Form of relativistic dynamics with world lines

N. Mukunda* and E. C. G. Sudarshan

Center for Particle Theory and Department of Physics, University of Texas, Austin, Texas 78712

(Received 18 August 1980)

In any Hamiltonian relativistic theory there are ten generators of the Poincaré group which are realized canonically. The dynamical evolution is described by a Hamiltonian which is one of the ten generators in Dirac’s generator formalism. The requirement that the canonical transformations reproduce the geometrical transformation of world points generates the world-line conditions. The Dirac identification of the Hamiltonian and the world-line conditions together lead to the no-interaction theorem. Interacting relativistic theories with world-line conditions should go beyond the Dirac theory and have eleven generators. In this paper we present a constraint dynamics formalism which describes an eleven-generator theory of N interacting particles using 8(N + 1) variables with suitable constraints. The (N + 1)th pair of four-vectors is associated with the uniform motion of a center which coincides with the center of energy for free particles. In such theories dynamics and kinematics cannot be separated out in a simple fashion.

I. INTRODUCTION

The problem of describing relativistic interacting point particles in the Hamiltonian formalism of dynamics has been with us for a long time. More than three decades ago Dirac⁵ proposed a very general formalism—the so-called generator formalism—for relativistic Hamiltonian dynamics. The basic idea is that one should have a phase-space structure on which the group of inhomogeneous Lorentz transformations—the Poincaré group φ—is realized by canonical transformations; and this requires that we have ten generators that give us a Poisson-bracket realization of the Lie algebra of φ. In Dirac’s work the question of dynamical evolution within one inertial frame was subsumed under the more general question of representing a change of inertial frame corresponding to any element of φ; so “equations of motion” were particular cases of more general equations describing the effect of infinitesimal Poincaré transformations. Thus both “Hamiltonian” and “interaction” were contained among the ten generators of φ.

Soon after Dirac’s work, Thomas,⁶ Bakamjian,⁷ and Foldy⁸ succeeded in constructing theories of interacting relativistic point particles in the generator formalism and their work was extended to quantum theory by Macfarlane, Jordan, and Sudarshan⁹ and by Mukunda.⁶ Thomas and Bakamjian gave up the concept of the objective reality of particle world lines. The idea here is that when the observations in two inertial frames θ and θ′ are related by the canonical transformation representing the appropriate element of φ, then in any particular state of motion, θ′ and θ do in fact “see” the same set of world lines in spacetime, i.e., the canonical and the geometrical (Lorentz) transformations are compatible when applied to the particle positions. We shall refer to this as the requirement of invariant world lines. It had already been recognized much earlier by Pryce⁵ that this requirement entailed conditions that definitely went beyond merely obeying the structure relations of φ in a canonical framework. These world-line conditions (WLC’s) were put into a convenient phase-space form in Poisson-bracket notation by Currie, Jordan, and Sudarshan.⁶,⁹ They succeeded in proving that, within the Dirac generator formalism, relativistic invariance, interaction, and invariant world lines were mutually incompatible for a system of point particles. This is the no-interaction theorem.

In response to the challenge represented by this theorem, there have recently been several papers that succeed in constructing models of relativistic interacting point particles within a constrained Hamiltonian formalism. Constrained Hamiltonian dynamics, originally devised by Dirac⁶ to handle singular Lagrangian systems, can be made the basis of a self-contained independently existing formalism for construction of theories. Its property of capturing the virtues of both the Lagrangian and conventional Hamiltonian forms of dynamics, and yet being more flexible than either, has been specially stressed by Komar.¹¹ The characteristic feature of the constraint formalism, if it does not arise from an underlying Lagrangian, is that the identification of the mathematical variables with physical quantities is delayed until the stage at which all needed constraint conditions have been given.

At first it appeared that the assumption underlying the no-interaction theorem that had apparently been given up in these models with interaction was the existence of invariant world lines. However, a more careful examination presented elsewhere has shown that the matter is much deeper than that: One has gone beyond the Dirac
FORM OF RELATIVISTIC DYNAMICS WITH WORLD LINES

We begin, however, with an artificially expanded $(8N + 8)$-dimensional phase space $\Gamma'$ with independent "four-vector" variables $Q_a, P_a, \xi_{ab}, \eta_{ab}, a = 1, 2, \ldots, N$. The only nonzero Poisson brackets (PB's) are taken to be

$$\{Q_a, P_b\} = \delta_{ab}, \quad \{\xi_{ab}, \eta_{bc}\} = \delta_{ac}\delta_{bc}.$$

The intention is that $Q$ should be the space-time position of a center, $P$ its constant four-momentum, $\xi$ the position of particle $a$ relative to the center in the frame in which the center is at rest, and $\eta_{ab}$ the momentum of particle $a$ in that frame. These intentions will be realized, and the system itself acquires definition, through a series of constraints.

