{\partial\beta}{\partial q_1} = \frac{\partial\gamma}{\partial q_1} = 0. \end{aligned} \tag{1.8}$$

These are immediately solved:

$$\begin{aligned} &\beta = \beta(p_1,q_2,p_2,\ldots), \quad \gamma = \gamma(p_1,q_2,p_2,\ldots), \\ &\int^\alpha du/\xi(u,\beta,\gamma) = q_1 + \psi(p_1,q_2,p_2,\ldots). \end{aligned} \tag{1.9}$$

This solution involves three arbitrary functions β, γ, and ψ of the variables $p_1, q_2, p_2, \ldots$ subject to the condition that β and γ be algebraically independent. The independence of α from β and γ is ensured by the q_1 dependence of α.

Now the condition that β and γ be mutually independent shows that we cannot embed Nambu's system into a Hamiltonian system with $n=1$, i.e., involving just one pair q_1, p_1, but we can certainly do so with $n=2$, i.e., using a four-dimensional phase space with variables q_1, p_1, q_2, p_2. Thus the assertion made in the Introduction is proved, and at the same time we see that there is a great deal of freedom in expressing Nambu's equations in Hamiltonian form.

To conclude this section, let us give two simple examples of the very general solution developed above. As the first possibility, we choose

$$\begin{aligned} &F = \beta(p_1,q_2,p_2) = p_1, \quad G = \gamma(p_1,q_2,p_2) = q_2, \\ &\int^\alpha du/\xi(u,p_1,q_2) = q_1 \Rightarrow x_3 = \alpha(q_1,p_1,q_2), \\ &H = p_1. \end{aligned} \tag{1.10}$$

In this solution, all the variables of interest involve only the three Hamiltonian variables q_1, p_1, q_2, and the following PB relations hold:

$$\{F,G\} = \{G,x_3\} = 0, \quad \{F,x_3\} \neq 0. \tag{1.11}$$

As a second possibility we could choose

$$\begin{aligned} &F = q_2, \quad G = p_2, \quad H = p_1, \\ &\int^\alpha du/\xi(u,q_2,p_2) = q_1 \qquad x_3 = \alpha(q_1,q_2,p_2) \end{aligned} \tag{1.12}$$

leading to

$$\{F,G\} = 1, \quad \{F,x_3\} \neq 0, \quad \{G,x_3\} \neq 0. \tag{1.13}$$

In this case, we see that F and G make up a canonically conjugate pair; the following section will show that this is just what occurs in the treatment of Nambu's equations in terms of a singular Lagrangian.

II. DERIVING NAMBU'S EQUATIONS

Consider the singular Lagrangian

$$L(q,\dot q)=F(q)\sum_{j=1}^{3}\dot q_j\frac{\partial G(q)}{\partial q_j} \tag{2.1}$$

linear in the velocities. This leads to the momenta

$$p_j=F(q)\frac{\partial G}{\partial q_j}$$

and to the equations of motion

$$\sum_{j=1}^{3}\dot q_j\frac{\partial(F,G)}{\partial(q_j,q_k)}=0\,.$$

The Nambu equations of motion (1.1) are compatible with these equations (with x replaced by q); as pointed out by Bayen and Flato,[2] the Nambu equations may be deduced from these equations if we allow for a redefinition of the time variable. Such a redefinition was to be anticipated in a singular Lagrangian linear homogeneous in the velocities,[6] and to reexpress its consequences in Hamiltonian form one must use Dirac's theory of constraints.[4]

We wish to reconcile such a treatment with the general result of Sec. III; we will show that when the entire analysis is done, the final phase space resulting from the Lagrangian (2.1) is indeed just four-dimensional, and we will exhibit a canonical system of coordinates for it.

