Massless particles of high spin

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The existence of a Lorentz-covariant energy-momentum tensor in a quantum field theory has certain strong implications about the matrix elements of this Lorentz tensor between one-particle states of high-spin ($j \geq 3/2$) zero-mass particles, composite or elementary. A similar result obtains for theories with a Lorentz-covariant current for zero-mass particles, composite or elementary with spin $j \geq 1$. If these results are taken together with a continuity requirement on the matrix elements of the energy-momentum tensor and the current, respectively, one ends with a contradiction which can be removed only by assuming that such particles do not exist. We reexamine this conclusion and recognize that theories of higher-spin free particles are consistent but the continuity of the matrix elements do not obtain. The reason is traced to the abrupt change in the sign of the polarization when the momentum transfer changes from spacelike to null values. In the general case with particles of $j \geq 3/2$ and $j \geq 1$, respectively, we prove that the divergence of any tensor and any vector vanish between one-particle states.

I. INTRODUCTION

In the construction of Lagrangian quantum field theories of fields with half-integral spin $> \frac{1}{2}$ secondary constraints arise which make the theory inconsistent[1] in a number of cases. For all higher-spin field theories in interaction there is the possibility of anomalous propagation.[2] For charged fields with spin one and mass zero Lagrangian field theory becomes inconsistent.[3] But in all these cases the spin is the spin of the fields; and the particles are elementary.

In a recent paper remarkable for its elegance and simplicity Weinberg and Witten[4] have developed arguments based on Lorentz invariance that suggest that in all theories with a Lorentz-covariant energy-momentum tensor, composite or elementary particles of spin $j \geqslant \frac{3}{2}$ are forbidden. They furnish a direct derivation based on Poincaré invariance only. In view of the many profound applications of the theory[4] (such as the possible origin of the graviton as a composite particle emerging from a Lorentz-covariant field theory or the possibility of untrapped zero-mass quanta of spin >1 in theories with Adler-Bell-Jackiw triangle anomalies in the amplitudes of conserved currents) we have reexamined the basis of the derivations. In particular we consider the free field theories with higher spin and examine the behavior of the energy-momentum tensor matrix elements between one-particle states as a function of the momentum transfer. We find that for spacelike momentum transfer these matrix elements vanish identically, *but the normalization of the diagonal matrix elements remains valid.*

We rederive the results of Weinberg and Witten for the matrix elements of a Lorentz-covariant tensor (Ref. 5) $\theta^{\mu\nu}$ between one-particle states of a zero-mass particle of helicity j. A one-particle state is labeled by the three-momentum $\vec{p}$ and the invariant helicity j. Write

$$F^{\mu\nu}(\vec{p}', \vec{p}, j) = \langle \vec{p}', j | \theta^{\mu\nu} | \vec{p}, j \rangle \ . \tag{1}$$

We distinguish the cases $p' \cdot p \neq 0$ and $p' \cdot p = 0$. If $p' \cdot p \neq 0$ since $p + p'$ is timelike, $p' \cdot p > 0$. There exists, then, a frame in which $\vec{p}' + \vec{p} = \vec{0}$. In this frame we may write

$$F^{\mu\nu}(p', p, j) = \langle -\vec{p}, j | \theta^{\mu\nu} | \vec{p}, j \rangle \ . \tag{2}$$

Under rotations around the axis $\vec{p}$ through the angle ϕ, we get the transformation laws

$$R(\phi) | \pm \vec{p}, j \rangle = \exp(\pm ij\phi) | \pm \vec{p}, j \rangle \ . \tag{3}$$

Hence

$$\exp(2ij\phi) F^{\mu\nu}(p', p, j) = R^{\mu}_{\rho}(\phi) R^{\nu}_{\sigma}(\phi) F^{\rho\sigma}(p', p, j) , \tag{4}$$

where $R^{\mu}_{\rho}(\phi)$ is the action of the rotation ϕ around $\vec{p}$ on four-vectors. But since this contains only linear combinations of 1, $\exp(\pm i\phi)$, and $\exp(\pm 2i\phi)$ it follows that $F^{\mu\nu}$ must vanish identically for all $j \geqslant \frac{3}{2}$ as long as $p' \cdot p > 0$.

