JMAPA,8003000331

Representation and properties of para-Bose oscillator operators. II. Coherent states and the minimum uncertainty states

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(Received 10 October 1979; accepted for publication 26 November 1979)

The energy, position, and momentum eigenstates of a para-Bose oscillator system were considered in paper I. Here we consider the Bargmann or the analytic function description of the para-Bose system. This brings in, in a natural way, the coherent states $|z;\alpha\rangle$ defined as the eigenstates of the annihilation operator $\hat{a}$. The transformation functions relating this description to the energy, position, and momentum eigenstates are explicitly obtained. Possible resolution of the identity operator using coherent states is examined. A particular resolution contains two integrals, one containing the diagonal basis $|z;\alpha\rangle\langle z;\alpha|$ and the other containing the pseudodiagonal basis $|z;\alpha\rangle\langle -z;\alpha|$. We briefly consider the normal and antinormal ordering of the operators and their diagonal and discrete diagonal coherent state approximations. The problem of constructing states with a minimum value of the product of the position and momentum uncertainties and the possible α dependence of this minimum value is considered.

PACS numbers: 03.65.Ca

1. INTRODUCTION

In paper[1] I of this work we have given a detailed study of the energy, position, and momentum eigenstates of a para-Bose oscillator system characterized by the commutation relation

$$[\tfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger),\hat{a}] = -\hat{a} \tag{1.1}$$

and by a parameter α denoting the minimum eigenvalue of the Hamiltonian $\frac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)$ ($\alpha=\frac{1}{2}$ being the normal Bose case). We had in particular considered the relationship between the matrix and the wave mechanical descriptions of the para-Bose operators. In the present paper we consider the Bargmann (or the entire function space) description, operator ordering, and construction of states with minimum value of the product of uncertainties in position–momentum variables and related matters. We begin in Sec. 2 by constructing the Bargmann representation using a suitably defined Hilbert space of entire analytic functions for the SL(2,R) Lie algebra relevant to us, and then for the para-Bose system. This brings in, in a natural way, the coherent states, i.e., the eigenstates of the para-Bose annihilation operator. The transformations relating this description to the energy, coordinate, and momentum descriptions will be explicitly obtained. Possible resolution of the identity operator using coherent states is examined. As is well known, in the normal Bose case a diagonal resolution of the identity operator does exist, viz.,

$$\hat{1} = \frac{1}{\pi}\int |z;\tfrac{1}{2}\rangle\langle z;\tfrac{1}{2}|\,d^2z\,. \tag{1.2}$$

However, it turns out that for other values of α, no such diagonal resolution exists. This is because a certain moment problem has no solution in the general case. An alternative resolution of the identity valid for all α will be developed and its uniqueness discussed. This resolution contains two integrals: one consisting of the diagonal basis $|z;\alpha\rangle\langle z;\alpha|$ and the other consisting of the pseudodiagonal basis $|z;\alpha\rangle\langle -z;\alpha|$. This second integral is of course absent in the normal Bose case $\alpha=\frac{1}{2}$. In Sec. 3, we discuss the possibilities of various operator descriptions such as the normal ordered, the antinormal ordered, and the diagonal and the discrete diagonal coherent state approximations. In Sec. 4, we consider the problem of constructing states with the minimum product of the uncertainties in position and momentum variables, and their α dependence. Section 5 comprises concluding remarks and some general questions.

2. BARGMANN REPRESENTATION OF PARA-BOSE OPERATORS

Para-Bose operators $\hat{a},\hat{a}^\dagger$, and $\hat{H}=\frac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)$ leave the representation space $\mathscr{D}_\alpha$ invariant. This space is spanned by the eigenstates $|n;\alpha\rangle, n=0,1,\cdots$ of $\hat{H}$ with the corresponding eigenvalues $n+\alpha$. The parameter α denotes the minimum eigenvalue of $\hat{H}$. Using the representation of the operator $\mathbf{\hat{a}}$ in space $\mathscr{D}_\alpha$, one can construct its eigenstate $|z;\alpha\rangle$ with eigenvalue z, where z is any complex number.[2] We call such a state the para-Bose coherent state in analogy with the normal Bose case. Instead of following this procedure for obtaining these states, we show that they appear in a natural way in the Bargmann description of $\mathscr{D}_\alpha$. We begin in Sec. 2A with the Bargmann[3] type description of the representation D_β of the SL(2,R) Lie algebra, using a Hilbert space of entire functions. This involves working with the eigenfunctions of J_-. This is used in Sec. 2B to construct a similar description of the para-Bose representation $\mathscr{D}_\alpha$. We are then directly led to the coherent states $|z;\alpha\rangle$. One of the outcomes of this procedure is a particular resolution of the identity in terms of the coherent states. The possibility of having a diagonal expression for the identity operator is examined in Sec. 2C.