The definitions of the momenta p_j conjugate to the q_j do not depend on the velocities $\dot q_j$ and so reduce to three primary constraints

$$\varphi_j(q,p)=p_j-F(q)\frac{\partial G(q)}{\partial q_j}\approx 0,\quad j=1,2,3 \tag{2.2}$$

which are "weak equations." Since L is linear homogeneous in the velocities, to begin with, the Hamiltonian is just a linear combination of the φ_j,

$$H=\sum_{j=1}^{3}v_j\varphi_j(q,p)\,, \tag{2.3}$$

where the v_j are the unsolved velocities. The starting Hamilton equations of motion are then

$$\begin{aligned}&\dot q_j\approx v_j\,,\\&\dot p_j\approx\sum_{k=1}^{3}v_k\frac{\partial}{\partial q_j}\left(F\frac{\partial G}{\partial q_k}\right),\end{aligned} \tag{2.4}$$

with, of course, the constraints (2.2). We now impose the condition that the primary constraints be maintained in time: These lead to the weak equations

$$\dot\varphi_j\approx\sum_{k=1}^{3}\{\varphi_j,\varphi_k\}v_k\approx 0\,. \tag{2.5}$$

Clearly, no secondary constraints can be generated. The matrix $\|\{\varphi_j,\varphi_k\}\|$ is necessarily singular, so that from Eq. (2.5) we can evaluate two of the unknown velocities in terms of the third, or alternatively express all three v_j in terms of a single unknown velocity v:

$$\begin{aligned}\{\varphi_j,\varphi_k\}&=\left\{p_j-F\frac{\partial G}{\partial q_j},\,p_k-F\frac{\partial G}{\partial q_k}\right\}\\&=\frac{\partial(F,G)}{\partial(q_j,q_k)}\,,\\\sum_k\{\varphi_j,\varphi_k\}v_k\approx 0&\Rightarrow v_j\approx v\frac{\partial(F,G)}{\partial(q_k,q_l)}\,.\end{aligned} \tag{2.6}$$

[Here (j,k,l) is a cyclic permutation of 1, 2, 3.] The final form for the Hamiltonian is

$$\begin{aligned}H&=\frac{v}{2}\epsilon_{jkl}\frac{\partial(F,G)}{\partial(q_k,q_l)}\varphi_j(q,p)\\&=v\epsilon_{jkl}p_j\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial q_l}\,.\end{aligned} \tag{2.7}$$

This result for H can be understood easily as follows. Since the number of (primary) constraints is three, there must be (at least) one first-class combination. Taking note of the value of the PB $\{\varphi_j,\varphi_k\}$ in Eq. (2.6), as well as the assumption in Eq. (1.3), we see that the system of three constraints $\varphi_j\approx 0$ can be replaced by an equivalent set

$$\varphi=\epsilon_{jkl}\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial q_l}\varphi_j\approx 0,\quad \varphi_1\approx 0,\quad \varphi_2\approx 0\,. \tag{2.8}$$

The first one, φ, is the first-class combination, and the final Hamiltonian in Eq. (2.7) appears as a multiple of it. The remaining two constraints, φ_1 and φ_2, are second class because

$$\{\varphi_1,\varphi_2\}=a(q_1,q_2,q_3)\neq 0\,. \tag{2.9}$$

At this stage, the motion is restricted to a three-dimensional surface $\varphi\approx\varphi_1\approx\varphi_2\approx 0$ in the six-dimensional phase space. But by using the

Dirac-bracket formalism we can explicitly dispense with the second-class constraints $\varphi_1 \approx \varphi_2 \approx 0$ and reduce the dimensionality of the phase space from six to four.[4] We set up the Dirac bracket of any two functions f and g as

$$\{f,g\}^* = \{f,g\} + (\{f,\varphi_1\}\{\varphi_2,g\} - \{f,\varphi_2\}\{\varphi_1,g\})/a(q) \tag{2.10}$$

The general equation of motion can be stated in the form

$$\frac{df}{dt} \approx \{f,H\}^*, \quad \varphi \approx 0\,; \tag{2.11}$$

once this is done, the conditions $\varphi_1 \approx 0$, $\varphi_2 \approx 0$ can be used to explicitly eliminate the variables p_1 and p_2, replacing them wherever they occur (including in the Hamiltonian) by functions of q alone:

$$p_1 = F(q)\frac{\partial G(q)}{\partial q_1}, \quad p_2 = F(q)\frac{\partial G(q)}{\partial q_2}\ . \tag{2.12}$$

The Hamiltonian is then a (weakly vanishing) function on a four-dimensional phase space, for which we may choose q_1, q_2, q_3, and p_3 as independent coordinates.