On the other hand, if $p' \cdot p = 0$, then $\vec{p}'$ and $\vec{p}$ are parallel. Hence

$$R(\phi) | \vec{p}, j \rangle = \exp(ij\phi) | \vec{p}, j \rangle , \quad R(\phi) | \vec{p}', j \rangle = \exp(ij\phi) | \vec{p}', j \rangle , \tag{5}$$

and

$$F^{\mu\nu}(p', p, j) = R^{\mu}_{\rho}(\phi) R^{\nu}_{\sigma}(\phi) F^{\rho\sigma} \ . \tag{6}$$

It follows that the components of $F^{\mu\nu}$ which are not along $\vec{p}$ (or $\vec{p}'$) or along the time axis must vanish; those along these directions do not necessarily vanish. For $p' \cdot p > 0$, $j \geqslant \frac{3}{2}$ we have

$$F^{\mu\nu}(p, p', j) = 0, \quad p \cdot p' > 0 \ . \tag{7}$$

For $j = 0, \frac{1}{2}, 1$ the expression (7) is not valid. In particular, for spinless particles

$$F^{\mu\nu}(p,p',0)=\frac{1}{(2\pi)^3(p^0p'^0)^{1/2}}\times[a_1(p'\cdot p)(p'^\mu p'^\nu+p^\mu p^\nu)+a_2(p'\cdot p)(p'^\mu p^\nu+p^\mu p'^\nu)]\,. \tag{8}$$

II. HIGHER-SPIN FIELD THEORIES

For free field theories of particles of zero mass and helicity j, $|j|\geq\frac{1}{2}$, we study the matrix elements of the current operator $j^\mu(x)$ and the stress tensor $\theta^{\mu\nu}(x)$ between one-particle states of momenta p,p' and the same helicity. Since p,p' are positive lightlike we have three classes of matrix elements. The first class corresponds to the diagonal matrix elements $(p-p'=0)$ which are the only ones to enter in the computation of the total charge, or total energy-momentum.[6,7] The second class corresponds to p,p' being parallel so that $p-p'$ is also lightlike. The third class corresponds to p,p' not being parallel so that $p+p'$ is timelike. In this case we may always choose the frame in which $\vec{p}+\vec{p}'=0$. In the first and second classes of configurations the same helicity implies the same polarization, while in the third class of configurations the same helicity implies opposite polarizations. However, in all cases we may choose to discuss the matrix elements in the standard (dynamically defined) Lorentz frame in which $\vec{p}$ and $\vec{p}'$ are along the third axis in space. In the following, we examine these matrix elements for the special cases of $j=\frac{1}{2}, 1, \frac{3}{2}$ and then generalize the results to particles of arbitrary spin. In all cases we shall find that the matrix element is continuous in going from configurations of the first class to those of the second, or among those of the second class. But the matrix elements are discontinuous between configurations of the third class and either of the other two classes of configurations. This result is of general validity whether the particles are interacting or not.

A. Spin $\frac{1}{2}$

The expression for the current density of chiral particles[8] (chosen for convenience to be positive chiral or left-handed) is

$$\tfrac{1}{2}\overline{\psi}(x)\gamma_\mu(1+\gamma_5)\psi(x)=j_\mu(x) \tag{9}$$

while the stress tensor is linear in the quantity

$$\tfrac{1}{2}\overline{\psi}(x)(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)(1+\gamma_5)\psi(x)=h_{\mu\nu}(x)\,. \tag{10}$$

But by virtue of the equation of motion

$$(p^0-\beta\gamma\cdot\boldsymbol{p})(1+\gamma_5)u(\vec{p})=0$$

the one-particle to one-particle matrix elements have the form $(\lambda=+\frac{1}{2})$

$$\langle\vec{p}\lambda|j^\mu(0)|\vec{p}'\lambda\rangle=u^\dagger(\vec{p},\lambda)\beta\gamma^\mu u(\vec{p}',\lambda)=\tfrac{1}{4}u^\dagger(\vec{p})(1+\vec{\sigma}\cdot\vec{p})\beta\gamma^\mu(1+\vec{\sigma}\cdot\vec{p}')u(p')\,. \tag{11}$$

For the first- and second-class configurations the matrix elements of $j^1(x)$ and $j^2(x)$ identically vanish while the $j^0(x)$ and $j^3(x)$ matrix elements do not vanish and are numerically equal. For the third-class configurations the matrix elements of $j^0(x)$ and $j^3(x)$ vanish identically while $j^1(x)$ and $j^2(x)$ have matrix elements which differ by a factor i.

We note that the current matrix elements are discontinuous in passing from the first- and second-class configurations to the third-class configurations. The current matrix elements are automatically conserved.

In the extension of these results to interacting theories we note that there is the possibility of creation of very soft pairs. For a massless photon mediating the creation of such photons we have a severe infrared problem; but even otherwise we have a soft-pair cloud surrounding each charged zero-mass particle.

