Relativistically interacting particles and world lines

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The action of the Poincaré group P on the phase space Γ of two interacting relativistic particles is studied. Trajectories are defined by introducing two fixation constraints which, together with two first-class dynamical constraints, form a second-class system. By a projection followed by an injection, a pair of world lines are defined in Minkowski space. A world-line condition is determined by the commutation of the Poincaré transformation with the projection-injection map. The results are compared with those of our preceding paper.

I. INTRODUCTION

Recently there have been a number of papers discussing classical relativistic interaction at a distance.[1-7] For many years the "no-interaction" theorems[8,9] developed in the 1960's dominated the thinking about this problem. The first break came with the work of Komar[1,2] and Todorov[5] who considered that the dynamics of the interaction should be described in terms of first-class constraints on a relativistic phase space. [These are constraints with vanishing Poisson Brackets (Pb) modulo the constraints.] There is to be one such constraint for each particle considered. In the course of this formulation one finds that one must give up the quest for a uniquely defined world line. While it seemed that their method should hold for an arbitrary number of particles, they gave only the example of two-particle interactions. Starting from a different set of ideas, Rohrlich[4] also introduced a theory of relativistic action at a distance which also used constrained systems on a relativistic phase space. His approach had the apparent advantage of showing immediately how to construct interactions involving many particles. However, in the following we shall limit our discussion to two particles for simplicity.

In the preceding article[7] (SMG), we have shown how to retain the notion of unique world lines for particles in interaction. We start with the formulation in terms of a constrained relativistic phase space, but then consider a reduced phase space with the use of Dirac brackets[10-13] (Db). On the reduced phase space, we construct a Hamiltonian function which reproduces the correct dynamics of the constrained system and which yields a unique trajectory. Finally, we address the problem of Poincaré invariance. The Poincaré generators have a representation in terms of the variables of the reduced phase space. The Poincaré group also acts geometrically on a Minkowski space into which the spatial positions of each particle can be mapped. Thus, for a two-particle system one has, for example, a twelve-dimensional reduced space with variables $(x_1^m, x_2^m, p_1^m, p_2^m)$, with $m=1,2,3$, and a four-dimensional Minkowski space with coordinates x^μ into which the particle trajectories are injected by the identity map on x_1^m and x_2^m. x_1^0 and x_2^0 are then determined by the auxiliary constraints which are needed in order to define the Db.

The question then arises whether the canonical Poincaré transformation on the reduced phase space followed by the injection into Minkowski space leads to the same world line as the geometrical action of the Poincaré group on the Minkowski space. In SMG we write the condition explicitly and show by example that sometimes the condition can be satisfied and sometimes not. However, it is not entirely clear from that work when the condition can or cannot be satisfied.

Here we wish to treat the problem of SMG without the use of the reduced phase space and, hence, without the use of the Db. At each stage the mapping to the reduced phase space commutes with the procedure presented below, but the geometrical picture and the computations are simpler in the relativistic language. However, there is an important difference (perhaps only semantic) between that work and that given here. In SMG we say explicitly that the dynamics is not defined until one states the auxiliary conditions needed to define the Db. Here we take the statement of the auxiliary conditions in the sense of the "fixation of coordinates" by Dirac in general relativity.[14] Therefore, the *fixation constraints*, as we shall call these auxiliary conditions, state an algorithm by which we are to describe the particle positions and an algorithm by which we define an evolution parameter for the particle positions. For example, in a two-particle system, an evolution parameter τ can always be chosen rather arbitrarily, but we must define through a fixation constraint when the position of two particles is at the same value of the parameter τ. Do we want the same τ to represent the particle separation in the rest frame of the total momentum, the rest frame of particle one, the rest frame of particle two, or some other condition such as $x_1^0 - x_2^0 = 0$? This decision cannot alter the physics of the problem and therefore is not dynamical.

A consequence of this point of view is that, even if one is willing to give up the invariance of the fixation constraints when necessary, a world-line condition can always be satisfied. This point is discussed in Sec. III.

The following section treats the two-body problem explicitly since the free particle is trivial and can be obtained by reduction. The world-line condition is discussed in Sec. III and some concluding remarks are given in the final section.