We now obtain agreement with the result of Sec. I by establishing that in the q_1, q_2, q_3, p_3 space we have precisely two independent canonical pairs (with respect to the Dirac bracket, of course). The brackets among q_j and p_3, evaluated using Eq. (2.10), are

$$\{q_j,q_k\}^* = -\epsilon_{jk3}/a(q)\,,$$

$$\{q_j,p_3\}^* = \delta_{j3} + \left[\delta_{j2}\frac{\partial}{\partial q_3}\left(F\frac{\partial G}{\partial q_1}\right) - \delta_{j1}\frac{\partial}{\partial q_3}\left(F\frac{\partial G}{\partial q_2}\right)\right]\Big/a(q)\,. \tag{2.13}$$

The 4×4 matrix made up of these fundamental brackets can be checked to be nonsingular on account of $a(q) \neq 0$. This is proof that it must be possible to construct two canonical pairs out of functions of q_j and p_3.

One could evidently choose one pair to be q_3, p_3; however, it is simpler to start from the following facts:

$$\{G,F\}^* = 1, \quad \{q_3,F\}^* = \{q_3,G\}^* = 0\,. \tag{2.14}$$

[These are straightforward consequences of Eq. (2.13).] We recognize that the two "Hamiltonians" of Nambu's scheme form a canonical pair, as in the second illustrative example of Sec. I. To complete the analysis of the four-dimensional phase space, we must find a conjugate π_3 to q_3, having vanishing Dirac bracket with F and G. Let us start from the equations

$$\{p_3,F\}^* = A(F,G,q_3), \quad \{p_3,G\}^* = B(F,G,q_3)\,. \tag{2.15}$$

The values of these brackets are functions of q_j alone, and since F, G, and q_3 are independent, we are permitted to write them in the above form. Now the Jacobi identity

$$\{p_3,\{F,G\}^*\}^* + \{F,\{G,p_3\}^*\}^* + \{G,\{p_3,F\}^*\}^* = 0 \tag{2.16}$$

implies, by virtue of Eq. (2.14),

$$\frac{\partial A}{\partial F} + \frac{\partial B}{\partial G} = 0\,, \tag{2.17}$$

which in turn guarantees the existence of a function $\Lambda(F,G,q_3)$ such that

$$A = \frac{\partial \Lambda}{\partial G}, \quad B = -\frac{\partial \Lambda}{\partial F}\,. \tag{2.18}$$

Putting this into Eq. (2.15), we see immediately that the variable

$$\pi_3 = p_3 - \Lambda(F,G,q_3) \tag{2.19}$$

satisfies all the conditions needed of a conjugate to q_3:

$$\{q_3,\pi_3\}^* = 1, \quad \{F,\pi_3\}^* = \{G,\pi_3\}^* = 0\,. \tag{2.20}$$

We have thus succeeded in exhibiting two independent canonical pairs G, F and q_3, π_3 in the four-dimensional phase space of the system obtained after converting the two second-class constraints into identities. Thus the description of the Nambu system of equations as a degenerate Lagrangian system ultimately amounts to embedding the former in a Hamiltonian scheme with two canonical pairs. The only difference from the treatment of the preceding section is that a redefinition of the time variable, dependent on the state of motion, is now needed.