For the matrix element of $h_{\mu\nu}(x)$ similar considerations apply. The $h_{00}(0)$, $h_{03}(0)$, $h_{33}(0)$ matrix elements are nonzero for the first two classes of configurations but vanish for the third-class configurations. The $h_{01}(0)$, $h_{02}(0)$, $h_{31}(0)$, $h_{32}(0)$ matrix elements have the opposite behavior. The $h_{11}(0)$, $h_{22}(0)$, and $h_{12}(0)$ matrix elements vanish identically.

B. Spin 1

The complexified form of the Maxwell field provides the wave field equations

$$\partial^\mu C_{\mu\nu}=0,\quad \partial_\mu B_\mu-\partial_\mu B_\nu=C_{\mu\nu}\,. \tag{12}$$

These field equations naturally do not determine B_μ which have a gauge freedom. Using the Coulomb gauge simplifies the calculations. The expression for the current may be taken to be

$$J_\mu(x)=C^*_{\mu\nu}(x)B_\nu(x)-B^*_\nu(x)C_{\mu\nu}(x)\,. \tag{13}$$

The field $C_{\mu\nu}$ decomposes into chiral components using the self-dual and the anti-self-dual components. The positive-energy positive-chiral spin-1 wave functions are transverse vectors satisfying

$$\vec{p}\cdot\vec{\eta}(\vec{p})=0,\quad \hat{p}\times\vec{\eta}(\vec{p})=i\vec{\eta}(\vec{p})\,. \tag{14}$$

The four-current contains both chirality-preserving and chirality-violating terms. If we calculate the matrix elements

$$\langle\vec{p},\lambda|J_\mu(0)|\vec{p}',\lambda\rangle$$

for the first two classes of configurations we get equal matrix elements for j_0 and j_3 and vanishing values for j_1 and j_2. For the third-class configurations all the matrix elements vanish. If we calculate the alternate matrix element

$$\langle \vec{p}, \lambda | j_\mu(0) | \vec{p}', -\lambda \rangle ,$$

the configurations interchange their roles. For the first two classes of configurations all matrix elements vanish. For the third-class configurations j_0 and j_3 have equal matrix elements but j_1 and j_2 have vanishing matrix elements. In this case the local Lagrangian theory is invariant under reflection and hence contains both chiralities. The third-class helicity-changing matrix elements are continuous with the helicity-preserving first- and second-class matrix elements.

The matrix elements of the tensor $h_{\mu\nu}(x)$ have a related behavior. Elementary calculations show that h_{00}, h_{03}, h_{33} alone have nonvanishing helicity-conserving matrix elements for the first two classes of configurations and their magnitudes are equal. h_{11}, h_{12}, h_{21} have nonvanishing helicity-flipping matrix elements for the same configurations. Their roles are interchanged for third-class configurations. These matrix elements are not continuous in passing from the first- and second-class configurations to the third-class configurations.

C. Spin $\frac{3}{2}$

The equations of motion follow from the Lagrangian[9]

$$\mathcal{L} = -\tfrac{1}{2} i \epsilon^{\rho\mu\nu\alpha} \overline{\Psi}_\rho \gamma_5 \gamma_\mu \partial_\nu \Psi_\alpha$$
$$= \Psi^\dagger A^\mu \partial_\mu \Psi \,. \tag{15}$$

This gives the chirality-invariant equation of motion

$$\epsilon^{\rho\mu\nu\alpha} \gamma_\mu \partial_\nu \Psi_\alpha = 0 \,.$$

One notes that in accordance with a general theorem[1] A^0 is not positive definite. It is singular; it annihilates the components Ψ_0. If we use a 3×3 matrix notation for the matrices acting on the vector indices for the Ψ_j we may write

$$A^0 = \tfrac{1}{2}\gamma_5\gamma_0 \vec{\gamma}\cdot\vec{S}$$
$$= \tfrac{1}{2}\vec{\sigma}\cdot\vec{S}, \tag{16}$$

where $\vec{S}$ are the 3×3 spin matrices for spin 1. The equations of motion yield the constraints

$$\gamma^\mu \cdot \Psi_\mu = 0 , \tag{17}$$

$$\partial^\mu \cdot \Psi_\mu = 0 , \tag{18}$$

and admits the gauge transformations

$$\Psi_\mu \to \Psi_\mu + \partial_\mu \phi, \;\; \gamma^\mu \partial_\mu \phi = 0 \,. \tag{19}$$

By choice of the radiation gauge

$$\Psi_0 = 0 \tag{20}$$

the constraint can be written in the form

$$\vec{S}\cdot\vec{p}\Psi = \Psi \,. \tag{21}$$

With this supplementary condition, out of the two eigenvalues $+1$, -2, $\vec{\sigma}\cdot\vec{S}$ assumes the positive eigenvalue. The equation of motion becomes

$$\vec{S}\cdot\vec{p}\Psi = \gamma_5 p_0 \vec{S}\cdot\vec{\sigma}\Psi \,. \tag{22}$$

The chirality invariance of the theory tells us that we may make the projections $\frac{1}{2}(1\pm\gamma_5)\Psi$. This selects out the eigenvalues $\pm p_0$ for $\vec{S}\cdot\vec{p}$.

