A criterion for reducibility of a relativistic wave equation

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In general when one writes a relativistic wave equation of the form \((-i\Gamma \cdot \partial + m)\psi(x) = 0\), that transforms covariantly under some representation \(\Lambda \rightarrow T(\Lambda)\) of \(SL(2, \mathbb{C})\), it is nontrivial to determine whether or not the equation is irreducible or to avoid ending up with a reducible equation; especially if \(T(\Lambda)\) contains repeating irreducible representations. In this paper a simple criterion is given by which one can determine whether or not an equation is irreducible. It is shown that if \(\Gamma_\mu\) have any invariant subspace at all, then that subspace must be a representation space of some combination of \(SL(2, \mathbb{C})\) representations in \(T(\Lambda)\). Knowing this, it is shown that a wave equation is reducible if and only if there exists some idempotent projector \(\tilde{P}\) such that \((1 - \tilde{P})\Gamma_\mu \tilde{P} = 0\) other than \(\tilde{P} = 0\) or \(I\). A method for constructing all possible admissible \(\tilde{P}\)'s is given. A simple example of the technique is also given.

I. INTRODUCTION

Relativistic wave equations of the form:

\[ (-i\Gamma \cdot \partial + m)\psi(x) = 0 \]

can be reducible or irreducible. The meaning of "reducible" in the context of relativistic wave equations is precisely formulated in the next section.

It turns out that reducible equations have particular properties that make theories based on such equations equivalent to simpler theories, both in the free field and interacting cases.\(^\dagger\)

\(^\dagger\) It is therefore important to know when a given equation is reducible, and hence, possibly equivalent to a simpler equation. The structure of reducible equations has been studied in Ref. 2.

When one constructs a wave equation, it is in general nontrivial to insure that the equation is reducible. The main concern of this paper is to formulate the simplest possible criterion by which one can determine whether a given wave equation is reducible or not. Such a criterion is formulated in the next section.

Finally, in Sec. III, a simple example is considered that illustrates the use of the criterion.

II. CRITERION FOR REDUCIBILITY

\[ (-i\Gamma_\mu \partial^\mu + m)\psi(x) = 0 \]

is a relativistic wave equation that transforms covariantly under a representation of \(SL(2, \mathbb{C})\), \(\Lambda \rightarrow T(\Lambda)\):

\[ T(\Lambda) = \overset{\pi}{\otimes} \alpha_j T_j(\Lambda). \]

The set of matrices \(\{\Gamma_\mu\}\) may be regarded as a set of linear transformations over a linear space \(R(N)\), where \(N\) is the number of rows (or columns) of \(\Gamma_\mu\).

\[ \{\Gamma_\mu\} = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}. \]

Definition 1: \(\{\Gamma_\mu\}\) is called a reducible set \(\iff\) there exists a proper subspace \(R_0 \subset R(N)\) such that

\[ \Gamma R_0 \subset R_1 \subset R(N) \]

\[ \nabla \Gamma \in \{\Gamma_\mu\}. \]

The subspace \(R_1\) is called an invariant subspace of \(\{\Gamma_\mu\}\) (IS of \(\Gamma_\mu\)).

Definition 2: Suppose \(R_j\) with \(j = 1, \ldots, L\) is a collection of all the invariant subspaces of \(\{\Gamma_\mu\}\), then

\[ R_0 = \bigcup_{j=1}^L R_j \]

is an invariant subspace of \(\{\Gamma_\mu\}\) called the maximal invariant subspace of \(\{\Gamma_\mu\}\).

The space \(R(N)\) is a representation space of \(\Lambda \rightarrow T(\Lambda)\), i.e., \(T(\Lambda)\) act as linear transformations on \(R(N)\).

Definition 3: If \(R_1\) is an IS of \(\{\Gamma_\mu\}\) and \( TR_1 = R_{\mu} \), then \(R_1\) is called an invariant \(SL(2, \mathbb{C})\) subspace of \(\{\Gamma_\mu\}\).

Lemma 1: \(R_0 \subset R(N)\) in Eq. (4) is an invariant \(SL(2, \mathbb{C})\) subspace of \(\{\Gamma_\mu\}\).