III. HAMILTON'S EQUATIONS FOR N DEGREES OF FREEDOM

One direction in which Nambu's equation (1.1) could be generalized is to consider a dynamical system made up of several kinematically independent triplets but governed by just two independent Hamiltonians. Another direction is to replace the triplet by an N-tuple of variables $x_1, x_2, \ldots, x_N$ and two Hamiltonians by $(N-1)$ independent Hamiltonians $H_1(x)$, $H_2(x)$, $\ldots$, $H_{N-1}(x)$. Then, Eq. (1.1) gives way to the system

$$\dot{x}_j=\frac{\partial(x_j, H_1, H_2, \ldots, H_{N-1})}{\partial(x_1, x_2, \ldots, x_N)}\ , \ j=1,2,\ldots,N. \tag{3.1}$$

We shall show that a conventional Hamilton system of equations of motion in a $2n$-dimensional phase space, governed by a single Hamiltonian $H(q_1, \ldots, p_n)$, can be put into the above Nambu form for $N=2n$.

The main result we need from canonical mechanics is the one already used in Sec. I (see Ref. 5): By means of a canonical transformation one can pass from $q_1, \ldots, p_n$ to $Q_1, \ldots, P_n$ such that the Hamiltonian becomes, say, P_1:

$$H(q_1, \ldots, p_n)=P_1. \tag{3.2}$$

$Q_2, \ldots, Q_n, P_1, \ldots, P_n$ are $(2n-1)$ independent constants of motion, and Hamilton's equations are very simple:

$$\dot{Q}_j=\delta_{j1}, \dot{P}_j=0. \tag{3.3}$$

But these same equations are reproduced if in

$$\dot{f}=\frac{\partial(f, Q_2, \ldots, Q_n, P_1, \ldots, P_n)}{\partial(Q_1, Q_2, \ldots, Q_n, P_1, \ldots, P_n)} \tag{3.4}$$

we set f equal to $Q_1, \ldots, Q_n, P_1, \ldots, P_n$ in turn. Therefore Hamilton's equations for the canonical coordinates, and by the derivation property for a general dynamical variable f, are equivalent to the Nambu form (3.4). One can now switch back to the original canonical coordinates q, p because the transformation $q, p \rightarrow Q, P$ has unit Jacobian (Liouville's theorem), and we see that Hamilton's general equation

$$\dot{f}(q,p)=\{f(q,p), H(q,p)\} \tag{3.5}$$

is equivalent to

$$\dot{f}=\frac{\partial(f, Q_2(qp), \ldots, Q_n(qp), H(qp), P_2(qp), \ldots, P_n(qp))}{\partial(q_1, \ldots, q_n, p_1, \ldots, p_n)} \tag{3.6}$$

This is the stated equivalence. However, it is true only locally in phase space since in general one cannot find $(2n-2)$ global constants of the motion to go with a given H.

IV. CONCLUDING REMARKS

We have examined the possible relationships between Nambu's generalized dynamics and conventional Hamiltonian ideas. For the case of a single Nambu triplet, an embedding in a 4-dimensional Hamiltonian scheme is always possible, with, however, the following being understood: This is locally possible and is highly nonunique. After such embedding, one could add a constraint to the Hamiltonian system, consistent with the earlier equations, and then only three of the four phase-space variables would be truly free.

The extension of this analysis to several triplets seems fraught with several difficulties. At any rate, the technique used in this paper does not seem well suited to an examination of this question, since the basic symmetry among the variables in the triplet was given up in replacing Eq. (1.1) by Eq. (1.5).

Again, since our analysis was only a local one in the phase space, we are unable to conclude anything direct about the quantization of Nambu's scheme. In particular, we cannot picture a quantized Nambu triplet (whatever that may mean) as consisting of three quarters of a quantum-mechanical system made up of two independent conjugate pairs of operators obeying the Heisenberg-Dirac commutation relations.

Finally, one may be tempted to say on the basis of Nambu's work that while classical dynamics is capable of easy generalization, quantum mechanics is not. However, such a statement does injustice to classical dynamics and springs from the notion that any system of differential equations can be taken to be "classical dynamics." On the contrary, the latter discipline has a great deal of characteristic structure—phase space, symplectic structure, canonical transformation theory, Lie algebraic structure of variables, etc.[7]—much of which is apparently given up in Nambu's generalizations. Therefore, Nambu's scheme differs as much from classical dynamics properly understood as a quantized Nambu scheme is likely to differ from conventional quantum mechanics.

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