II. THE TWO-PARTICLE SYSTEM

Two-particle relativistic dynamics is described by two dynamical constraints[2] on Γ, a sixteen-dimensional phase space of the particle positions and momenta $(x_1^\mu, x_2^\mu, p_1^\mu, p_2^\mu)$, $\mu = 0,1,2,3$:

$$K_1 \equiv p_1{}^2 - m_1{}^2 - 2\mu V(r) = 0\,,\qquad K_2 \equiv p_2{}^2 - m_2{}^2 - 2\mu V(r) = 0\,, \tag{2.1a}$$

or by taking the sum and difference,

$$K_+ \equiv p^2 + p^2 - 2m_1{}^2 - 2m_2{}^2 - 8\mu V(r) = 0\,,\qquad K_- \equiv p_\mu - m_1{}^2 + m_2{}^2 = 0\,, \tag{2.1b}$$

with

$$P_\mu \equiv p_{1\mu} + p_{2\mu}\,,\quad p_\mu \equiv p_{1\mu} - p_{2\mu}\,,\quad X^\mu = \tfrac{1}{2}(x_1^\mu + x_2^\mu)\,,\quad x^\mu = \tfrac{1}{2}(x_1^\mu - x_2^\mu)\,,\quad r^2 = -4[x^2 - (x^\mu p_\mu/p)^2]\,.$$

It is easy to show that the constraints are first class:

$$[K_1, K_2] = [K_+, K_-] = 0\,. \tag{2.2}$$

Although it makes no difference to the general argument, it is better to use K_1 and K_2 since we wish to identify particle world lines.

The generators of a Poincaré transformation on Γ can be written as

$$G = a^\mu P_\mu + \tfrac{1}{2}\omega_{\mu\nu} J^{\mu\nu}\,,\qquad J^{\mu\nu} = x_1^\mu p_1^\nu - x_1^\nu p_1^\mu + x_2^\mu p_2^\nu - x_2^\nu p_2^\mu\,. \tag{2.3}$$

It is easy to see that K_1 and K_2 are Poincaré invariant—they change neither their form nor their value when acted on by any element $g_\Gamma \in \mathcal{P}_\Gamma$, the realization of the Poincaré group on Γ.

The two dynamical constraints define a 14-dimensional constraint hypersurface Σ in Γ. Because of the invariance of K_1 and K_2, Σ is mapped onto itself by every $g_\Gamma \in \mathcal{P}_\Gamma$. Starting from any point in Σ, the constraints generate a two-dimensional surface which Komar calls a *generalized trajectory*. The constraint hypersurface Σ is filled with these generalized trajectories which we shall call $S_{\underline{a}}$, where $\underline{a}$ is a collective index specifying the values of 12 functions on Γ which have vanishing Pb with respect to K_1 and K_2. These form a complete set of observables.[3] The observables are constant on the surfaces $S_{\underline{a}}$ and are an independent set of constants of the motion which defines a specific generalized trajectory. Introducing the space of surfaces $S_{\underline{a}}$ as the reduced phase space leads to the frozen formalism of Bergmann and Komar.[15]

A trajectory in phase space is obtained by choosing a linear combination of the constraints as a Hamiltonian:

$$H = \alpha K_1 + \beta K_2\,. \tag{2.4}$$

Because of the invariance of K_1 and K_2, H itself may change only through α and β. In any case, the trajectory lies entirely in one surface $S_{\underline{a}}$ and is determined by $(i = 1,2)$

$$\frac{dx_i^\mu}{d\tau} = [x_i^\mu, H]\,,\qquad \frac{dp_i^\mu}{d\tau} = [p_i^\mu, H]\,. \tag{2.5}$$

The choice of α and β determines the evolution parameter τ and one other condition on the phase-space variables. One can see this by choosing two fixation constraints,

$$\chi_1(x_1, x_2, p_1, p_2) = 0\,, \tag{2.6a}$$

$$\chi_2(x_1, x_2, p_1, p_2, \tau) = 0\,, \tag{2.6b}$$

such that the determinant of their Pb's with K_1 and K_2 is not degenerate. Then require that these fixation constraints are satisfied along the trajectory,

$$[\chi_1, H] \equiv \alpha[\chi_1, K_1] + \beta[\chi_1, K_2] = 0\,, \tag{2.7a}$$

$$\frac{\partial\chi_2}{\partial\tau} + [\chi_2, H] \equiv \frac{\partial\chi_2}{\partial\tau} + \alpha[\chi_2, K_1] + \beta[\chi_2, K_2] = 0\,. \tag{2.7b}$$

α and β are then uniquely determined. (Given α and β, it is not clear to what extent χ_1 and χ_2 are determined.)