 0022-2488/81/010078-13$1.00

A. Bargmann description of SL(2,R) representation D_β

In Sec. 2 of part I, we introduced the para-Bose operators and the relations satisfied by them. The para-Bose operator algebra is determined in terms of the Hamiltonian (I2.2):

$$\hat{H} = \tfrac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \tag{2.1}$$

and the commutation relation [Eq. (I2.1)]

$$[\hat{a},\hat{H}] = \hat{a} . \tag{2.2}$$

We have seen that the operators $\hat{J}_0$, $\hat{J}_1$, and $\hat{J}_2$ defined as

$$\hat{J}_0 = \tfrac{1}{2}\hat{H}, \quad \hat{J}_1 = \tfrac{1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2}), \quad \hat{J}_2 = (i/4)(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{2.3}$$

obey commutation relations which correspond to the SL(2,R) Lie algebra [Eq. (I2.8)]. The eigenvectors of $\hat{J}_0$, viz., $|n;\beta\rangle$, form a complete orthonormal basis:

$$\hat{J}_0|n;\beta\rangle = (n+\beta)|n;\beta\rangle , \tag{2.4}$$

$$\langle n';\beta|n;\beta\rangle = \delta_{n',n} . \tag{2.5}$$

We ask for a realization of the representation D_β of the SL(2,R) Lie algebra in which the vector $|n;\beta\rangle$ is realized as the nth power of a complex variable ω and $\hat{J}_+ \equiv \hat{J}_1 + i\hat{J}_2$ is realized as a simple multiplication by ω:

$$|n;\beta\rangle \to \mu_n \omega^n , \tag{2.6}$$

$$J_+ \to \omega . \tag{2.7}$$

A set of constants μ_n are introduced since $\hat{J}_+$ has definite nontrivial matrix elements. Equation (I2.13) leads to a recursion relation for μ_n:

$$\mu_{n+1} = [(n+1)(n+2\beta)]^{-1/2}\mu_n . \tag{2.8}$$

With the choice $\mu_0 = 1$, we are led to a solution

$$\mu_n = \left[\frac{\Gamma(2\beta)}{n!\Gamma(n+2\beta)}\right]^{1/2} . \tag{2.9}$$

Equation (2.4) implies that $\hat{J}_0$ is realized according to

$$\hat{J}_0 \to \beta + \omega\frac{d}{d\omega} . \tag{2.10}$$

The form of $\hat{J}_- = \hat{J}_1 - i\hat{J}_2$ in this realization is obtained using Eqs. (I2.13) and (2.9):

$$\hat{J}_- \to \omega\frac{d^2}{d\omega^2} + 2\beta\frac{d}{d\omega} . \tag{2.11}$$

A general vector $|g\rangle$ in D_β now determines the function $g(\omega)$ as follows:

$$|g\rangle = \sum_{n=0}^{\infty} g_n |n;\beta\rangle$$
$$\to g(\omega) = \sum_{n=0}^{\infty} g_n \mu_n \omega^n . \tag{2.12}$$

If $|g\rangle$ has a finite norm, i.e., if $\{g_n\}$ is l_2, the behavior of μ_n for large n ensures that $g(\omega)$ is an entire analytic function of ω. Thus, D_β has been realized in a Hilbert space of entire functions. The inner product in this realization can be exhibited in the form

$$\langle g'|g\rangle = \sum_{n=0}^{\infty} g_n'^* g_n = \int d^2\omega\, K(\omega;\beta) g'^*(\omega) g(\omega) . \tag{2.13}$$

Here

$$d^2\omega \equiv du\, dv \quad (\omega = u + iv) , \tag{2.14}$$

and the integration extends over the entire complex ω plane. Taking $|g\rangle$ to be the vector $|n;\beta\rangle$ and using Eq. (2.7) along with the orthonormality of $|n;\beta\rangle$, we find that K is a function of $|\omega|$ only and that it obeys the relation

$$\int_0^\infty d|\omega| K(|\omega|;\beta)|\omega|^{2n+1} = (2\pi\mu_n^2)^{-1}$$
$$= n!\Gamma(n+2\beta)[2\pi\Gamma(2\beta)]^{-1} . \tag{2.15}$$

In writing the last line (2.15), we have substituted for μ_n from Eq. (2.9). A solution for $K(|\omega|;\beta)$ exists in terms of the Bessel function $K_\nu(x)$ [see Ref. 4, p. 684, formula (6.561. 16)].

$$K(|\omega|;\beta) = 2(\pi\Gamma(2\beta))^{-1}|\omega|^{2\beta-1}K_{2\beta-1}(2|\omega|) . \tag{2.16}$$

We show in Appendix A that this is a unique positive solution.

