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DISCRETE STATES BURIED IN THE CONTINUUM

This paper is dedicated to the memory of Professor B. Jouvet who saw clearly the close connection between renormalizable theories like the Lee model and the nonrenormalizable four-spinor interaction at a time when most physicists would not seriously entertain even the possibility.

ABSTRACT. It is possible to have discrete normalizable eigenstates of energy with eigenvalues overlapping the continuous spectrum. A simple collection of examples exhibiting this behaviour is presented in this paper.

INTRODUCTION

For most systems of interest in scattering theory there is a continuous spectrum corresponding to those states which represent the scattered particles being far from the scattering center and thus being presumably unaffected by the interaction. In addition to these (usually degenerate) continuum states there may be one or more states with discrete energy eigenvalues which correspond to bound states which represent the component parts which are held together and do not separate outside each other's range of interaction. In general, such states have energies less than the range of the continuous spectrum. For a (local) potential of finite range we can deduce this from the recognition of the phenomenon of quantum tunneling. In such cases we would not have eigenstates of discrete energy but rather a resonance corresponding to eigenstates of complex energy in the analytic continuation of the theory. These considerations appear so natural that we are tempted to consider that discrete eigenstates should not have energies which overlap the continuous spectrum.

There is a trivial sense in which this expectation can be violated: if one or more quantum numbers are conserved by the Hamiltonian. In this case we can consider the various sectors labelled by these quantum

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numbers to be distinct families. In the following discussion we shall ignore the possibility of a discrete state in one sector overlapping the continuum in another.

The first example of a local potential with a (normalizable) discrete energy eigenstate buried in the continuum was discovered by Wigner and Von Neumann. The potential in this case is a non-monotonic function that does not vanish at infinity. I know that there are other local potentials known with a single discrete 'bound state' in the continuum.

In this paper, I shall show that one can construct simple Hamiltonians which exhibit one or more discrete states overlapping with the continuum. None of these are local potential systems but are reminiscent of quantum field theory. In all these cases one can compute the scattering amplitude, and at each energy where there is a discrete state, there is a characteristic behaviour of the scattering amplitude.

The plan of the paper is as follows. In Section 2 we outline a Friedrich model theory. To simplify the algebraic manipulations as much as possible we go to the S-wave sector and realize the state space as a function space over the (unperturbed) energy. The spectrum and the (generalized) eigenvectors are obtained in closed form. The question of the existence of discrete energy eigenstates buried in the continuum is posed and solved. Section 3 deals with a system with more than one discrete energy states buried in the continuum. The possibility of repulsive separable potentials generating a discrete state in the continuum is explored in Section 4. With a view to illustrate the essential ideas most simply we have absorbed the phase volume into the normalizations to keep the kinematic complications to a minimum.

2. The Friedrich–Lee Example

Consider a collection of harmonic oscillators with every oscillator mode coupled to a base mode. The Hamiltonian of such a system is of the form

$$H = a_0^+ a_0 m + \sum_j a_j^+ a_j w_j + \sum_j f_j (a_j^+ a_0 + a_0^+ a_j). \tag{1}$$

The number operator

$$N = a_0^+ a_0 + \sum_j a_j^+ a_j \tag{2}$$

is a constant of the motion having nonnegative integer eigenvalues. We shall be interested in the special sector with $N = 1$. The state is then specified by the amplitudes ψ_0, ψ_j. We can now take the limit when the discrete index j takes on continuously many values with the corresponding natural frequencies taking the continuum of values

$$0 < W < \infty. \tag{3}$$

The states are now represented by a vector with a component ψ_0 and a function $\psi(W)$. The scalar product of two vectors ψ, ϕ is given by

$$(\psi, \phi) = \int_0^{\infty} dW\, \psi(W)^* \phi(W) + \psi_0^* \phi_0 \tag{4}$$

The action of the Hamiltonian on the vectors is given by the equations.

$$\begin{aligned}(H\psi)_0 &= m\psi_0 + \int_0^{\infty} dW f(W)\psi(W), \\ (H\psi)(W) &= f(W)\psi_0 + W\psi(W).\end{aligned} \tag{5}$$

This is the familiar Friedrich–Lee model with the explicit continuum (idealized) eigenvectors

$$\begin{aligned}{}_{\lambda}\psi_0 &= f(\lambda)/\alpha(\lambda + i\varepsilon), \\ {}_{\lambda}\psi(W) &= \delta(\lambda - W) + f(W)f(\lambda)/(\lambda - W + i\varepsilon)\alpha(\lambda + i\varepsilon),\end{aligned} \tag{6}$$

with

$$\alpha(Z) = Z - m - \int_0^{\infty} dW (Z - W)^{-1} f^2(W). \tag{7}$$

If m is such that

$$\alpha(0) = -m + \int_0^{\infty} dW . W^{-1} . f^2(W) > 0 \tag{8}$$

there exists a real negative number M such that

$$\alpha(M) = 0. \tag{9}$$

Then there is a discrete state with the eigenvector:

$$\begin{aligned}{}_M\psi_0 &= (\alpha'(M))^{-1/2}, \\ {}_M\psi(W) &= f(W)(\alpha'(M))^{-1/2}/(M - W).\end{aligned} \tag{10}$$

