A criterion for reducibility of a relativistic wave equation*

E. C. G. Sudarshan, M.A.K. Khalil, and W. J. Hurley

Center for Particle Theory, Department of Physics, University of Texas at Austin, Austin, Texas 78712
(Received 18 June 1976; revised manuscript received 30 June 1976)

In general when one writes a relativistic wave equation of the form $(-i\Gamma\cdot\partial+m)\psi(x)=0$, that transforms covariantly under some representation $\Lambda\rightarrow T(\Lambda)$ of SL(2,$\mathbb{C}$), it is nontrivial to determine whether or not the equation is irreducible or to avoid ending up with a reducible equation; especially if $T(\Lambda)$ contains repeating irreducible representations. In this paper a simple(st) criterion is given by which one can determine whether or not an equation is irreducible. It is shown that if Γ_μ have any invariant subspace at all, then that subspace must be a representation space of some combination of SL(2,$\mathbb{C}$) representations in $T(\Lambda)$. Knowing this, it is shown that a wave equation is reducible if and only if there exists some idempotent projector $\tilde{P}$ such that $(1-\tilde{P})\Gamma_0\tilde{P}=0$ other than $\tilde{P}=0$ or I. A method for constructing all possible admissable $\tilde{P}$'s is given. A simple example of the technique is also given.

I. INTRODUCTION

Relativistic wave equations of the form:

$$(-i\Gamma\cdot\partial+m)\psi(x)=0$$

can be reducible or irreducible. The meaning of "reducible"[1] in the context of relativistic wave equations is precisely formulated in the next section.

It turns out that reducible equations have particular properties that make theories based on such equations equivalent to simpler theories, both in the free field and interacting cases.[2,3] It is therefore important to know when a given equation is reducible, and hence, possibly equivalent to a simpler equation. The structure of reducible equations has been studied in Ref. 2.

When one constructs a wave equation, it is in general nontrivial to insure that the equation is irreducible. The main concern of this paper is to formulate the simplest possible criterion by which one can determine whether a given wave equation is reducible or not. Such a criterion is formulated in the next section.

Finally, in Sec. III, a simple example is considered that illustrates the use of the criterion.

II. CRITERION FOR REDUCIBILITY

$$(-i\Gamma_\mu\partial^\mu+m)\psi(x)=0 \tag{1}$$

is a relativistic wave equation that transforms covariantly under a representation of SL(2,$\mathbb{C}$), $\Lambda\rightarrow T(\Lambda)$:

$$T(\Lambda)=\bigoplus_{j=1}^{n}\alpha_j T_j(\Lambda). \tag{2}$$

The set of matrices $\{\Gamma_\mu\}$ may be regarded as a set of linear transformations over a linear space $R(N)$, where N is the number of rows (or columns) of Γ_μ,

$$\{\Gamma_\mu\}=\{\Gamma_0,\Gamma_1,\Gamma_2,\Gamma_3\}.$$

Definition 1: $\{\Gamma_\mu\}$ is called a *reducible set* $\Longleftrightarrow\exists$ a proper subspace $R_1\subset R(N)\ni$

$$\Gamma R_1\subseteq R_1\subset R(N)\quad\forall\Gamma\in\{\Gamma_\mu\}. \tag{3}$$

The subspace R_1 is called an *invariant subspace of* $\{\Gamma_\mu\}$ (IS of $\{\Gamma_\mu\}$).

Definition 2: Suppose R_j with $j=1,\ldots,L$ is a collection of all the invariant subspaces of $\{\Gamma_\mu\}$; then

$$R_0=\bigcup_{j=1}^{L}R_j \tag{4}$$

is an invariant subspace of $\{\Gamma_\mu\}$ called the *maximal invariant subspace of* $\{\Gamma_\mu\}$.

The space $R(N)$ is a representation space of $\Lambda\rightarrow T(\Lambda)$, i.e., $T(\Lambda)$ act as linear transformations on $R(N)$.