Let us now view $g(\omega)$ as the inner product of the ket $|g\rangle$ with a ket $|\omega^*;\beta\rangle$ labeled by ω^*:

$$g(\omega) = N_\beta(|\omega|)\langle\omega^*;\beta|g\rangle , \tag{2.17}$$

where $N_\beta(|\omega|)$ is some real positive function of $|\omega|$ to be adjusted later for proper normalization [cf. Eq. (2.20) below]. The action of $\hat{J}_+$ given in Eq. (2.7) implies that the bras (kets) are the eigenstates of $\hat{J}_+(\hat{J}_-)$

$$\langle\omega^*;\beta|\hat{J}_+ = \omega\langle\omega^*;\beta| , \quad \hat{J}_-|\omega;\beta\rangle = \omega|\omega;\beta\rangle . \tag{2.18}$$

Taking $|g\rangle$ to be $|n;\beta\rangle$ and using Eqs. (2.12) and (2.17) we find that

$$\langle\omega^*;\beta|n;\beta\rangle = \mu_n\omega^n[N_\beta(|\omega|)]^{-1} ,$$

which implies, on taking the Hermitian adjoint of this equation and substituting for μ_n from Eq. (2.9), that (cf. Barut and Girardello, Ref. 3).

$$|\omega;\beta\rangle = [N_\beta(|\omega|)]^{-1}\sum_{n=0}^{\infty}\left[\frac{\Gamma(2\beta)}{n!\Gamma(n+2\beta)}\right]^{1/2}\omega^n|n;\beta\rangle . \tag{2.19}$$

The ket $|\omega;\beta\rangle$ is obviously a finite norm vector, and in fact we may choose $N_\beta(|\omega|)$ such that $|\omega;\beta\rangle$ is normalized. Using the orthonormality of $|n;\beta\rangle$ we find from Eq. (2.19) that

$$\langle\omega;\beta|\omega;\beta\rangle = [N_\beta(|\omega|)]^{-2}\sum_{n=0}^{\infty}\frac{\Gamma(2\beta)|\omega|^{2n}}{n!\Gamma(n+2\beta)}$$
$$= [N_\beta(|\omega|)]^{-2}\Gamma(2\beta)|\omega|^{1-2\beta}I_{2\beta-1}(2|\omega|) ,$$

where $I_\nu(x)$ is the modified Bessel function. Hence, we take

$$N_\beta(|\omega|) = \{\Gamma(2\beta)|\omega|^{1-2\beta}I_{2\beta-1}(2|\omega|)\}^{1/2} . \tag{2.20}$$

It may however be noted that the eigenvectors of $\hat{J}_-$ are not orthogonal. We find that

$$\langle|\omega';\beta|\omega;\beta\rangle = \{N_\beta(|\omega'|)N_\beta(|\omega|)\}^{-1}\Gamma(2\beta)(\omega'^*\omega)^{1/2-\beta}$$
$$\times I_{2\beta-1}(2(\omega'^*\omega)^{1/2}) . \tag{2.21}$$

The use of Eqs. (2.16), (2.17), and (2.20) in the inner product expression (2.13) leads to the resolution of the identity operator in the space of the representations D_β:

$$\hat{1}_\beta = \frac{2}{\pi}\int d^2\omega\, I_{2\beta-1}(2|\omega|)K_{2\beta-1}(2|\omega|)|\omega;\beta\rangle\langle\omega;\beta| . \tag{2.22}$$

The subscript β on $\hat{1}$ indicates the space wherein this resolution holds. It is shown in Appendix A that a resolution of the identity in the form

$$\hat{1}_\beta = \int F(\omega)|\omega;\beta\rangle\langle\omega;\beta|d^2\omega \quad (2.23)$$

is unique as long as we restrict $F(\omega)$ to be a positive definite function.

B. Bargmann description of $\mathscr{D}_\alpha$

There are two ways in which a similar description of the para-Bose representation $\mathscr{D}_\alpha$ can be constructed. One is to use the representation just constructed for D_β and use the fact that $\mathscr{D}_\alpha$ is the direct sum of D_β and $D_{\beta+1/2}$ [cf. Eq. (I2.16)]. Alternatively, we may start afresh and require that $\hat{a}^\dagger$ be realized as multiplication by a complex number z while $|n;\alpha\rangle$ is realized essentially as the nth power of z. We follow here the first method.

$\mathscr{D}_\alpha$ is realized in a space which is the direct sum of two spaces carrying D_β and $D_{\beta+1/2}$:

$$\mathscr{D}_\alpha = D_\beta \oplus D_{\beta+1/2}\ , \quad \alpha = 2\beta.$$

In each of the constituent spaces we can set up the eigenvectors of J_-. Equation (I2.19) shows that $\hat{a}$ acts on these states as follows:

$$\hat{a}|\omega;\beta\rangle = [N_{\beta+1/2}(|\omega|)/N_\beta(|\omega|)](2/\alpha)^{1/2}\omega|\omega;\beta+\tfrac{1}{2}\rangle\ , \quad (2.24a)$$

$$\hat{a}|\omega;\beta+\tfrac{1}{2}\rangle = [N_\beta(|\omega|)/N_{\beta+1/2}(|\omega|)](2\alpha)^{1/2}|\omega;\beta\rangle\ . \quad (2.24b)$$