On $\Gamma'$ the Poincaré group $\theta$ can be canonically realized in this way:

$$Q_a = Q_a + Q_a, \quad P_a = P_a + P_a, \quad \xi_{ab} = \xi_{ab} + \eta_{ab} \eta_{ab}.$$

Note that $Q$ alone responds to a space-time translation. The ten infinitesimal generators of the above canonical transformations, providing a PB realization of the Lie algebra of $\theta$, are

$$J_{\mu\nu} = Q_{\mu} P_{\nu} - Q_{\nu} P_{\mu} + \sum_{a=1}^{N} (\xi_{a\mu} \eta_{a\nu} - \xi_{a\nu} \eta_{a\mu}),$$

for the homogeneous Lorentz group, and $P_{\mu}$ itself for the translations. Let us immediately reduce the number of variables from $8N + 8$ to $6N + 8$ by imposing the $2N$ independent second-class constraints

$$P \cdot \xi = 0, \quad P \cdot \eta = 0, \quad a = 1, \ldots, N.$$

The nontrivial parts of $\xi, \eta$ therefore transform as vectors under the little group of $\theta$ with respect to $P_{\mu}$. These conditions define a constraint surface $\Gamma$ in $\Gamma'$. It is clear that the mappings (2) carry $\Gamma$ into itself. We can explicitly get rid of the extra variables by passing from the PB's (1) to the Dirac brackets (DB's) corresponding to the constraints (4). Since we shall hereafter not have to make any reference to $\Gamma'$ or the PB's (1), we shall use the symbol $\{ , \}$ for the DB's obtained at this stage. The nonvanishing DB's are

$$\{Q_a, Q_b\} = -\frac{1}{p^2} \sum_a (\xi_{a\mu} \eta_{a\nu} - \xi_{a\nu} \eta_{a\mu}), \quad \{Q_a, P_b\} = \delta_{ab},$$

$$\{Q_a, \xi_{ab}\} = -\frac{\xi_{ab}}{p^2}, \quad \{Q_a, \eta_{ab}\} = -\eta_{ab} \frac{P_{ab}}{p^2},$$

$$\{\xi_{ab}, \eta_{ab}\} = \delta_{ab} \left( \eta_{ab} - \frac{P_{ab}}{p^2} \right).$$

From now on it is understood that the conditions (4) hold as identities. Based on these brackets (5), the ten variables $J_{\mu\nu}, P_{\mu}$ continue to yield a
realization of the Lie algebra of $\mathfrak{g}$; so a finite element $(\lambda, a)$ in $\mathfrak{g}$ is realized by a transformation $R(\lambda, a)$ on $\Gamma$ which is canonical with respect to the brackets (5). The action of $R(\lambda, a)$ on $q$, $p$, $\xi$, $\eta$ is exactly as before, namely, it is given by Eq. (2), even though these are not all independent variables.

At this point we introduce a pair of variables $(\eta_a, p_a)$ for each value of $a$, with the intention that in the final theory these shall be interpreted as “four-position” and “four-momentum” of particle number $a$:

\begin{equation}
q_a = Q + \xi_a, \tag{6a}
\end{equation}

\begin{equation}
p_a = \eta_a + (m^2 - \eta^2_a + V_a)^{1/2} P/(p^2)^{1/2}, \quad a = 1, 2, \ldots, N. \tag{6b}
\end{equation}

These variables do not have any special canonical bracket properties. The $V_a$ are a set of “interaction potentials,” to be further described shortly. To preserve covariance under $R(\lambda, a)$, we only demand that each $V_a$ be invariant under the action of $R(\lambda, a)$; we also exclude any dependence of $V_a$ on $P$ for a reason to be explained later; thus each $V_a$ can be any Lorentz-invariant function of $\xi, \eta$. Then each $q_a$ transforms like $q$, i.e., like space-time position, and each $p_a$ like $P$, under $\mathfrak{g}$.