Equation (2.6a) is the important fixation for determining the trajectory. That is true because (2.7a) guarantees that the surface $\chi_1 = 0$ will intersect each generalized trajectory S_a in a line. Equation (2.6b) specifies the kind of clock we are using to define the evolution parameter τ. Therefore it determines the rate in τ at which the trajectory is transversed.

Having chosen a fixation constraint as in (2.7), we get a family of trajectories, one in each surface S_a. What happens if we now apply a Poincaré transformation? First of all, because K_1 and K_2 have vanishing Pb's with G, the two-surfaces are mapped into two-surfaces, $S_a \to S_b$. What about the family of trajectories? If χ_1 is invariant,

$$[\chi_1, G] = 0 , \tag{2.8}$$

then the surface $\chi_1 = 0$ is mapped into itself and the trajectory $l_a \in S_a$ into the trajectory $l_b \in S_b$. To correlate the rates at which the trajectories are traversed, we require that

$$\frac{\partial \chi_2}{\partial \tau} \delta\tau + [\chi_2, G] = 0 . \tag{2.9}$$

This says that χ_2 can always be defined to be an invariant function of the phase-space variables and τ provided τ changes its value in accordance with (2.9).

Now suppose that χ_1 is not an invariant. Then the hypersurface $\chi_1 = 0$ is mapped into the hypersurface $\chi_1{}' = 0$ determined by

$$\delta\chi_1 = [\chi_1, G] \neq 0 . \tag{2.10}$$

In this case the trajectory l_a will be mapped into the trajectory $l'_b \in S_b$. However, consider the fixation transformation generated by

$$K = aK_1 + bK_2 . \tag{2.11}$$

It maps each generalized trajectory into itself. In particular, it maps S_b into itself. There is enough freedom in the functions a and b to generate a transformation so that any two trajectories in S_b can be mapped into each other. In the infinitesimal case which we are considering, we may choose a and b so that

$$[\chi_1, K + G] = 0 . \tag{2.12}$$

Thus, the transformation K maps S_b into itself and $l'_b \to l_b$. Now, to first order χ_2 will also change, so that altogether we have

$$\frac{\partial \chi_2}{\partial \tau} \delta\tau + [\chi_2, K + G] = 0 . \tag{2.13}$$

Equations (2.12) and (2.13) say that under the combined infinitesimal transformation $K+G$, χ_1 and χ_2 are invariant—unchanged in form and value. The Hamiltonian is also unchanged by the total process, although it is changed by G or K alone.

The conclusions to be drawn from this discussion are the following: (1) If the fixation constraint χ_1 is an invariant, then trajectories are mapped into trajectories by a Poincaré transformation. However, the evolution parameter may have to change its value in accordance with Eq. (2.9). (2) If the fixation constraint is not an invariant, then the Poincaré transformation must be followed by a fixation transformation if trajectories are to be mapped into trajectories in the phase space. In this case τ may also change, but in accordance with (2.13).

Note that the trivial invariance of χ_2, either through (2.9) or (2.13), explains why the world-line condition for the free particle can always be satisfied. There is only one fixation constraint needed and it involves τ.

III. WORLD LINES IN MINKOWSKI SPACE

So far we have only discussed the mapping of trajectories in the phase space. To get a world line for each of the particles in Minkowski space M one must carry out a mapping from the phase space Γ into Minkowski space. This map is defined by a projection

$$\Pi : \ (x_1^\mu, x_2^\mu, p_1^\mu, p_2^\mu) \to (x_1^\mu, x_2^\mu) \tag{3.1}$$

followed by the injection

$$\begin{aligned} \iota : \ & x_1^\mu(\tau) \to x^\mu = x_1^\mu(\tau) , \\ & x_2^\mu(\tau) \to x^\mu = x_2^\mu(\tau), x^\mu \in M . \end{aligned} \tag{3.2}$$