The state $|\omega;\beta\rangle$ is orthogonal to the state $|\omega';\beta+\tfrac{1}{2}\rangle$:

$$\langle\omega;\beta|\omega';\beta+\tfrac{1}{2}\rangle = 0\ . \quad (2.25)$$

An eigenstate $|z,\alpha\rangle$ of $\hat{a}$ with an arbitrary complex number z as the eigenvalue

$$\hat{a}|z;\alpha\rangle = z|z;\alpha\rangle \quad (2.26)$$

can now be constructed as a linear superposition of $|\omega;\beta\rangle$ and $|\omega;\beta+\tfrac{1}{2}\rangle$. We find that the $|z;\alpha\rangle$ satisfying Eq. (2.26) is given by

$$|z;\alpha\rangle = \mathscr{N}_\alpha(|z|)[N_\beta(\tfrac{1}{2}|z|^2)|\tfrac{1}{2}z^2;\beta\rangle + N_{\beta+\frac{1}{2}}(\tfrac{1}{2}|z|^2)(z/\sqrt{2\alpha})|\tfrac{1}{2}z^2;\beta+\tfrac{1}{2}\rangle]\ . \quad (2.27)$$

The overall real positive factor $\mathscr{N}_\alpha(|z|)$ is to be so chosen that $|z;\alpha\rangle$ is properly normalized:

$$\langle z;\alpha|z;\alpha\rangle = 1\ . \quad (2.28)$$

We therefore write

$$\mathscr{N}_\alpha(|z|) = \left[\{N_\beta(\tfrac{1}{2}|z|^2)\}^2 + \{N_{\beta+1/2}(\tfrac{1}{2}|z|^2)\}^2\frac{|z|^2}{2\alpha}\right]^{-1/2} \quad (2.29)$$

and from Eq. (2.20) we find that

$$\mathscr{N}_\alpha(|z|) = [2^{\alpha-1}\Gamma(\alpha)\mathscr{F}_\alpha(|z|^2)]^{-1/2}\ , \quad (2.30)$$

where $\mathscr{F}_\alpha(z)$ is the function introduced in paper I [Eq. (I3.16)], viz.,

$$\mathscr{F}_\alpha(z) = z^{1-\alpha}(I_{\alpha-1}(z) + I_\alpha(z))\ . \quad (2.31)$$

On substituting from Eq. (2.19) into (2.27) we find that

$$|z;\alpha\rangle = \mathscr{N}_\alpha(|z|)\sum_{n=0}^{\infty}\left\{\frac{\Gamma(2\beta)}{n!\Gamma(n+2\beta)}\right\}^{1/2}(\tfrac{1}{2}z^2)^n \times[|n;\beta\rangle + (z/2^{1/2})(n+2\beta)^{-1/2}|n;\beta+\tfrac{1}{2}\rangle]\ . \quad (2.32)$$

Recalling from paper I [Eq. (I(2.17)] that

$$|l;\beta\rangle = |2l;\alpha\rangle \text{ and } |l;\beta+\tfrac{1}{2}\rangle = |2l+1;\alpha\rangle\ , \quad (2.33)$$

we obtain the following expansion for the coherent state $|z;\alpha\rangle$ in terms of the eigenstates $|n;\alpha\rangle$ of the Hamiltonian $\hat{H}$ (cf. Ref. 2):

$$\begin{aligned}|z;\alpha\rangle &= \mathscr{N}_\alpha(|z|)\sum_{n=0}^{\infty}\left[\frac{\Gamma(\alpha)}{n!\Gamma(n+\alpha)}\right]^{1/2}\{(z/2^{1/2})^{2n}|2n;\alpha\rangle + (z/2^{1/2})^{2n+1}(n+\alpha)^{-1/2}|2n+1;\alpha\rangle\}\\ &= \{2^{\alpha-1}\mathscr{F}_\alpha(|z|^2)\}^{-1/2}\times\sum_{n=0}^{\infty}\left\{\left[\frac{n}{2}\right]!\Gamma\left(\left[\frac{n+1}{2}\right]+\alpha\right)\right\}^{-1/2}\times(z/2^{1/2})^n|n;\alpha\rangle\ .\end{aligned} \quad (2.34)$$

Here $[K]$ stands for the integral part of K i.e., the largest integer smaller than or equal to K.

Equation (2.34) gives the coherent states of the para-Bose representation $\mathscr{D}_\alpha$. Several important properties readily follow from here. These states are not mutually orthogonal. The inner product of $|z;\alpha\rangle$ with $|z';\alpha\rangle$ is given by

$$\langle z';\alpha|z;\alpha\rangle = \mathscr{F}_\alpha(z'^*z)/\{\mathscr{F}_\alpha(|z|^2)\mathscr{F}_\alpha(|z'|^2)\}^{1/2}\ . \quad (2.35)$$

It is interesting to note that the entire function $\mathscr{F}(x)$ also appears in the momentum eigenfunction $\langle x;\alpha|k;\alpha\rangle$ Eq. (IB.15)].