The set of continuum eigenstates form a complete set if (9) is not satisfied provided $f(W)$ is nonvanishing for $0 < W < \infty$; if (9) is satisfied we will have to include (10) to obtain the complete set:

$$\int_0^\infty d\lambda\, {}_\lambda\psi(W)_\lambda\psi(W')^* + {}_M\psi(W)_M\psi^*(W') = \delta(W - W'),$$

$$\int_0^\infty d\lambda\, {}_\lambda\psi(W)_\lambda\psi_0^* + {}_M\psi(W)_M\psi_0^* = 0,$$

$$\int_0^\infty d\lambda\, {}_\lambda\psi_{0\,\lambda}\psi_0^* + {}_M\psi_{0\,M}\psi_0^* = \qquad (11)$$

If the value of m exceeds $\int_0^\infty dW.W^{-1}.f^2(W)$ there would be no discrete state at all. We would, instead have a resonance indicative of an unstable particle in the calculated scattering amplitude:

$$T(W) = f^2(W)/\alpha(W + i\varepsilon) \qquad (12)$$

which would correspond to a complex zero of $\alpha(Z)$ near the real axis.

Let us now raise the question: Can we have a discrete (normalizable) eigenstate for an eigenvalue $M > 0$? Such a solution cannot exist if $f(W)$ is nonvanishing for all W in the range $0 < W < \infty$. Therefore we relax this condition and allow for $f(W)$ to have an isolated zero in this domain. This would make it possible for $\alpha(Z)$ to have a zero in this range.

We therefore consider $f(W)$ to be such that the following twin conditions hold:

$$f(M) = 0, \qquad \alpha(M) = 0. \qquad (13)$$

Then there is a discrete solution of the form (10) with $M > 0$. Clearly $T(M) = 0$.

There is also a continuum of solutions given by (6). These solutions together with the discrete solution still form a complete set.

It is to be noted that the eigenvalue M is now degenerate: there is, in addition to (10), a (nonnormalizable) solution belonging to the continuum with

$$\psi(W) = \delta(M - W), \qquad \psi_0 = 0 \qquad (14)$$

which corresponds to an uncoupled ('plane wave') excitation. We note that if $f(W)$ vanishes faster than $(M - W)$ as $W \to M$, then (14) and (10) are orthogonal.

Since $\alpha(Z)$ defined by (7) is such that the real part of

$$\alpha'(Z) = 1 + \int_0^\infty dW f^2(W)(W - Z)^{-2}$$

is nonnegative along the real axis, it follows that $\alpha(Z)$ can have at most only one zero. Therefore, there cannot be more than one discrete state whatever be the nature and magnitude of the coupling function $f(W)$.

If we need more than one such discrete state we must generalize the model.

3. Model with many discrete eigenstates

A generalization of the model presented in the previous section is furnished by replacing the Hamiltonian (1) by

$$H = \sum_\alpha a_\alpha^+ a_\alpha m_\alpha + \sum_j a_j^+ a_j W_j + \sum_{\alpha j} f_{\alpha j}(a_j^+ a_\alpha + a_\alpha^+ a_j)$$

and proceeding to the continuum limit

$$0 < W < \infty$$

for W_j. The states are now represented by vectors with discrete components ψ_α and a complex-valued function $\psi(W)$ with the scalar product:

$$(\psi, \phi) = \sum_\alpha \psi_\alpha^* \phi_\alpha + \int_0^\infty dW \psi(W)^* \phi(W). \tag{17}$$

The action of (16) on a vector leads to

$$(H\psi)_\alpha = m_\alpha \psi_\alpha + \int_0^\infty dW f_\alpha(W) \psi(W)$$

$$(H\psi)(W) = W\psi(W) + \sum_\alpha f_\alpha(W)\, \psi_\alpha. \tag{18}$$

The eigenstates can now be worked out. The continuum solutions with

$0 < \lambda < \infty$ are of the form

$$\psi(W) = \delta(\lambda - W) + (\lambda - W + i\varepsilon)^{-1} \sum_\alpha f_\alpha(W)\psi_\alpha,$$

so that

$$\int_0^\infty \mathrm{d}W f_\alpha(W)\psi(W) = \sum_\beta \left\{ \int_0^\infty \mathrm{d}W f_\alpha(W) f_\beta(W)(\lambda - W + i\varepsilon)^{-1} \right\} \psi_\beta + f_\alpha(\lambda)$$
$$= \sum_\beta F_{\alpha\beta}(\lambda + i\varepsilon)\psi_\beta + f_\alpha(\lambda), \tag{19}$$

and

$$(\lambda - m_\alpha)\psi_\alpha = \sum_\beta F_{\alpha\beta}(\lambda + i\varepsilon)\psi_\beta + f_\alpha(\lambda)\theta(\lambda) \tag{20}$$

$$(\lambda - W)\psi(W) = \sum_\beta f_\beta(W)\psi_\beta. \tag{21}$$

Since (20) gives a set of simultaneous inhomogeneous linear equations, we can write down the solutions in terms of the determinants of those coefficients. The essential point is that the function

$$\Delta(Z) = \det\{Z\delta_{\alpha\beta} - m_\alpha\delta_{\alpha\beta} - F_{\alpha\beta}(Z)\} = \det D_{\alpha\beta}(z) \tag{22}$$

plays the role previously assigned to the function $\alpha(Z)$. Unlike $\alpha(Z)$, this new function $\Delta(Z)$ can have multiple zeros atmost equal in number to the range of the index α in (16)–(20).