The calculation of the matrix elements of the current

$$J_\mu(x) = \tfrac{1}{2}\epsilon^{\rho\mu\nu\alpha}\overline{\Psi}_\rho\gamma_5\gamma_\nu\Psi_\alpha \tag{23}$$

between various one particle configurations is straightforward. In this case j_0 and j_3 have equal matrix elements for the first- and second-class configurations. For third-class configurations they vanish identically. The matrix elements of j_1 and j_2 vanish for all configurations. For the tensor $h_{\mu\nu}(x)$ defined by

$$h^{\mu\nu}(x) = \frac{i}{2}\epsilon^{\rho\mu\beta\alpha}\Psi_\rho\gamma_5\gamma_\beta\partial^\nu\Psi_\alpha \tag{24}$$

the matrix elements may be calculated to obtain identically zero values for all matrix elements except for h_{00}, h_{03}, h_{33} which are all numerically equal and nonzero only for the first- and second-class configurations. Both the current and the stress tensor transform nontrivially under the gauge transformations of Eq. (19) illustrating the results of Witten and Weinberg.[4] But the conclusions about the vanishing of a number of the matrix elements are unaffected by gauge transformations.

D. Higher spins

The general pattern is now clear. The wave functions satisfy first-order equations admitting suitable gauge invariance and supplementary conditions. If we write $\vec{J}$ for the total spin angular momentum of the wave function (carried by the tensor and spinor indices) the supplementary conditions take the form

$$(\vec{J}\cdot\hat{p}+j)(\vec{J}\cdot\hat{p}+j-1)\cdots(\vec{J}\cdot\hat{p}-j+1)\Psi = 0$$

for the two choices of chiralities. The j_0, j_3, h_{00}, h_{03}, h_{33} matrix elements are nonvanishing for the first- and second-class configurations but vanish identically for the third-class configurations. All the other matrix elements also vanish identically.

III. DISCUSSION

In all these cases we see that the continuity requirement in the form

$$\lim_{\substack{p'\to p\\ p\cdot p'>0}} \langle p', j\,|\,\theta^{\mu\nu}(0)\,|\,p, j\rangle \to \frac{f}{(2\kappa)^3 p^0}p^\mu p^\nu \qquad (25)$$

is *not* satisfied, but for $p'\to p$, $p'\cdot p=0$, the limit is satisfied. Similar comments apply to the limits

$$\lim_{p'\to p} \langle p', j\,|\,j^\mu(x)\,|\,p, j\rangle \to \frac{gp^\mu}{(2\pi)^3 p^0}\,. \qquad (26)$$

Weinberg and Witten do not require such a continuity. Instead they require that the current- and stress-tensor matrix elements be evaluated in terms of limits of matrix elements as $p'-p$ approaches zero, and that this corresponds to the method by which charges, energies, and momenta are actually determined. We also see that the one-particle to one-particle matrix elements of $\partial_\mu h^{\mu\nu}$ and $\partial_\mu j^\mu$ for zero-mass particles of spin $j\geq\frac{3}{2}$ and $j\geq 1$, respectively, vanish whether or not the tensor $h^{\mu\nu}$ and the vector j^μ are differentially conserved. This is true whether the particles are composite or elementary.[10]

In conclusion, we find that theories of higher-spin free particles are consistent but the continuity of the matrix elements does not obtain. The reason is traced to the abrupt change in the sign of the polarization when the momentum transfer changes from spacelike to null values. In the general case with particles of $j\geq\frac{3}{2}$ and $j\geq 1$, respectively, we also prove that the divergence of any tensor and any vector vanishes between one-particle states.

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[5]This tensor need not be the energy-momentum tensor nor need it be conserved. But as Weinberg and Witten point out, the energy-momentum tensor is the universal source of gravitation.

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[10]There is a stronger result. The helicity-conserving one-particle to one-particle matrix elements of $\partial_\mu j^\mu$ vanish for zero-mass particles of spin $\geq\frac{1}{2}$; and those of $\partial_\mu\,\theta^{\mu\nu}$ for spin ≥ 1.