PENCILS OF RAYS IN WAVE OPTICS

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Every partially coherent field of illumination may be associated with a collection of generalized pencils of rays which, in turn, furnish a representation of the wave field. The hybrid Wolf function which represents the radiant energy of a thin pencil is shown to be a real, but not necessarily positive definite quantity. Such generalized pencils are used to represent certain familiar interferograms and recognize elementary results.

Preamble. The picture of light as a collection of pencils of rays has been used in elementary optics in connection with the inverse square law of photometry and in the context of the formation of shadows and partial shadows. Most of radiometry and radiative transfer theory has used this picture [1]. Yet light consists of waves; and the phenomena of interference, diffraction and polarization bear testimony to this. The question whether pencils of rays could be used for representing a wave field was raised by Gabor [2] but we owe to Wolf [3] [‡1] the first definitive treatment of the problem in the context of radiometry.

This letter addresses itself to the general question of pencils of light rays in wave optics. It improves on, and completes, the work initiated by Wolf and shows that by generalizing the concept of pencils of rays to include tamasic rays ("dark light") we are able to obtain a simple picture of interferograms and diffraction patterns. The intrinsic positivity of photometric intensity is preserved. For ease of presentation and in the interests of brevity the polarization of light is ignored in this letter: light is treated in terms of scalar waves.

The two-point function in the far field. Let $\phi_\nu(xyz) = e^{i\nu t}\phi(xyzt)$ be the analytic signal corresponding to a unique frequency ν. For most practical purposes a sufficiently accurate (far field) representation of the propagation is provided by the Huygens' construction:

$$\phi_\nu(\boldsymbol{r}) = i\nu \iint dx'\, dy'\, |\boldsymbol{r}-\boldsymbol{r}'|^{-1} \exp(i\nu|\boldsymbol{r}-\boldsymbol{r}'|)\, \phi_\nu(\boldsymbol{r}') \,. \tag{1}$$

On the other hand the state of illumination is given by the two-point coherence function

$$\Gamma(x_1y_1z_1; x_2y_2z_2) = \langle \phi^+(x_1y_1z_1)\, \phi(x_2y_2z_2)\rangle \,. \tag{2}$$

This definition is accurate both in classical optics [6] and in quantum optics [7]. Using eqs. (1) and (2) for quasi-monochromatic light we obtain the far field two-point function:

$$\Gamma(x_1y_1z_1; x_2y_2z_2) = \frac{\nu^2}{z_1z_2}\iiiint dx_1'\, dy_1'\, dx_2'\, dy_2'\, \Gamma(x_1'y_1'0; x_2'y_2'0)$$
$$\times \exp\{-(i\nu/z_1)[(x_1-x_1')^2+(y_1-y_1')^2+z_1^2]^{1/2}\} \exp\{(i\nu/z_2)[(x_2-x_2')^2+(y_2-y_2')^2+z_2^2]^{1/2}\} \,. \tag{3}$$

‡1 The function $W(\boldsymbol{k},\boldsymbol{r})$ by eq. (5) is not quite identical with the function introduced by Wolf. His definition corresponds to the choice of a normal-ordered phase space function while the definition used in this paper corresponds to the Weyl–Wigner ordering. The definition (5) is reminiscent of the definition of the Wigner distribution for a wave function [4]. But Wolf's function refers to the illumination in a field and is a different quantity. The double Fourier transform of $W(\boldsymbol{k},\boldsymbol{r})$ (called the ambiguity function) has been studied by Papoulis [5].