At this stage, since $\Gamma$ has dimension $6N+8$, we are in need of eight more independent constraints. Only then can we make contact with a system of $N$ particles. The definition of $p_a$, together with the notion that $P$ is the total four-momentum, suggests immediately four independent constraints we may impose:

\begin{equation}
C = \sum_a \eta_a = 0, \quad D = (P^2)^{1/2} - \sum_a (m^2 - \eta^2_a + V_a)^{1/2} = 0. \tag{7}
\end{equation}

Note that the component of the $C$ equation in the direction of $P_a$ vanishes by virtue of (4) so that there are only three independent $C$ constraints and one $D$ constraint. Then it will indeed be true that the $p_a$ all add up to $P$. Now the theory of constrained systems reminds us that if $C, D$ form a first-class system, one naturally has room for the introduction of four more constraints as foils to these. Then our counting of degrees of freedom would be just right. The three independent functions in $C$ do have vanishing brackets with one another. Let us now demand that the brackets $[P, C]$ also vanish. The simplest way to achieve this is to make each $[C, V_a]$ vanish. From the brackets (5) we see that this can be arranged if each $V_a$ depends only on the differences of $t'$'s, and on the $\eta$'s, in a Lorentz-invariant way. This restriction on $V_a$ will now be assumed.

The region in $\Gamma$ where the four independent constraints (7) hold is a region $\Sigma$ of dimension $6N+4$. of $C$ and $D$, $R(\lambda, a)$ maps $\Sigma$ onto itself: namely, we have

\begin{equation}
\{J_{a\nu} \text{ or } P_a, \ C \text{ or } D\} = 0. \tag{8}
\end{equation}

Let us now consider the canonical transformations generated by $C, D$. Because of the first-class property, (i) these transformations map $\Sigma$ onto itself, (ii) they form a four-parameter group. In the present case, this group happens to be Abelian since the brackets among $C$ and $D$ vanish identically, i.e., strongly. Starting with some point in $\Sigma$, we can apply all the canonical transformations in this Abelian group and thus build up the “orbit” of that point under this group: This will be a region of dimension four, contained entirely within $\Sigma$, and we will call it a “sheet.” We can display a sheet $S$ graphically in this way: Let $X$ collectively symbolize all the coordinates $Q, P, \xi, \eta$ of a point on $\Sigma$. Set up the differential equation

\begin{equation}
\frac{dX(\sigma)}{d\sigma} = [X(\sigma), D] u + [X(\sigma), C] v, \quad P \cdot v = 0, \quad X(0) = X. \tag{9}
\end{equation}

If we imagine solving these equations for all possible choices of the four free multipliers $u$, $v$ and collect together all the points of $\Sigma$ that have been reached in this way from the given $X$, we get the sheet $S$ containing $X$. [It is of course understood that the arguments of $D$, $C$ in Eq. (9) are the variables $X(\sigma)$ themselves.]

Each sheet $S$ is determined by any one of the points it contains, and $\Sigma$ is seen to be the union of disjoint sheets. It follows that the sheets form a $6N$-parameter family. Because of (8) one also sees that the transformation $R(\lambda, a)$ maps each sheet $S$ entirely onto another sheet $S'$. One can now consider whether it is physically reasonable to imagine that each sheet $S$, as a whole, corresponds to one state of motion of an $N$-particle system. If it were so, then one could say that the change of inertial frame $\theta \rightarrow \theta' = (\lambda, a) \theta$ is implemented by the action of $R(\lambda, a)$ on the sheets. However, we discard this possible interpretation for this reason: If a sheet $S$ be chosen, and all points in it be used in order to reconstruct a set of world lines in space-time, relative to an inertial frame $\theta$, we will not end up with a line but a whole region of space-time for each particle. $S$ is of dimension four, so as a point $X$ varies over it, the variables $q_a^\lambda(X)$ for each $a$ form a four-parametric set of points in space-time. Our aim is to define a “state of motion” for the system in such a way that it will let us construct in an unambiguous way, in each frame $\theta$,
a set of $N$ world lines. Once such a definition has been made, we can then develop a WLC to
test if these are invariant world lines or not.
It is clear that this can be achieved by picking,
on each sheet $S$, a one-dimensional curve $\mathcal{C}$ in
some way, and then parametrizing the points on
$\mathcal{C}$ by an evolution parameter $\tau$. To fix $\mathcal{C}$ on $S$
requires three independent constraints $x_r$,
$r = 1, 2, 3$, dependent only on the phase-space vari-
bles; and a fourth one $x_4$ with explicit $\tau$ depend-
cence can then be added on to parametrize $\mathcal{C}$:

$$x_r(Q, P, \xi, \eta) = 0, \quad r = 1, 2, 3$$

$$x_4(Q, P, \xi, \eta, \tau) = 0.$$  \hspace{1cm} (10)