Definition 3: If R_s is an IS of $\{\Gamma_\mu\}$ and $TR_s=R_s$, then R_s is called an *invariant* SL(2,$\mathbb{C}$) *subspace of* $\{\Gamma_\mu\}$.

Lemma 1: $R_0\subset R(N)$ in Eq. (4) is an invariant SL(2,$\mathbb{C}$) subspace of $\{\Gamma_\mu\}$.

Proof: $\Gamma_\mu R_0\subseteq R_0\quad\forall\Gamma_\mu$.

Recall that

$$T^{-1}\Gamma_\mu T=\Lambda_\mu{}^\nu\Gamma_\nu, \tag{5}$$

$$\Gamma_\mu TR_0\subseteq T\Lambda_\mu{}^\nu\Gamma_\nu R_0. \tag{6}$$

Suppose ϕ_0 is any vector in R_0;

$$\Gamma_\mu T\phi_0=T\Lambda_\mu{}^\nu\Gamma_\nu\phi_0, \tag{7}$$

now

$$\Lambda_\mu{}^\nu\Gamma_\nu\phi_0\in R_0 \tag{8}$$

since for any value of $\mu=0,1,2,3$ the right-hand side of (7) is a linear combination of Γ_ν acting on ϕ_0, and each $\Gamma_\nu\phi_0\in R_0$ hence (8) follows. Now according to (7)

$$\Gamma_\mu T\phi_0=T\phi_0',\quad\phi_0,\phi_0'\in R_0$$

or

$$\Gamma_\mu TR_0\subseteq TR_0. \tag{9}$$

Therefore, TR_0 is also an invariant subspace of Γ_μ, but R_0 is a maximal invariant subspace of Γ_μ, hence

$$TR_0\subseteq R_0. \tag{10}$$

Recall that T are nonsingular transformations so

$$TR_0=R_0.\quad\blacksquare \tag{11}$$

In the following discussion a criterion for determining whether or not $\{\Gamma_\mu\}$ is a reducible set, will be formulated. For later convenience the bases for $R(N)$ will be chosen to be completely reducible bases (CRB's). A CRB is any basis in which $T(\Lambda)$ is block diagonal, and each block corresponds to an irreducible representation, $T_j(\Lambda)$ of SL(2, $\mathbb{C}$), in Eq. (2). Recall that altogether there are M irreducible representations where $M = \sum_{j=1}^{n} \alpha_j$ [Eq. (2)], there are α_j copies of the irreducible representation $T_j(\Lambda)$ for each j; so $T(\Lambda)$ is $M \times M$ in block form. Similarly $R(N)$ is a direct sum of M subspaces, each being a representation space of some irreducible representation of SL(2, $\mathbb{C}$) in (2),

$$R(N) = \bigoplus_{j=1}^{n} \alpha_j R_{(j)}. \quad (12)$$

One can now define SL(2, $\mathbb{C}$) *projectors* represented by Hermitian matrices in some CRB, that are idempotent and act on $R(N)$ such that

$$P_\alpha R(N) = R_{[\alpha]}, \quad (13)$$

where $R_{[\alpha]}$ is an SL(2, $\mathbb{C}$) subspace of $R(N)$, i.e., a direct sum of some $R_{(j)}$ in (2),

$$R_{[\alpha]} = \bigoplus_{j=1}^{n} a_j R_{(j)}, \quad (14a)$$

$$0 \leq a_j \leq \alpha_j. \quad (14b)$$

The subscript $[\alpha]$ denotes all the different combinations of a_j that satisfy (14b). P_α can be written in $n \times n$ block form, where each block j corresponds to the connection of the α_j representations T_j and is thus an $\alpha_j \times \alpha_j$ block matrix that is idempotent. The P_α do not mix vectors from spaces corresponding to different representations of SL(2, $\mathbb{C}$) in (2), but can mix vectors corresponding to the same representation of which there are α_j copies for a given T_j in (2). In this form it is obvious that

$$[P_\alpha, T(\Lambda)] = 0 \quad \forall \Lambda \in \mathrm{SL}(2, \mathbb{C}). \quad (15)$$