We now search for discrete solutions of the eigenvalue problem

$$H\psi = \lambda\psi. \tag{23}$$

These may arise if $\Delta(Z) = 0$ for $Z < 0$ and are then analogous to the bound states of the model studied in the last section: the second term on the right-hand side of (20) is then missing and the equations are homogeneous in the ψ_β. As long as the determinant of $D_{\alpha\beta}$ vanishes but none of its principal minors have their determinant vanish the ψ_α are uniquely determined except for a multiplicative constant; this constant can be chosen for each solution by requiring the eigenvectors to be normalized.

If it turns out that for any such solution $\lambda = M > 0$, the coupling functions $f_\alpha(M)$ all vanish then, in complete analogy with (10) we have a discrete solution

$$\sum_\alpha {}_M\psi_\alpha f_\alpha(W)/(M - W) = {}_M\psi(W), \tag{24}$$

$$\sum_{\beta} F_{\alpha\beta}(M)_M\psi_\beta = (\lambda - m_\alpha)_M\psi_\alpha. \tag{25}$$

By suitable choice of parameters we can have a number of genuine 'bound state' solutions with $M < 0$, a number of discrete states buried in the continuum and possibly a number of resonances.

4. Repulsive potentials leading to discrete states

Another generalization of the model Hamiltonian (1) is obtained by choosing

$$H = \sum_j a_j^+ a_j W_j \pm \sum_{jk} f_j f_k (a_j^+ a_k + a_k^+ a_j). \tag{26}$$

The $\pm$ signs correspond to repulsive and attractive potentials. Restricting attention to the sector where

$$\Sigma a_j^+ a_j = 1 \tag{27}$$

and proceeding to the continuum limit with $0 < W < \infty$ we get the separable potential model with states represented by functions $\psi(W)$ with scalar product

$$(\psi,\phi) = \int_0^\infty \mathrm{d}W \psi^*(W)\phi(W) \tag{28}$$

The action of the Hamiltonian is

$$(H\psi)(W) = W\psi(W) \pm \int_0^\infty \mathrm{d}W' f(W')\psi(W') f(W). \tag{29}$$

The continuum solutions with eigenvalues

$$0 < \lambda < \infty$$

have the form

$$_\lambda\psi(W) = \delta(\lambda - W) \pm \frac{f(W) \int_0^\infty \mathrm{d}W' f(W')\psi(W'}{(\lambda - W + i\varepsilon)}$$

Defining

$$D(Z)=1\mp\int_0 \mathrm{d}W f^2(W)/(Z-W). \tag{30}$$

We can write down the complete solution

$$_{\lambda}\psi(W)=\delta(\lambda-W)\pm f(\lambda)f(W)(\lambda-W+i\varepsilon)^{-1}D^{-1}(\lambda+i\varepsilon). \tag{31}$$

The question naturally arises if there are any discrete solutions. From (29)–(31) we can see that this is possible in general only if

$$D(M)=0, \qquad M<0. \tag{32}$$

Since $\pm D'(Z)>0$ for $Z<0$ and $D(-\infty)=1$, it follows that unless we choose attractive potentials we cannot have a bound state. When we do have a bound state it has the representative wave function

$$_{M}\psi(W)=\frac{f(W)}{M-W}(-D'(M))^{-1/2}. \tag{33}$$

For a repulsive potential there are no bound states, but there is still the possibility of a discrete state buried in the continuum. The condition for the existence of such a state is

$$D(M)=0, \qquad f(M)=0. \tag{34}$$

The explicit solution is

$$_{M}\psi(W)=f(W)(M-W)^{-1}(D'(M))^{1/2}. \tag{35}$$

There is also a continuum solution for this energy

$$_{\lambda}\psi(W)|_{\lambda=M}=\delta(M-W) \tag{36}$$

which will become orthogonal to the discrete solution if $f(W)$ vanishes faster than $M-W$ as $W\to M$.

It is curious that a repulsive potential leads to a discrete normalizable state.

5. Concluding remarks

The discussion in this paper was calculated to show that suitable interactions can lead to discrete normalizable energy eigenstates with energy eigenvalues buried in the continuum. It is possible to design a number of such systems with one or more such states.

In all such cases there is an energy degenerate continuum solution which coincides with the continuum solution for the noninteracting Hamiltonian. Consequently, the scattering amplitude of the interacting system always vanishes at this energy for all the systems considered. This is in marked contrast to the usual case of an unstable state which leads to a reasonance.

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