For these constraints to do what we want them to,
it is necessary that along with the $C, D$ they form a
second-class set. If for the four independent condi-
tions contained in (7) we write $B_r$, we must en-
Sure that

$$\text{det}(x_r, B_s) \neq 0.$$  \hspace{1cm} (11)

We can then define each curve $\mathcal{C}$ to represent one
state of motion. The rest of the sheet $S$ contain-
ing $\mathcal{C}$ can be discarded as being of no physical
consequence. With this definition of the term
state of motion, we can see that in every such
state, in a frame $\theta$, a definite set of $N$ world
lines may be drawn: Along a $\mathcal{C}$, the variables
(6a) appear in the form $Q^\mu(\tau)$.

Let the inverse to the matrix (11) be written as
($\alpha_{rs}$):

$$\alpha_{rs}(x_r, B_s) = \delta_{rs}.$$  \hspace{1cm} (12)

If we can explicitly eliminate the constraints $x, B$,
we will then be left with exactly $6N$ independent
variables. To this end we pass from the brackets
(5) to the (final) Dirac brackets

$$\{f, g\} = \{f, g\}$$

$$-\alpha_{rs}(\{f, B_r\} \{x_s, g\} - \{f, x_s\} \{B_r, g\})$$

$$-\{f, B_r\} \alpha_{rs}(x_s, x_r) \alpha_{rt}(B_t, g).$$  \hspace{1cm} (13)

We now note several important things: Because of
(8), the $J_{\mu\nu}, P_\mu$ continue to furnish a realiza-
tion of the Lie algebra of $\Phi$ if we compute their
DB's $\{\cdot, \cdot\}$ rather than their brackets according to
(5). They may therefore be seen to build up a realiza-
tion of $\Phi$ by transformations $R^*(\Lambda, \alpha)$ which
are canonical with respect to the DB $\{\cdot, \cdot\}^*$. The
relationship between $R(\Lambda, \alpha)$ and $R^*(\Lambda, \alpha)$, for
the same $(\Lambda, \alpha)$ in $\Phi$, is this: $R^*(\Lambda, \alpha)$ maps $\Sigma$ onto $\Sigma$ as
$R(\Lambda, \alpha)$ does; if $R(\Lambda, \alpha)$ takes a sheet $S$ into $S'$,
$R^*(\Lambda, \alpha)$ also carries $S$ to $S'$; beyond this, $R^*(\Lambda, \alpha)$
will automatically carry the $\mathcal{C}$ determined on $S$
by (10) onto the $\mathcal{C}$' determined on $S'$, while
preserving the value of $\tau$. This last characterization
is meaningless for $R(\Lambda, \alpha)$.

The DB (13) is a nondegenerate bracket for a
generalized phase space of $6N$ dimensions. It is
cumbersome to exhibit explicitly a set of $6N$
independent variables spanning this space, but
there is no need to do so either; we can work with all
the variables $Q, P, \xi, \eta$, and $Q_\alpha$, with $q_\alpha$ and
$p_\alpha$ being derived quantities given by Eqs. (6). The
constraints (4), (7), and (10) are understood to
hold as identities, and these are consistent with
the DB (13). From this point, we physically iden-
tify $R^*(\Lambda, \alpha)$ as representing the change of inertial
frame $\theta \rightarrow \theta^\prime = (\Lambda, \alpha) \theta$. It is a transformation can-
onical with respect to the final physical brackets
(13).