Proof: \(\Gamma R_0 \subset R_0 \nabla \Gamma R_0 \subset R_0 \).

Recall that

\[ T^{-1} \Gamma_\mu = \Lambda_\mu^{-1} \Gamma_\nu, \]

\[ \Gamma_0 R_0 \subset T \Lambda_\mu^{*} \Gamma_\mu R_0. \]

Suppose \(\phi_0\) is any vector in \(R_0\);

\[ \Gamma_\mu T \phi_0 = T \Lambda_\mu T \phi_0, \]

now

\[ \Lambda_\mu^{*} \Gamma_\mu \phi_0 \in R_0 \]

since for any value of \(\mu = 0, 1, 2, 3\) the right-hand side of (7) is a linear combination of \(\Gamma_\mu\) acting on \(\phi_0\), and each \(\Gamma_\mu \phi_0 \in R_0\) hence (8) follows. Now according to (7)

\[ \Gamma_\mu T \phi_0 = T \phi_0, \phi_0, \phi_0 \in R_0 \]

or

\[ \Gamma_\mu T R_0 \subset T R_0. \]

Therefore, \(T R_0\) is also an invariant subspace of \(\{\Gamma_\mu\}\), but \(R_0\) is a maximal invariant subspace of \(\{\Gamma_\mu\}\), hence

\[ TR_0 \subset R_0. \]

Recall that \(T\) are nonsingular transformations so

\[ TR_0 = R_0. \]

In the following discussion a criterion for determining whether or not \(\{\Gamma_\mu\}\) is a reducible set, will be formulated. For later convenience the bases for \(R(N)\) will be chosen to be completely reducible bases (CRB's). A CRB is any basis in which \(T(\Lambda)\) is block diagonal, and each block corresponds to an irreducible representa—
tion, \( T_j(\Lambda) \) of \( \text{SL}(2, \mathbb{C}) \), in Eq. (2). Recall that altogether there are \( M \) irreducible representations where \( M = \sum_{j=1}^{\infty} m_j \), there are \( m_j \) copies of the irreducible representation \( T_{m_j}(\lambda) \) for each \( j \); so \( T(\Lambda) \) is \( M \times M \) in block form. Similarly \( R(N) \) is a direct sum of \( M \) subspaces, each being a representation space of some irreducible representation of \( \text{SL}(2, \mathbb{C}) \) in (2),

\[
R(N) = \bigoplus_{j=1}^{M} \alpha_j R_{m_j}.
\]

One can now define \( \text{SL}(2, \mathbb{C}) \) projectors represented by Hermitian matrices in some CRB, that are idempotent and act on \( R(N) \) such that

\[
P_\sigma R(N) = R_{m_\sigma},
\]

where \( R_{m_\sigma} \) is an \( \text{SL}(2, \mathbb{C}) \) subspace of \( R(N) \), i.e., a direct sum of some \( R_{m_j} \), in (2),

\[
R_{m_\sigma} = \bigoplus_{j=1}^{M} \alpha_j R_{m_j},
\]

\[
0 \leq \alpha_j \leq \alpha_j.
\]

The subscript \([\sigma]\) denotes all the different combinations of \( \alpha_j \) that satisfy (14b). \( P_\sigma \) can be written in an \( n \times n \) block form, where each block \( j \) corresponds to the connection of the \( \alpha_j \) representations \( T_j \) and is thus an \( \alpha_j \times \alpha_j \) block matrix that is idempotent. The \( P_\sigma \) do not mix vectors from spaces corresponding to different representations of \( \text{SL}(2, \mathbb{C}) \) in (2), but can mix vectors corresponding to the same representation of which there are \( \alpha_j \) copies for a given \( T_j \) in (2). In this form it is obvious that

\[
\{P_\sigma, T(\Lambda)\} = 0 \quad \forall \Lambda \in \text{SL}(2, \mathbb{C}).
\]

Since \( R_0 \) is an \( \text{SL}(2, \mathbb{C}) \) subspace of \( R(N) \), there exists projectors of the type described above, \( P_0 \), such that

\[
P_0 R(N) = R_0
\]

and every vector \( \phi_0 \in R_0 \) can be written as \( P_0 \phi \) for some vector \( \phi \in R(N) \).