To an element $g \in \mathcal{P}$, the Poincaré group, there exists a canonical transformation g_Γ on the phase space Γ, and a space-time mapping g_M on the Minkowski space M. It is clear that the following "commutation" relationship holds:

$$(\iota\Pi) g_\Gamma = g_M (\iota\Pi) . \tag{3.3}$$

In this sense world lines are trivially mapped into world lines in Minkowski space. However, unless the fixation constraint χ_1 is itself invariant, trajectories will not be mapped into trajectories. Recall that the reason for this is that while the functional form of χ_1 may be unchanged by the canonical transformation, its value changes. (One may choose the value to remain zero, then its functional form must change.) Therefore, the canonical transformation maps $\chi_1 = 0$ into some other surface unless χ_1 is invariant. In the previous section we argued that we could always find a transformation which mapped trajectories into trajectories by following the Poincaré transformation g_Γ by a canonical transformation k generated by K defined in Eq. (2.11). However, if Eq. (3.3) holds, then

$$(\iota\Pi)kg_\Gamma = g_M(\iota\Pi) \quad (3.4)$$

can hold only if k is the identity transformation. Therefore, we have (1) *for* χ_1 *an invariant, we can map trajectories into trajectories and world lines into world lines*; (2) *for* χ_1 *not an invariant, we can map trajectories into trajectories or world lines into world lines, but not both.*

This explains the results of SMG. There we choose to use a reduced phase space Γ_R and take the constraints $K_1 = K_2 = \chi_1 = \chi_2 = 0$ into account by using Db's instead of Pb's. The Db's are defined so that every function on the phase space has a vanishing bracket with any of the constraints. This means that every canonical transformation defined with the Db's leaves the constraints invariant. Therefore, in SMG we are always examining whether Eq. (3.4) holds. Not surprisingly, we find that it does if χ_1 is invariant on Γ not on Γ_R.

IV. CONCLUSION

In the above we have discussed the formulation of the relativistic two-body problem in phase space. Following the work of Komar we have defined the dynamics through two Poincaré-invariant first-class constraints on the phase space. Determining a one-dimensional trajectory in the phase space requires adding two additional constraints which together with the original constraints form a second-class system. We then define a projection followed by an injection into Minkowski space. The injection produces a world line for each of the particles in Minkowski space. One then examines the relationship between a Poincaré transformation on the phase space and on Minkowski space. One finds that if the fixation constraints (needed to form the second-class system) are invariant, then, on the phase space, trajectories are mapped into trajectories and, after projection and injection into Minkowski space, world lines are mapped into world lines.

On the other hand, if the additional constraints are not invariant, one can choose either to have trajectories mapped into trajectories or world lines into world lines, but not both. In this case the mapping of trajectories into trajectories requires that the Poincaré transformation on the phase space be followed by a canonical transformation which restores the original noninvariant fixation constraint. This latter transformation can be compared with what is normally done in electrodynamics. Usually one works in the Lorentz gauge which is Poincaré invariant. But sometimes it is convenient to use the Coulomb or radiation gauge. These conditions are not invariant under a Poincaré transformation, and to maintain these conditions requires a gauge transformation following the Poincaré transformation.

If one wishes to consider an n-body problem, one starts with n first-class constraints. A fixation of coordinates requires $(n-1)$ fixation constraints which do not depend on an evolution parameter τ, and one which does. To have a mapping of trajectories into trajectories and world lines into world lines, the $(n-1)$ constraints must be invariant under Poincaré transformations on Γ. The arguments are exactly the same as before.

The question remain whether any of this discussion is relevant to quantization of a system interacting particles. Komar and Rohrlich argue that it is not, although Rohrlich's approach seems to straddle the issue. In SMG we take the position that since the dynamics is not determined until the fixation is set, quantization depends on the choice. The point of view taken here is that the fixation is the same as choosing a gauge in electrodynamics. Therefore, it may be simpler to carry out the quantization with a specific fixation, but physical observables should be fixation independent. Komar has constructed a complete set of observables which have vanishing Pb's with the dynamical constraints which generate the transformation from one fixation to another. Therefore, one has a complete set of observables which are independent of the fixation.

The main interest in this problem for those concerned with general relativity is as a relativistic model with a finite number of degrees of freedom with which to study the quantization of theories with constraints. And it is from this point of view that further study on this problem will continue. Apart form the difference in philosophy or semantics, the specific way in which we carry out the reduction in SMG may be of interest in general relativity. There we point out that any set of reduced variables is acceptable to define Γ_R provided that their Db's do not involve the evolution parameter τ explicitly. For then one can find a Hamiltonian $\mathcal{H}$ on the reduced phase space which will generate trajectories on Γ_R which are the projections of the trajectories in Γ. In this way one can avoid the dilemma of the reduced phase space leading to a frozen formalism. It will be interesting to see whether such a Hamiltonian can be constructed in the canonical theory of general relativity.

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