Along the curve $\mathcal{C}$ on an $S$, the equation of
motion (9) with unspecified multipliers $w, v$ be-
moves more definite: We must assign to $w$ and the
$v$'s such values as will maintain the constraints
(10) all along $\mathcal{C}$. Since $x_4$ alone carries explicit
$\tau$ dependence we find

$$\frac{dx_r}{d\tau} = 0 = \frac{\partial x_r}{\partial \tau} + \{x_r, B_s\} v_s = 0$$

$$\Rightarrow v_r = -\alpha_{rs} \frac{\partial x_s}{\partial \tau} = -\alpha_{rs} \frac{\partial x_s}{\partial \tau}.$$  \hspace{1cm} (14)

For a general dynamical variable $f$ the equation of
motion along $\mathcal{C}$ would be

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, B_r\} v_r$$

$$= \frac{\partial f}{\partial \tau} - \{f, B_r\} \alpha_{rs} \frac{\partial x_s}{\partial \tau}.$$  \hspace{1cm} (15)

We now recall a general result that tells us that
this equation of motion can always be put into
Hamiltonian form using the DB (13). If in (15) it
is understood that $f$ is some function of $Q, P,$
$\xi, \eta$, and $\tau$; it is this last explicit $\tau$ dependence
that gives the term $\partial f/\partial \tau$ in (15). It is a fact that
one can find $6N$ independent variables (in infini-
tely many ways) which will form a basic set with
respect to the DB (13) and with the further proper-
ty that their DB's with one another are independent
of $\tau$ when expressed in terms of themselves again.
Since $x_4$ carries explicit $\tau$ dependence, these $6N$
quantities might be explicitly dependent on $\tau$ as
well, when constructed as functions of $Q, P,$
$\xi, \eta$. If an $f$ occurring in (15) is now expressed in terms
of such a set of $6N$ quantities and $\tau$ (and this can be
done since all the constraints $B_r, \chi_r$ are opa-
tive), the meaning of the term "explicit dependence
on $\tau$" may change. Using the sign $\partial f/\partial \tau$ for the
partial derivative in this new sense, we are now
assured that a Hamiltonian $\mathcal{C}$ will exist so that in
terms of the DB (13) the general equation of motion
is
A physically well-motivated choice of the $\chi_\theta$ will be given in Sec. III; subject to that, the physical system of $N$ interacting point particles can at this point be defined. It is a system whose $6N$ independent phase-space variables are the $8N+8$ variables $Q, P, \xi_\theta, \eta_\theta$ modulo the constraints (4), (7), and (10) which can in fact be used as identities. Brackets among general dynamical variables are computed starting from Eq. (13). We have ten generators $J_\mu, P_\mu$ to implement changes of inertial frame, and one generator $\mathcal{K}$ to determine dynamical evolution in one frame. The DB's among the former reproduce the Lie algebra of $\mathfrak{g}$. The physical interpretation of $\tau$ depends on the form of $\chi_\theta$. In any state of motion, in a given frame $\theta$, a definite set of $N$ world lines can be drawn. All the generators $J_\mu, P_\mu$ (and $\mathcal{K}$ too) are constants of motion. From the fact that each $V_\nu$ is restricted to be a Lorentz-invariant [in the sense of the transformation rules (2)] function of the $\eta_\theta$ and the differences of the $\xi_\theta$, we easily find

$$\{Q_\mu, D\} = P_\mu/(P^2)^{1/2}, \quad \{Q_\mu, C_\nu\} = 0. \quad (17)$$

It follows that in every state of motion, $Q_\mu$ traces a straight world line parallel to $P_\mu$.

III. INVARIANT WORLD LINES—THE WLC

We now take up the question: Are the world lines, whose existence has been secured, invariant? The answer depends on the set of constraints $\chi_\theta$.

Let $\theta, \theta'$ be two inertial frames related by an infinitesimal element of $\mathfrak{g}$ in this way:

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu} x_\nu + a^{\mu}, \quad |\omega|, |a| \ll 1. \quad (18)$$