Since R_0 is an SL(2, $\mathbb{C}$) subspace of $R(N)$, there exists projectors of the type described above, P_0, such that

$$P_0 R(N) = R_0 \quad (16)$$

and every vector $\phi_0 \in R_0$ can be written as $P_0 \phi$ for some vector $\phi \in R(N)$.

Lemma 2: $\{\Gamma_\mu\}$ reducible $\Longleftrightarrow \exists$ some $\tilde{P}_0$ satisfying (16) that is idempotent such that

$$(I - \tilde{P}_0)\Gamma_\mu \tilde{P}_0 = 0. \quad (17)$$

Proof: $\tilde{P}_0$ is not required to be Hermitian. Suppose $\{\Gamma_\mu\}$ is reducible, then by Lemma 1 $\exists$ an SL(2, $\mathbb{C}$) subspace R_0 of $R(N) \ni \Gamma_\mu R_0 \subseteq R_0$. Choose a particular basis for $R(N)$ where $\phi \in R(N)$ is in the form

$$\phi = \left[\frac{\chi_0}{\psi}\right], \quad (18)$$

where every vector of R_0 can be written

$$\phi_0 \in R_0, \quad \phi_0 = \left[\frac{\chi_0}{0}\right]. \quad (19)$$

Now in this basis Γ'_μ still has the property

$$\Gamma'_\mu R_0 \subseteq R_0,$$

since this property is basis independent. Now clearly one may pick a $\tilde{P}_0^J$ in this basis such that

$$\tilde{P}_0^J = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \quad (20)$$

I is a $q \times q$ identity matrix where $q = \dim R_0$. Note that (20) is an idempotent operator (actually in its Jordan canonical form). Apply the matrix $[\Gamma'_\mu \tilde{P}_0^J - \tilde{P}_0^J \Gamma'_\mu \tilde{P}_0^J]$ to an arbitrary vector $\phi \in R(N)$,

$$\Gamma'_\mu \tilde{P}_0^J \phi - \tilde{P}_0^J \Gamma'_\mu \tilde{P}_0^J \phi = \Gamma'_\mu \phi_0 - \tilde{P}_0^J (\Gamma'_\mu \phi_0) = 0.$$

The last step follows because $\Gamma'_\mu \phi_0 = \phi'_0 \in R_0$ and in this basis $\tilde{P}_0^J \phi'_0 = \phi'_0$ for any vector $\phi'_0 \in R_0$. The only matrix that maps every vector $\phi \in R(N)$ into 0 is the zero matrix

$$\Gamma'_\mu \tilde{P}_0^J - \tilde{P}_0^J \Gamma'_\mu P_0^J = 0. \quad (21)$$

Since an idempotent projector exists in one basis satisfying (21) (by construction) such an operator exists in all bases; i.e., for any $\Gamma_\mu = V \Gamma'_\mu V^{-1}$, the operator $V \tilde{P}_0^J V^{-1}$ is such an operator. In particular a $\tilde{P}_0$ exists in all CRB's satisfying (17). The quality of $\tilde{P}_0$ being idempotent is preserved by all nonsingular transformations but the hermiticity is not. So in general for any CRB, only $\tilde{P}_0^2 = \tilde{P}_0$ will be required.