Finally, we derive the coordinate and momentum representations of the coherent state. From Eqs. (2.34) and (I3.10) we find that

$$\langle x;\alpha|z;\alpha\rangle = \exp(-\tfrac{1}{2}x^2)|x|^{\alpha-1/2}\{2^{\alpha-1}\mathscr{F}_\alpha(|z|^2)\}^{-1/2}\times\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2^n\Gamma(n+\alpha)}\times\left[L_n^{\alpha-1}(x^2) + \frac{zx}{2^{1/2}}(n+\alpha)\right]L_n^\alpha(x^2)\ , \quad (2.36)$$

where L_n^α is the associated Laguerre polynomial. On making use of the generating function relation [Ref. 4, p. 1038 formula (8.975.3)]

$$(xz)^{-(1/2)\alpha}e^zJ_\alpha(2(xz)^{1/2}) = \sum_{n=0}^{\alpha}\{\Gamma(n+\alpha+1)\}^{-1}L_n^\alpha(x)z^n\ , \quad (2.37)$$

and on simplification, we may rewrite Eq. (2.36) as

$$\langle x;\alpha|z;\alpha\rangle = 2^{(\alpha-1)/2}|x|^{\alpha-(1/2)}\exp[-\tfrac{1}{2}(x^2+z^2)]\times\{\mathscr{F}_\alpha(|z|^2)\}^{-1/2}\mathscr{F}_\alpha(\sqrt{2}xz)\ . \quad (2.38)$$

Again, from Eqs. (2.34), (2.37), and (I3.13) we obtain on simplification

$$\langle k;\alpha|z;\alpha\rangle = 2|k|^{\alpha-(1/2)}\exp[-\tfrac{1}{2}(k^2+z^2)]\times\{\mathscr{F}_\alpha(|z|^2)\}^{-1/2}\mathscr{F}_\alpha(\sqrt{2}ikz). \quad (2.39)$$

To set up the representation $\mathscr{D}_\alpha$ in a Hilbert space of entire functions, we must associate with every vector $|f\rangle$ an entire function $f(z)$. This is achieved using coherent states in the following manner:

$$|f\rangle \rightarrow f(z) = [2^{\alpha-1}\Gamma(\alpha)\mathscr{F}_\alpha(|z|^2)]^{1/2}\langle z^*;\alpha|f\rangle\ . \quad (2.40)$$

In particular, from Eq. (2.34) we find that the vector $|n;\alpha\rangle$ is realized as

$$|n;\alpha\rangle = \{\Gamma(\alpha)\}^{1/2}\left\{\left[\frac{n}{2}\right]!\Gamma\left(\left[\frac{n+1}{2}\right]+\alpha\right)\right\}^{-1/2} \times (z/\sqrt{2})^n . \quad (2.41)$$

Further, since the coherent states are the eigenstates of $\hat{a}$, or equivalently

$$\langle z^*;\alpha|\hat{a}^\dagger = z\langle z^*;\alpha| , \quad (2.42)$$

it is evident that $\hat{a}^\dagger$ is realized as multiplication by z. In fact, this requirement along with the relations [Eqs. (I2.20 c,d)]

$$\hat{a}^\dagger|2n;\alpha\rangle = (2n+2\alpha)^{1/2}|2n+1;\alpha\rangle , \quad (2.43a)$$

$$\hat{a}^\dagger|2n+1;\alpha\rangle = (2n+2)^{1/2}|2n+2;\alpha\rangle , \quad (2.43b)$$

would lead us directly to the numerical factors present in Eq. (2.41). On the other hand, the action of $\hat{a}$ is different on even and odd entire functions of z. From Eqs. (2.41) and (I2.20 a,b), viz.,

$$\hat{a}|2n;\alpha\rangle = (2n)^{1/2}|2n-1;\alpha\rangle , \quad (2.44a)$$

$$\hat{a}|2n+1;\alpha\rangle = (2n+2\alpha)^{1/2}|2n;\alpha\rangle , \quad (2.44b)$$

we find that the action of $\hat{a}$ on an even entire function is $df(z)/dz$ whereas its action on an odd entire function is $[d/dz)+(2\alpha-1)/z]f(z)$. Hence, if we write $f(z)$ as a sum of an even part and an odd part

$$f(z) = f_+(z) + f_-(z) , \quad (2.45)$$

then $\hat{a}$ is realized by

$$\hat{a} \to \frac{d}{dz} + \frac{(\alpha-1/2)}{z}(1-\hat{P}) , \quad (2.46)$$

where $\hat{P}$ is the parity operator

$$\hat{P}f(z) = f(-z) . \quad (2.47)$$

Alternatively, if we express $f(z)$ as a column vector

$$\begin{pmatrix} f_+(z) \\ f_-(z) \end{pmatrix} , \quad (2.48)$$

then $\hat{a}^\dagger$ and $\hat{a}$ are realized by matrix operators

$$\hat{a}^\dagger \to \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}, \quad \hat{a} \to \begin{pmatrix} 0 & \dfrac{d}{dz} + \dfrac{2\alpha-1}{z} \\ \dfrac{d}{dz} & 0 \end{pmatrix} \quad (2.49)$$