This is a geometrical connection between the coordinates $x, x'$ assigned in $\theta, \theta'$ to a single event in space-time. Let the $N$-particle system be in a state of motion which, in $\theta$, leads to the set of world lines $q^a_\mu(\tau)$: The single point on $\mathcal{C}$ with parameter value $\tau$ gives us one point on each particle's world line, and as $\tau$ varies along $\mathcal{C}$ these $N$ space-time points trace $N$ paths in space-time. Points in space-time sharing the same $\tau$ need not be simultaneous in the physical sense in $\theta$. Now by the definition of the canonical formalism, the space-time coordinates to be used in $\theta'$ to build up world lines in that frame, for the same state of motion, are given by

$$q'^{\mu}_{a}(\tau) = q^{\mu}_{a}(\tau) + \{G, q^{\mu}_{a}(\tau)\}^{*},$$

$$G = \frac{1}{2} \omega^{\mu\nu} J_{\mu\nu} + a^{\mu} P_{\mu}. \quad (19)$$

This is the action of the infinitesimal transformation $R^*(\Lambda, a)$ for the element (18) in $\mathfrak{g}$. The set of world lines drawn in $\theta'$ will be the same as those which were drawn in $\theta$ if each $q^a_{\mu}(\tau)$ is the geometrical transform, according to (18), of $q^a_{\mu}(\tau) + \delta_{a,\tau}$ for some $\delta_{a,\tau}$. We must only require that the two sets of lines in $\theta$ coincide; it need not happen that $N$ points sharing a common $\tau$ value in $\theta'$ do so also in $\theta$. The WLC is thus the requirement that there be $N$ infinitesimal expressions $\delta_{a,\tau}$, each linear in $\omega^{\mu\nu}$ and $a^{\mu}$, such that

$$\{G, q^{a}_{\mu}\}^{*} = \omega^{\mu\nu} q_{\nu\mu} + a^{\mu} + (\frac{\partial q^{(a)}_{\mu}}{\partial \tau} + \{q^{(a)}_{\mu}, \mathcal{K}\}^{*}) \delta_{a,\tau}, \quad (20)$$

$$a = 1, 2, \ldots, N.$$ 

This is the condition that the geometrical and canonical rules of transformation for space-time positions be compatible. It is written exclusively in terms of the physical brackets and involves all the eleven generators $J_{\mu}, P_{\mu}, \mathcal{K}$.

To check the validity of (20) in any given case, and since $\mathcal{K}$ is difficult to exhibit, we reexpress the WLC in a form in which just the brackets (5) appear:

$$\{q^{a}_{\mu}, B_{\nu}\}{q^{b}_{\sigma}, \chi_{\nu}} = \{q^{a}_{\mu}, B_{\nu}\}{q^{b}_{\sigma}, \chi_{\nu}} \frac{\partial \chi_{\mu}}{\partial \tau} \delta_{a,\tau}, \quad (21)$$

$$a = 1, 2, \ldots, N.$$ 

It is clear that suitable expressions $\delta_{a,\tau}$ obeying these WLC can definitely be found, if we choose $\chi_1, \chi_2, \chi_3$ to be explicitly invariant or form a covariant set under $R(\Lambda, a)$, and $\chi_4$ alone breaks this invariance. For then we have the possibility

$$\delta_{a,\tau} = \{G, \chi_{a}\} \frac{\partial \chi_{a}}{\partial \tau}, \quad a = 1, 2, \ldots, N \quad (22)$$

and the invariance of world lines is guaranteed.

One simple choice of $\chi_4$, expressing our original intention that $Q$ should in some sense be the center of the system, is

$$\chi_4 = \sum_{a} \epsilon_a \xi_a, \quad \epsilon_a > 0, \quad \sum_{a} \epsilon_a = 1,$$

$$\chi_4 = P \cdot Q - \tau. \quad (23)$$

There are just four independent $\chi$'s here, and the $\epsilon_a$ could be either constants or dynamical variables at this stage. $Q$ then emerges as the center in the sense

$$Q = \sum_{a} \epsilon_a q_{a}. \quad (24)$$

We restrict the $\epsilon_a$ to be Lorentz scalars, with respect again to $R(\Lambda, a)$. They could depend in any way on the $\eta$'s, but if we permit them to depend on the $\xi$'s only through the $V_\alpha$, then because
the brackets $\{V_a, C_b\}$ vanish we will have the particularly simple relation

$$\{X_a, C_b\} = 0.$$  

With the choice (23) for the $\chi$'s the WLC (21) are all obeyed if we take

$$\delta_a \tau = -a \cdot P, \quad a = 1, 2, \ldots, N.$$  

Note that $\tau$ is measured in units of action.