Now assume that some SL(2, $\mathbb{C}$) invariant operator $\tilde{P}_0$ exists such that (17) holds. Suppose ϕ is any vector in $R(N)$, then $\tilde{P}_0 \phi = \phi_0 \in R_0$ and by (17),

$$\Gamma_\mu P_0 \phi = \tilde{P}_0 \Gamma_\mu \tilde{P}_0 \phi \iff \Gamma_\mu \phi_0 = \tilde{P}_0 \Gamma_\mu \phi_0.$$

Now $\Gamma_\mu \phi_0 \in R(N)$ since $\Gamma_\mu : R(N) \to R(N)$. The right-hand side of the above equation,

$$\tilde{P}_0 \Gamma_\mu \phi_0 = \tilde{P}_0 \phi' = \phi'_0 \in R_0,$$

so one can see that $\Gamma_\mu : \phi_0 \to \phi'_0$, ϕ_0, $\phi'_0 \in R_0$. Since every vector $\phi_0 \in R_0$ can be written as $\tilde{P}_0 \phi$ for some $\phi \in R(N)$ one concludes that Γ_μ maps every vector $\phi_0 \in R_0$ into some other vector of R_0, therefore $\Gamma_\mu R_0 \subseteq R_0$ and $\{\Gamma_\mu\}$ is a reducible set. ∎

Lemma 3: $(1 - \tilde{P}_0)\Gamma_\mu \tilde{P}_0 = 0 \iff (1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0 = 0$.

Proof: It is obvious that $(1 - \tilde{P}_0)\Gamma_\mu \tilde{P}_0 = 0$ $\Longrightarrow (1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0 = 0$.

On the other hand, suppose $(1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0 = 0$, then since $\Gamma_i = i\Gamma_0 N_i - iN_i \Gamma_0$ where N_i are the generators of the boosts in the i direction for the representation $\Lambda \to T(\Lambda)$, one notices that

$$(1 - \tilde{P}_0)\Gamma_i \tilde{P}_0 = i[(1 - \tilde{P}_0)\Gamma_0 N_i \tilde{P}_0 - (1 - \tilde{P}_0)N_i \Gamma_0 \tilde{P}_0].$$

Since

$$[\tilde{P}_0, N_i] = 0,$$
$$(1 - \tilde{P}_0)\Gamma_i \tilde{P}_0 = i[(1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0 N_i - N_i (1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0] = 0.$$

The conclusion one can draw is that $\{\Gamma_\mu\}$ is a reducible set $\iff$ there exists an idempotent SL(2, $\mathbb{C}$) projector $\tilde{P}_0$ such that

$$(1 - \tilde{P}_0)\Gamma_0 \tilde{P}_0 = 0. \quad (22)$$

$\tilde{P}_0$, being an SL(2, $\mathbb{C}$) projector, is of the following form in the CRB indicated:

$$\tilde{P}_0 = \begin{array}{c} \begin{array}{cccc} \alpha_1 T_1 & \alpha_2 T_2 & \cdots & \alpha_n T_n \end{array} \\ \left[\begin{array}{|c|c|c|c|} \hline \tilde{P}_0^{11} & & & \\ \hline & \tilde{P}_0^{22} & & \\ \hline & & \ddots & \\ \hline & & & \tilde{P}_0^{nn} \\ \hline \end{array}\right] \end{array} \begin{array}{c} \alpha_1 T_1 \\ \alpha_2 T_2 \\ \vdots \\ \alpha_n T_n \end{array},$$

$[\tilde{P}_0^{jj}]^2 = \tilde{P}_0^{jj}$ for each $j = 1, \ldots, n$.

If one finds that no such projector exists other than I or 0, then one may conclude that $\{\Gamma_\mu\}$ is an irreducible set.

Relativistic wave equations where $\{\Gamma_\mu\}$ is a reducible set are called *reducible wave equations*.