›r most cases of interest we can make the approximations

$$_1-z_2)^2 \ll (z_1+z_2)^2\,, \quad (x-x')^2+(y-y')^2 \ll z^2$$

d writing

$$= \tfrac{1}{2}(x_1+x_2)\,, \quad y=\tfrac{1}{2}(y_1+y_2)\,, \quad z=\tfrac{1}{2}(z_1+z_2)\,,$$

$$x_1-x_2 \quad \eta=y_1-y_2\,, \quad \zeta=z_1-z_2\,,$$

we have

$$\Gamma(x+\tfrac{1}{2}\xi\, y+\tfrac{1}{2}\eta\, z+\tfrac{1}{2}\zeta; x-\tfrac{1}{2}\xi\, y-\tfrac{1}{2}\eta\, z-\tfrac{1}{2}\zeta) = (\nu^2/z^2)\exp\{i\nu[\zeta+(x\xi+y\eta)/z]\}$$
$$\times \iiiint d\xi'\, d\eta'\, dx'\, dy'\, \Gamma(x'+\tfrac{1}{2}\xi'\, y'+\tfrac{1}{2}\eta'\, 0; x'-\tfrac{1}{2}\xi'\, y'-\tfrac{1}{2}\eta'\, 0)$$
$$\times \exp[(i\nu/z)(x'\xi'+y'\eta')]\exp[(-i\nu/z)(\xi x'+\eta y')]\,. \tag{4}$$

This is the general formula to which we will have to make reference more than onc‹

Generalized pencils of rays. With *any* two-point function $\Gamma(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{\sigma}, \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{\sigma})$ we associate a collection of *generalized* pencils of rays $W(\boldsymbol{k},\boldsymbol{r})$ through every point $\boldsymbol{r}$ in the field of illumination according to

$$W(\boldsymbol{k},\boldsymbol{r}) = (2\pi)^{-3}\int d^3\sigma\, e^{-i\boldsymbol{k}\cdot\boldsymbol{\sigma}}\,\Gamma(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{\sigma}; \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{\sigma})\,. \tag{5}$$

Since by definition (2) Γ is a hermitian kernel W is automatically *real* but not necessarily positive. We shall see below that this lack of positivity is a generalization which is essential in that it accounts for typically wave-optical phenomena. We interpret $W(\boldsymbol{k},\boldsymbol{r})$ to represent the intensity of each pencil of rays per unit solid angle with frequency $\nu=|\boldsymbol{k}|$ along the direction of $\boldsymbol{k}$ in the neighborhood of the point $\boldsymbol{r}$.

The integral of Wolf's function $W(\boldsymbol{k},\boldsymbol{r})$ with respect to the orientation of $\boldsymbol{k}$ is the spectral photometric flux at the point $\boldsymbol{r}$. By definition this is equal to $\Gamma(\boldsymbol{r};\boldsymbol{r})$ which is nonnegative despite the possible nonpositivity of $W(\boldsymbol{k},\boldsymbol{r})$.

We have now the essential ingredients for the complete identification of a wave field with collections of pencils of rays. In this letter we shall confine our attention to four illustrative cases depending upon the illumination at $z=0$.

(i) Incoherent illumination. In this case

$$\Gamma(x'+\tfrac{1}{2}\xi'\, y+\tfrac{1}{2}\eta'\, 0; x'-\tfrac{1}{2}\xi'\, y'-\tfrac{1}{2}\eta'\, 0) = \rho(x'y')\,\delta(\xi')\,\delta(\eta')\,. \tag{6}$$

Hence

$$W(k_1k_2k_3, xyz) = \iint dx'\, dy'\, \rho(x'y')\,\delta(k_3-\nu)\,\delta(k_1-\nu(x-x')/z)\,\delta(k_2-\nu(y-y')/z)\,. \tag{7}$$

This corresponds to a collection of pencils of rays with total strength $\rho(x'y')$ emanating from each point $x'y'$ in the primary illumination and passing through the points xyz.

(ii) Coherently illuminated double slit. If the two slits are in phase we can calculate the two-point function by elementary techniques:

$$\Gamma(x+\tfrac{1}{2}\xi\, y+\tfrac{1}{2}\eta\, z+\tfrac{1}{2}\zeta; x-\tfrac{1}{2}\xi\, y-\tfrac{1}{2}\eta\, z-\tfrac{1}{2}\zeta)$$
$$= \exp(-i\nu\zeta)\{\exp[i\nu(x+a)\,\xi/z]+\exp[i\nu(x-a)\,\xi/z]+2\exp(i\nu\xi/z)\cos(2\nu ax/z)\}\,. \tag{8}$$