To pin down the $\epsilon_a$ further, we examine the limiting case when there are no interactions and all $N$ particles are free. It is reasonable to demand that this situation correspond to $V_a = 0$. In that case, each $P_a$ is a constant of motion and from Eqs. (6), (4), and (23) we find

$$P_a^2 = m_a^2, \quad P \cdot q_a = \tau, \quad a = 1, 2, \ldots, N.$$  

Let us check how the $q_a$ vary with respect to $\tau$.

In the equation of motion (15), written as

$$df = \frac{\partial f}{\partial \tau} + \{f, D\} \cdot w + \{f, C\} \cdot v, \quad P \cdot v = 0,$$

$w$ and the $v$'s can easily be determined. Since $V_a = 0$, both $\eta_a$ and $\epsilon_a$ are constants of motion; and furthermore,

$$\{X_a, D\} = (P_a^2)^{1/2}, \quad \{X_a, C_b\} = 0,$$

$$\{\xi_a, D\} = \frac{\eta_a}{(m_a^2 - \eta_a^2)^{1/2}}, \quad \{\xi_a, C_b\} = \frac{P_a P_b}{P^2}.$$  

From all these relations we find

$$\frac{d\xi_a}{d\tau} = 0 \Rightarrow w = \frac{1}{(P \cdot v)^2},$$

$$\frac{d\eta_a}{d\tau} = 0 \Rightarrow v = \frac{1}{(P \cdot v)^2} \sum \frac{\epsilon_a \eta_a}{(m_a^2 - \eta_a^2)^{1/2}}.$$  

Therefore, the equation of motion for $\xi_a$, and hence for $q_a$, is

$$i_\epsilon = \frac{1}{(P \cdot v)^2} \frac{\eta_a}{(m_a^2 - \eta_a^2)^{1/2}} + v,$$

$$q_\mu = \frac{1}{(P \cdot v)^2} \frac{\eta_\mu}{(m_a^2 - \eta_a^2)^{1/2}} + \frac{P_\mu}{(P \cdot v)^2} + v_\mu.$$  

For the free case, each $q_a$ must trace a world line parallel to the corresponding (constant) $P_a$, so we must arrange for $v_\mu$ to vanish as a consequence of all the constraints in hand. Given the forms of $C_a$ in Eq. (7) and $v_\mu$ in Eq. (30), we are led to the natural choice

$$\epsilon_a = \frac{(m_a^2 - \eta_a^2)^{1/2}}{\sum_a (m_a^2 - \eta_a^2)^{1/2}}, \quad a = 1, 2, \ldots, N.$$  

$Q$ is thus seen as the center of energy (rather than center of mass) of the collection of $N$ free particles. For the interacting case we have two alternatives: (i) We may either retain the choice (22) or (ii) we may make the specific choice

$$\epsilon_a = \frac{(m_a^2 - \eta_a^2 + V_a)^{1/2}}{\sum_a (m_a^2 - \eta_a^2 + V_a)^{1/2}}.$$  

More generally, we may choose

$$\epsilon_a = \frac{\phi_a(\eta_a, m_a, V)}{\sum \phi_a(\eta, m, V)},$$

$$\phi_a(\eta, m, 0) = (m_a^2 - \eta_a^2)^{1/2}.$$  

In either case, the bracket relations (25) are valid. These two possible choices for the $\epsilon_a$ must be viewed as leading to two distinct physical models.

For this physical system, the following interpretation emerges. In any given state of motion, as seen in the fixed inertial frame $\theta$, the parameter $\tau$ is the time in the center-of-momentum frame. That frame is inertial but dependent on the particular state of motion. $\xi_a$ is the spatial part of the position of particle $a$, relative to the center $Q$, in the center-of-momentum frame; $\eta_a$ is the spatial part of the momentum of particle $a$ in that frame. The single-particle variables $q_a, p_a$ have no specific significance in the sense of canonical-bracket properties with one another. Their important properties are their Poincaré transformation laws and the numerical values of the $q_a$ in any state of motion. These values lead to invariant world lines. In the free case each $P_a$ is on its mass shell and stays constant, while $q_a$ traces the appropriate straight line uniformly. In any state, even in the presence of interactions, the center $Q$ traces a straight world line

$$Q(\tau) = Q(0) + \tau P / P^2, \quad P \cdot Q(0) = 0.$$  

IV. DISCUSSION AND OUTLOOK

Dynamical theory of direct particle interactions within a relativistic framework involves a number of demanding conditions. For $N$ particles, interacting or otherwise, we need $6N$ phase-space variables. Relativistic invariance is best incorporated using $2N$ four-vectors for the particles together with $2N$ invariant conditions eliminating $2N$ variables. In this paper we have followed the suggestion of Foldy and Rohrlich to introduce explicitly at the start an $(N + 1)$th pair of four-vectors $Q, P$. The $2N$ second-class transversality conditions (4) immediately reduce the number of independent variables from $8N + 8$ to $6N + 8$; we then need to get rid of eight more variables. Only then do we arrive at a physical system identifiable with $N$ particles.