III. AN EXAMPLE

Consider any two irreducible, interlocking representations of $SL(2, \mathbb{C})$ denoted A and B. For illustrating the technique consider any wave equation that can be constructed so as to transform under $T(\Lambda) = A \oplus B \oplus B$, then Γ_0 is the following:

$$\Gamma_0 = \begin{array}{c} \begin{array}{ccc} A & B & B \end{array} \\ \begin{array}{|c|c|c|} \hline & aD_1 & bD_1 \\ \hline cD_2 & & \\ \hline dD_2 & & \\ \hline \end{array} \end{array} \begin{array}{c} A \\ B \\ B \end{array}, \tag{23}$$

where a, b, c, and d are complex numbers, assumed to be nonzero (no requirements of unique mass or spin are imposed). The most general allowed $\tilde{P}$ is

$$\tilde{P} = \begin{array}{c} \begin{array}{ccc} A & B & B \end{array} \\ \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & \alpha & \beta \\ \hline & \gamma & \rho \\ \hline \end{array} \end{array} \begin{array}{c} A \\ B \\ B \end{array} \qquad \begin{array}{|c|c|} \hline \alpha & \beta \\ \hline \gamma & \rho \\ \hline \end{array}^{\,2} = \begin{array}{|c|c|} \hline \alpha & \beta \\ \hline \gamma & \rho \\ \hline \end{array}, \tag{24}$$

where α, β, γ, and ρ are complex multiples of the appropriate dimensional identity matrices.

The criterion $(1 - \tilde{P})\Gamma_0 \tilde{P} = 0$ yields

$$(1 - \alpha)c - \beta d = 0, \tag{25a}$$

$$-\gamma c + (1 - \rho)d = 0. \tag{25b}$$

If $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \rho \end{smallmatrix}\right)$ is a nontrivial idempotent operator ($\neq 0, I$), then

$$\alpha + \rho = 1, \tag{26a}$$

$$\alpha\rho - \beta\gamma = 0. \tag{26b}$$

The condition $(1 - \tilde{P})\Gamma_0 \tilde{P} = 0$ and Eqs. (25) can be rewritten, using (26a), as

$$\rho c - \beta d = 0, \quad \alpha d - \gamma c = 0,$$

but (26b) assures us that there is always a nontrivial solution

$$\beta = \frac{(1 - \alpha)c}{d}, \quad \gamma = \alpha \frac{d}{c}.$$

So, whatever be the specific nonzero values of c, d (and any values of a, b) there exists a family of projectors (one for each choice of α),

$$\tilde{P} = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & \alpha & (1 - \alpha)c/d \\ \hline 0 & \alpha(d/c) & (1 - \alpha) \\ \hline \end{array}, \tag{3.5}$$

such that $\tilde{P}^2 = \tilde{P}$ and $\tilde{P} \neq 0$ or I, and $(1 - \tilde{P})\Gamma_0 \tilde{P} = 0$. In case $c = 0$, choose $\alpha = 0$; similarly if $d = 0$, choose $\alpha = 1$ (c and d cannot both be zero). Therefore, any equation transforming under $A \oplus B \oplus B$ must be a reducible equation. The structure of such equations, and general theorems regarding the condition on $T(\Lambda)$ when one is forced into reducible equations are discussed elsewhere.[3,4]

An example of the use of these results to prove a given equation to be irreducible can be found in the references.[5]

ACKNOWLEDGMENT

We are grateful to Professor N. Mukunda for critical reading of the manuscript.

*Research supported in part by the Energy Research and Development Administration E(40-1)3992.

[1]M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964).

[2]M. A. K. Khalil, "Reducible Relativistic Wave Equations," CPT, University of Texas preprint ORO 260, 1976 (to be published).

[3]M. A. K. Khalil, "Relativistic Wave Equations," unpublished Doctoral Dissertation (University of Texas, Austin, 1976).

[4]M. A. K. Khalil, "The Structure of Barnacled Relativistic Wave Equations," CPT, University of Texas preprint CPT 261, 1975 (to be published).

[5]M. A. K. Khalil, "Properties of a 20-component Spin 1/2 Relativistic Wave Equation," CPT, University of Texas preprint ORO 253, 1976 (to be published in Phys. Rev. D).