In the space $\mathscr{D}_\alpha$, a vector $|f\rangle$ is determined by the l_2 sequence $\{f_n\}$:

$$|f\rangle = \sum_{n=0}^{\infty} f_n|n;\alpha\rangle . \quad (2.50)$$

Now since $\mathscr{D}_\alpha$ is the direct sum of D_β and $D_{\beta+(1/2)}$, $\beta = \alpha/2$, the even members $\{f_{2l}\}$ define a vector lying in the subspace carrying the representation D_β, while the odd members $\{f_{2l+1}\}$ determine a vector lying in the complimentary space $D_{\beta+(1/2)}$. Each of these in turn gives an entire function of ω via Eq. (2.12) and a similar one with $\beta + \frac{1}{2}$ in place of β. Thus, we have

$$|f\rangle = \sum_{l=0}^{\infty} f_{2l}|l;\beta\rangle \oplus \sum_{l=0}^{\infty} f_{2l+1}|l;\beta+\tfrac{1}{2}\rangle \to f_1(\omega), f_2(\omega) , \quad (2.51)$$

$$f_1(\omega) = [\Gamma(\alpha)]^{1/2}\sum_{l=0}^{\infty} f_{2l}[n!\Gamma(n+\alpha)]^{-1/2}\omega^l , \quad (2.52)$$

$$f_2(\omega) = [\Gamma(\alpha+1)]^{1/2}\sum_{l=0}^{\infty} f_{2l+1}[n!\Gamma(n+\alpha+1)]^{-1/2}\omega^l . \quad (2.53)$$

From Eq. (2.27) and the relations (2.17) and (2.40) we readily find that the pair of entire functions of f_1 and f_2 are related to the single entire function $f(z)$ by the equation

$$f(z) = f_1(\tfrac{1}{2}z^2) + [z/(2\alpha)^{1/2}]f_2(\tfrac{1}{2}z^2) . \quad (2.54)$$

Thus, f_1 and f_2 determine the even and the odd parts, respectively, of f:

$$f_+(z) = f_1(\tfrac{1}{2}z^2) , \quad (2.55a)$$

$$f_-(z) = (z/(2\alpha)^{1/2})f_2(\tfrac{1}{2}z^2) . \quad (2.55b)$$

We can now develop an expression for the inner product in the Hilbert space of entire functions of z carrying the para-Bose representation $\mathscr{D}_\alpha$. We begin with

$$\langle f'|f\rangle = \int d^2\omega[K(\omega;\beta)f_1'(\omega)^*f_1(\omega) + K(\omega;\beta+\tfrac{1}{2})f_2'(\omega)^*f_2(\omega)] , \quad (2.56)$$

obtained using Eq. (2.13) within D_β and a similar equation with $D_{\beta+(1/2)}$. $K(\omega;\beta)$ is given by Eq. (2.16). We now make the change of variable

$$\omega = z^2/2, \quad d^2\omega = |z|^2 d^2z , \quad (2.57)$$

and also allow for the fact that ω covers the complex plane twice while z covers it once. Using Eqs. (2.16) and (2.55), we then get:

$$\begin{aligned}\langle f'|f\rangle &= \left(\frac{2^{1-\alpha}}{\pi\Gamma(\alpha)}\right)\int d^2z|z|^{2\alpha}[K_{\alpha-1}(|z|^2)f'_+(z)^*f_+(z) + K_\alpha(|z|^2)f'_-(z)^*f_-(z)] \\ &= \frac{1}{2\alpha\pi\Gamma(\alpha)}\int d^2z|z|^{2\alpha}[(K_{\alpha-1}(|z|^2)+K_\alpha(|z|^2)) \times f'(z)^*f(z) + (K_{\alpha-1}(|z|^2)-K_\alpha(|z|^2)) \times f'(z)^*f(-z)] . \end{aligned} \quad (2.58)$$

Substituting for $f(z)$ and $f'(z)$ from Eq. (2.40) and observing that $|f\rangle$ and $|f'\rangle$ are arbitrary, we obtain the following resolution of the identity operator in $\mathscr{D}_\alpha$:

$$\hat{1}_\alpha = \frac{1}{2\pi}\int d^2z|z|^{2\alpha}\mathscr{F}_\alpha(|z|^2)[\{K_{\alpha-1}(|z|^2+K_\alpha(|z|^2)\} \times |z;\alpha\rangle\langle z;\alpha|] + \{K_{\alpha-1}(|z|^2)-K_\alpha(|z|^2)\} \times |z;\alpha\rangle\langle -z;\alpha| . \quad (2.59)$$