This is done in two stages. First, the four constraints (7) allowing us to identify $P$ with the sum
of the $p$ are adopted. These four are first-class constraints. At this stage with the $2N$ second-class constraints (4) eliminated using the brackets (5), the four first-class constraints define four-dimensional sheets. These sheets are relativistically invariant.\textsuperscript{11,13,15} But such a system is not identifiable with a collection of particles; instead, we should obtain a one-parameter curve on each sheet describing the dynamical evolution.

To accomplish this we impose four more constraints which together with the four constraints on $P$ make a second-class system of eight constraints, at least one of which is dependent on the evolution parameter $\tau$. Once these have been imposed we do have a one-parameter curve on each sheet describing the dynamical evolution of a system of $N$ particles.

The Hamiltonian $\mathcal{H}$ is the generator of the $\tau$ evolution and is thus closely related to the fourth first-class constraint and its $\tau$-dependent conjugate. In general it is not a generator of the Poincaré group, and in the present theory it is most convenient to choose it not to be a Poincaré generator.

An alternate method of making the reduction from $8N$ phase-space variables to $6N$ phase-space variables in a relativistically invariant way and to introduce interactions has been discussed by us in a recent paper\textsuperscript{14} along the lines suggested by Komar.\textsuperscript{11} There is no need to introduce the eight new phase-space variables $Q, P$. Instead, the starting variables are four-vectors $q_a, p_a$ with first-class constraints of the form

$$K_a = p_a^2 - m_a^2 + V_a(q, p) = 0, \quad a = 1, 2, \ldots, N.$$ 

If we take a point in the $2N$-dimensional phase space the canonical transformations generated by the $N$ quantities $K_a$ produce an $N$-dimensional sheet. These sheets are disjoint, and lead to a breakup of the entire phase space into $e^\lambda$ such sheets. The theory so developed for two particles is the same as the theory developed by the formalism here. But the two schemes are not fully equivalent when we go beyond two-particle systems. But that formalism is similar to the present theory in having eleven generators.

Canonical formalism was often associated with a Lagrangian theory and a Hamiltonian variational principle. But this is not necessarily so; the canonical formalism and the associated equations of "motion" can stand on their own. The theory of constraint dynamics can also be formulated independent of any Lagrangian.\textsuperscript{1} The present study does not involve any Lagrangian; the manifest covariance of the system is implemented in terms of an invariant set of constraints. The worldline conditions are implementable provided the Hamiltonian constraint is relativistically invariant.

The "coordinate" $Q$ describes the uniform motion under the dynamical evolution and may therefore be identifiable with the relativistic center of energy. This happens to be also the proper definition of the center for a collection of free particles to realize the correct relation between velocity and momentum. For interacting particles there is no unique choice of how the center is to be constructed; rather, we have a variety of possible choices.

The choice of constraints here is defined invariantly but dynamically; reduction to the unconstrained set of variables is therefore not kinematical but dynamical. This reflects itself in the Poisson-bracket relations (5) and the final brackets (13). Consequently, the framework developed in this paper makes use of a form of relativistic dynamics going beyond Dirac's forms of relativistic dynamics.\textsuperscript{1}

\textbf{ACKNOWLEDGMENT}

This work was supported by the Department of Energy and the National Science Foundation.

---

\textsuperscript{*}Permanent address: Indian Institute of Science, Bangalore 560012.

\textsuperscript{†}Present address: Physics Department, Duke University, Durham, North Carolina, 27706.

\textsuperscript{1}P. A. M. Dirac, Rev. Mod. Phys. \textbf{21}, 392 (1949).

\textsuperscript{2}L. H. Thomas, Phys. Rev. 85, 368 (1952).


(1948).


16 N. Mukunda, Phys. Scr. 21, 801 (1980).