Lemma 2: \( \{\Gamma_\sigma\} \) reducible \( \iff \exists \) some \( \tilde{P}_0 \) satisfying (16) that is idempotent such that

\[
(I - \tilde{P}_0)\Gamma_\sigma\tilde{P}_0 = 0.
\]

Proof: \( \tilde{P}_0 \) is not required to be Hermitian. Suppose \( \{\Gamma_\sigma\} \) is reducible, then by Lemma 1 \( \exists \text{an } \text{SL}(2, \mathbb{C}) \) subspace \( R_0 \) of \( R(N) \) \( \ni \Gamma_\sigma R_0 \subseteq R_0 \). Choose a particular basis for \( R(N) \) where \( \phi \in R(N) \) is in the form

\[
\phi = \begin{bmatrix}
\phi_0 \\
\phi_1
\end{bmatrix},
\]

where every vector of \( R_0 \) can be written

\[
\phi_0 \in R_0, \quad \phi_0 = \begin{bmatrix}
\phi_0 \\
\phi_1
\end{bmatrix}.
\]

Now in this basis \( \Gamma_\sigma \) still has the property

\[
\Gamma_\sigma R_0 \subseteq R_0,
\]

since this property is basis independent. Now clearly one may pick a \( \tilde{P}_0 \) in this basis such that

\[
\tilde{P}_0 = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}.
\]

\( J \) is a \( q \times q \) identity matrix where \( q = \text{dim} R_0 \). Note that (20) is an idempotent operator (actually in its Jordan canonical form). Apply the matrix \( \{\Gamma_\sigma R_0 - \tilde{P}_0 \Gamma_\sigma R_0\Gamma_\sigma R_0\} \) to arbitrary vector \( \phi \in R(N) \),

\[
\Gamma_\sigma \tilde{P}_0 \phi = \Gamma_\sigma \tilde{P}_0 \phi = \Gamma_\sigma \phi - \Gamma_\sigma \phi = 0.
\]

The last step follows because \( \Gamma_\sigma \phi = \phi_0 \in R_0 \) and in this basis \( \tilde{P}_0 \phi = \phi_0 \phi_0 \) for any vector \( \phi_0 \in R_0 \). The only matrix that maps every vector \( \phi \in R(N) \) into 0 is the zero matrix

\[
\tilde{P}_0 \phi_0 = 0.
\]

Since an idempotent projector exists in one basis satisfying (21) (by construction) such an operator exists in all bases; i.e., for any \( \Gamma_\sigma = \Gamma_\sigma \Gamma_\sigma \Gamma_\sigma \), the operator \( \Gamma_\sigma \Gamma_\sigma \Gamma_\sigma \) is such an operator. In particular a \( \tilde{P}_0 \) exists in all CRB's satisfying (17). The quality of \( \tilde{P}_0 \) being idempotent is preserved by all nonsingular transformations but the hermiticity is not. So in general for any CRB, only \( \tilde{P}_0 = \tilde{P}_0 \) will be required.

Now assume that some \( \text{SL}(2, \mathbb{C}) \) invariant operator \( \tilde{P}_0 \) exists such that (17) holds. Suppose \( \phi \) is any vector in \( R(N) \), then \( \tilde{P}_0 \phi = \phi_0 \in R_0 \) and by (17),

\[
\Gamma_\sigma \phi_0 = \Gamma_\sigma \phi = \phi_0 \Rightarrow \Gamma_\sigma \phi_0 = \tilde{P}_0 \Gamma_\sigma \phi_0.
\]

Now \( \Gamma_\sigma \phi_0 \in R(N) \) since \( \Gamma_\sigma : R(N) \rightarrow R(N) \). The right-hand side of the above equation,

\[
\tilde{P}_0 \Gamma_\sigma \phi_0 = \tilde{P}_0 \phi = \phi_0 \in R_0,
\]

so one can see that \( \Gamma_\sigma : \phi_0 \phi_0 \phi_0 \phi_0 \phi_0 \in R_0 \). Since every vector \( \phi_0 \in R_0 \) can be written as \( \tilde{P}_0 \phi \) for some \( \phi \in R(N) \) one concludes that \( \Gamma_\sigma \) maps every vector \( \phi_0 \in R_0 \) into some other vector of \( R_0 \), therefore \( \Gamma_\sigma R_0 \subseteq R_0 \) and \( \{\Gamma_\sigma\} \) is a reducible set.