The appearance of the "nondiagonal" terms may be unexpected, but this formula has the virtue of being valid for all α and that the functions appearing in the integrand are all well behaved. For $\alpha = \frac{1}{2}$, we observe that $K_{1/2}(x) = K_{-1/2}(x)$ and Eq. (2.59) reduces to the diagonal resolution of the identity operator

$$\hat{1}_{1/2} = \frac{1}{\pi}\int |z;\tfrac{1}{2}\rangle\langle z;\tfrac{1}{2}|d^2z . \quad (2.60)$$

It is interesting to note as shown in Appendix A that a resolution of the identity in the form

$$\hat{1}_\alpha = \int F_1(z)|z;\alpha\rangle\langle z;\alpha||d^2z + \int F_2(z)|z;\alpha\rangle\langle -z;\alpha|d^2z \quad (2.61)$$

is also unique as long as we require that $F \pm F_2$ are positive definite functions.

It may further be observed that the nondiagonal nature of the representation (2.59) disappears if we rewrite it in terms of the eigenstates of $\hat{J}_- = \frac{1}{2}\hat{a}^2$, viz., $|\omega;\beta\rangle, |\omega,\beta+\frac{1}{2}\rangle$. Essentially, this appears in Eq. (2.56). Viewed differently, we may define

$$|z_\pm,\alpha\rangle = \frac{1}{\sqrt{2}}\{|z;\alpha\rangle \pm |-z;\alpha\rangle\} \tag{2.62}$$

and rewrite Eq. (2.59) in the form

$$\hat{1}_\alpha = \frac{1}{2\pi}\int d^2z|z|^{2\alpha}\mathscr{F}_\alpha(|z|^2)\{K_{\alpha-1}(|z|^2)|z_+;\alpha\rangle\langle z_+;\alpha| + K_\alpha(|z|^2)|z_-;\alpha\rangle\langle z_-;\alpha|\}, \tag{2.63}$$

which is essentially a "diagonal" representation. The states $|z_\pm;\alpha\rangle$ are orthogonal to each other, but are not properly normalized. They are in fact proportional to $|\omega;\beta\rangle$ and $|\omega;\beta+\frac{1}{2}\rangle$, respectively.

One may observe that the operator $\hat{R}_1$ introduced in Eq. (I2.22) has the effect of changing $|z;\alpha\rangle$ to $|-z;\alpha\rangle$:

$$\hat{R}_1|z;\alpha\rangle = |-z;\alpha\rangle, \tag{2.64}$$

so that

$$\hat{R}_1|z_\pm;\alpha\rangle = \pm|z_\pm;\alpha\rangle. \tag{2.65}$$

C. Diagonal coherent state representation of the identity operator in $\mathscr{D}_\alpha$

We now consider the question whether a diagonal resolution of the identity in terms of the coherent states exists in $\mathscr{D}_\alpha$:

$$\hat{1}_\alpha = \int d^2z\,\chi(z;\alpha)|z;\alpha\rangle\langle z;\alpha|. \tag{2.66}$$

On taking the matrix elements of Eq. (2.66) between the number states $|m;\alpha\rangle$ and $|n;\alpha\rangle$, it is readily seen from the orthogonality of these states that $\chi(z;\alpha)$, if it exists, depends on z through $|z|$ only and does not depend on the phase of z. We write

$$\rho = \tfrac{1}{2}|z|^2 \tag{2.67}$$

and take χ to be a function of ρ. We also write

$$\mathscr{K}(\rho;\alpha) = 2^{2-\alpha}\pi\{\Gamma(\alpha)\mathscr{F}_\alpha(2\rho)\}^{-1}\chi(\rho;\alpha). \tag{2.68}$$

For $m = n$, we then obtain using Eq. (2.34) the following moments of $\mathscr{K}$:

$$\begin{aligned}\int_0^\infty \rho^n\mathscr{K}(\rho;\alpha)d\rho &= \{\Gamma(\alpha)\}^{-1}\left[\frac{n}{2}\right]!\,\Gamma\left(\left[\frac{n+1}{2}\right]+\alpha\right),\\ &= \{\Gamma(\alpha)\}^{-1}l!\,\Gamma(l+\alpha), \quad n=2l,\\ &= \{\Gamma(\alpha)\}^{-1}l!\,\Gamma(l+1+\alpha), \quad n=2l+1.\end{aligned} \tag{2.69}$$

If Eq. (2.69) has a solution, then Eq. (2.66) is established. It follows from the results of Appendix A [uniqueness of representation (2.59)] that Eq. (2.69) has no solution if $\mathscr{K}(\rho;\alpha)$ was restricted to a positive definite function except for the case $\alpha = \frac{1}{2}$, (when $\mathscr{K} = 2e^{-2\rho}$). Hence, in Eq. (2.69), we have to give up the positivity of $\mathscr{K}$.