The Wolf function corresponding to this is

$$W(k_1k_2k_3, xyz) = \delta(k_2)\,\delta(k_3-\nu)[\delta(k_1-\nu(x+a)/z)+\delta(k_1-\nu(x-a)/z)+2\cos(2\nu ax/z)\,\delta(k_1-\nu x/z)]\,. \tag{9}$$

This corresponds to two cylindrically symmetrical pencils of rays from either slit source together with a generalized pencil of rays from the *line midway between the slits* possessing a pronouced angular dependence. The latter, therefore, acts as a fictitious generalized source. This dependence is sinusoidal with respect to the angle of rotation relative to the axis of the fictitious generalized source. If the two slits were out of phase by an angle ϕ the last term on the right-hand side would have the factor $2\cos(2\nu ax/z + \phi)$ in place of $2\cos(2\nu ax/z)$. It is interesting to note that while the generalized pencil contains tamasic rays (from Sanskrit, tamas = darkness) with negative energy flux, the total photometric flux at any point would be nonnegative.

The inclusion of tamasic rays in addition to luminous rays is an essential component to make up the interferogram and, hence, a general wave-optical field of illumination. The rectilinear propagation of both the luminous and the tamasic rays gives the familiar linear scaling laws of the interference pattern.

(iii) Multiple slit interferogram. When there are more than two slits or point sources the calculation becomes somewhat more complicated. We now have $N(N+1)/2$ terms replacing the right-hand side of eq. (9). N of these terms correspond to the geometric optics for the N sources, but the $N(N-1)/2$ terms are generalized fictitious source contributions with their characteristic angular dependences. These pencils originate at the points midway between sources and contain both luminous and tamasic rays.

A special case of such a multiple interference pattern is the diffraction grating. Our methods enable us to construct a geometric theory of the behavior of a diffraction grating.

(iv) Diffraction by a slit. As a limiting case where the number of sources becomes infinite we consider coherent illumination of an aperture of width $2a$. The two-point function can be computed to be

$$\Gamma(x + \tfrac{1}{2}\xi\, y + \tfrac{1}{2}\eta\, z + \tfrac{1}{2}\zeta; x - \tfrac{1}{2}\xi\, y - \tfrac{1}{2}\eta\, z - \tfrac{1}{2}\zeta)$$

$$\quad {}^{2} \exp[i\nu(\zeta \quad x\xi/z)] \int_{-a} \mathrm{d}\xi' \exp(i\nu x'\xi'/z) \frac{z}{\nu x} \sin(\nu x|\xi'|/z)$$

$$= 2\nu^{-2}x^{-2} \exp[i\nu(\zeta - x\xi/z)]\,[\cos(\nu a\xi/z) \quad \cos(2\nu ax/z)]$$

The corresponding Wolf function is:

$$W(k_1k_2k_3, xyz) \quad 2(\nu x)^{-2}\,\delta(k_2)\,\delta(k_3 - \nu)$$
$$\times\,[\delta(k_1 - \nu(x+a)/z) + \delta(k_1 - \nu(x-a)/z) - 2\cos(2\nu ax/z)\,\delta(k_1 - x\nu/z)]\,. \qquad (10)$$

This pattern is reminiscent of the two-slit pencils (9). The essential difference is the $(1/x^2)$ envelope modifying the nature of the generalized pencil with its central hump of double width.

Remarks. In summary, in a number of typically wave phenomena [8] we have shown how to view the wave-optical field as a collection of generalized pencils of rays containing both luminous and tamasic rays. The tamasic rays are absent when the initial illumination is incoherent, thus once again showing the intimate relation between coherence properties of light and superpositions leading to interferometry.

In conclusion, it may be noted that the quasimonochromaticity assumption only serves to simplify the calculation and could be relaxed. More important is the recognition that since all quantities that are dealt with in this letter are linear in the two-point function, the classical and quantum treatments are indistinguishable [9]: so, if we choose, the entire discussion may be considered to be within the framework of quantum optics.

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