Lemma 3: \( (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0 \iff (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0 \).

Proof: It is obvious that \( (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0 \)

\[
\Rightarrow (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0.
\]

On the other hand, suppose \( (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0 \), then since \( \Gamma_\sigma = \Gamma_\sigma \Gamma_\sigma \Gamma_\sigma \Gamma_\sigma \), where \( \Gamma_\sigma \) are the generators of the boosts in the \( i \) direction for the representation \( \Lambda \rightarrow T(\Lambda) \), one notices that

\[
(1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = t((1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 N_1 - (1 - \tilde{P}_0) N_1 \Gamma_\sigma \tilde{P}_0) = 0.
\]

Since

\[
[\tilde{P}_0, N_1] = 0,
\]

\[
(1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = t((1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 N_1 - N_1 (1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0) = 0.
\]

The conclusion one can draw is that \( \{\Gamma_\sigma\} \) is a reducible set \( \iff \) there exists an idempotent \( \text{SL}(2, \mathbb{C}) \) projector \( \tilde{P}_0 \) such that

\[
(1 - \tilde{P}_0) \Gamma_\sigma \tilde{P}_0 = 0.
\]

\( \tilde{P}_0 \), being an \( \text{SL}(2, \mathbb{C}) \) projector, is of the following form

\[\text{SL}(2, \mathbb{C})\]
in the CRB indicated:

\[
\begin{array}{cccccc}
\alpha_1 T, & \alpha_2 T, & \ldots & \alpha_n T, \\
\bar{P}_0 & \bar{P}_1 & \bar{P}_2 & \ldots & \bar{P}_n \\
\alpha_1 T & & & & \\
\alpha_2 T & & & & \\
\vdots & & & & \\
\alpha_n T & & & & \\
\end{array}
\]

\[\bar{P}_j = \bar{P}_j^0\] for each \( j = 1, \ldots, n.\)

If one finds that no such projector exists other than \( \Gamma_0 \) or \( \Gamma_1 \), then one may conclude that \( \{ \Gamma_n \} \) is an irreducible set.

Relativistic wave equations where \( \{ \Gamma_n \} \) is a reducible set are called reducible wave equations.

III. AN EXAMPLE

Consider any two irreducible, interlocking representations of \( SL(2, \mathbb{C}) \) denoted \( A \) and \( B \). For illustrating the technique consider any wave equation that can be constructed so as to transform under \( T(\Lambda) = A \oplus B \oplus B \), then \( \Gamma_0 \) is the following:

\[
\begin{array}{ccc}
\alpha & \beta & A \\
\gamma & \rho & B \\
\end{array}
\]

\[\Gamma_0 =
\begin{array}{ccc}
aD_1 & bD_1 & A \\
cD_2 & dD_2 & B \\
\end{array}
\]

(23)

where \( a, b, c, \) and \( d \) are complex numbers, assumed to be nonzero (no requirements of unique mass or spin are imposed). The most general allowed \( \bar{P} \) is

\[
\begin{array}{ccc}
\alpha & \beta & A \\
\gamma & \rho & B \\
\end{array}
\]

\[\bar{P} =
\begin{array}{ccc}
1 & 0 & 0 \\
\alpha & (1-\alpha)c/d & 0 \\
0 & (a(d/c) & (1-\alpha) \\
\end{array}
\]

(3.5)

such that \( \bar{P} = \bar{P} \) and \( \bar{P} \neq 0 \) or \( I \), and \( (1 - \bar{P}) \Gamma_0 \bar{P} = 0. \) In case \( c = 0 \), choose \( \alpha = 0 \); similarly if \( d = 0 \), choose \( \alpha = 1 \) \((c \) and \( d \) cannot both be zero). Therefore, any equation transforming under \( A \oplus B \oplus B \) must be a reducible equation. The structure of such equations, and general theorems regarding the condition on \( T(\Lambda) \) when one is forced into reducible equations are discussed elsewhere.\(^4\)

An example of the use of these results to prove a given equation to be irreducible can be found in the references.\(^5\)

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