Introducing a new variable σ, we may convert the moment conditions (2.69) into the equation

$$\int_0^\infty \mathscr{K}(\rho;\alpha)e^{i\rho\sigma}d\rho = \{\Gamma(\alpha)\}^{-1}\sum_{l=0}^{\infty}\left\{\frac{l!\,\Gamma(l+\alpha)}{(2l)!}(i\sigma)^{2l} + \frac{l!\,\Gamma(l+1+\alpha)}{(2l+1)!}(i\sigma)^{2l+1}\right\}, \tag{2.70}$$

valid within the radius of convergence of the power series on the right hand side, i.e., $|\sigma| < 2$. We rewrite Eq. (2.70) using the hypergeometric function

$$F(a,b;c;u) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{l=0}^{\infty}\frac{\Gamma(a+l)\Gamma(b+l)}{l!\,\Gamma(c+l)}u^l \tag{2.71}$$

in the form

$$\int_0^\infty \mathscr{K}(\rho;\alpha)e^{i\rho\sigma}d\rho = h(\sigma;\alpha), \tag{2.72}$$

where

$$h(\sigma;\alpha) = F(\alpha,1;\tfrac{1}{2};-\tfrac{1}{4}\sigma^2) + i\alpha\sigma F(\alpha+1,1;\tfrac{3}{2};-\tfrac{1}{4}\sigma^2). \tag{2.73}$$

We can now state a precise condition that will determine, for each α, whether $\mathscr{K}$ exists. Equation (2.72) allows us to analytically continue the right hand side of Eq. (2.70) outside the circle $|\sigma| = 2$. Since the integral on the left hand side of Eq. (2.72) runs from 0 to ∞, a solution to our problem exists if and only if the right-hand side of Eq. (2.73) is free of singularities in the upper half of the complex σ plane. In general, the hypergeometric function (2.71) has a branch point at $u = 1$, with a cut conventionally drawn along the real axis from $u = 1$ to $u = \infty$, and in the cut plane it has no singularities. Thus, the function on the right hand side of Eq. (2.73) in general has a branch point on the positive imaginary axis at $\sigma = 2i$, with a cut from $\sigma = 2i$ to $\sigma = i\infty$. (The branc point at $\sigma = -2i$ is not relevant to us here.) Let us first calculate the discontinuity across the cut.

We must evaluate the limits of $h(\sigma,\alpha)$ as σ approaches a point on the positive imaginary axis beyond $2i$ from the right and from the left half-planes. For this we must use standard continuation formulas to deal with the hypergeometric function outside the circle of convergence of its power series definition. For the moment, assume $\alpha \neq 1,2,3,\cdots$. Then the relevant formulas are [Ref. 4, p. 1043, formula (9.132.2)]

$$\begin{aligned}F(\alpha,1;\tfrac{1}{2};u) &= [\pi^{1/2}\Gamma(1-\alpha)/\Gamma(\tfrac{1}{2}-\alpha)](-u)^{-\alpha}F(\alpha,\alpha+\tfrac{1}{2};\alpha;1/u) + [2u(1+\alpha)]^{-1}F(1,\tfrac{3}{2};2-\alpha;1/u),\\ F(\alpha+1,1;\tfrac{3}{2};u) &= \tfrac{1}{2}\pi^{1/2}[\Gamma(-\alpha)/\Gamma(\tfrac{1}{2}-\alpha)](-u)^{-\alpha-1}F(\alpha+1,\alpha+\tfrac{1}{2};\alpha+1;1/u)\\ &\quad - \tfrac{1}{2}(u\alpha)^{-1}F(1,\tfrac{1}{2};1-\alpha;1/u), \quad |\arg(-u)| < \pi.\end{aligned} \tag{2.74}$$

We can now calculate the jump of h across its cut. With y a real number greater than 2, we find that

$$\begin{aligned}h(iy-\epsilon;\alpha) - h(iy+\epsilon;\alpha) &= [\pi^{1/2}\Gamma(1-\alpha)/\Gamma(\tfrac{1}{2}-\alpha)](e^{i\pi\alpha}-e^{-i\pi\alpha})(y^2/4)^{-\alpha}F(\alpha,\alpha+\tfrac{1}{2};\alpha;4/y^2)\\ &\quad - \alpha y[\pi^{1/2}\Gamma(-\alpha)/\{2\Gamma(\tfrac{1}{2}-\alpha)\}](e^{i\pi(\alpha+1)}-e^{-i\pi(\alpha+1)})(y^2/4)^{-\alpha-1}F(\alpha+1,\alpha+\tfrac{1}{2};\alpha+1;4/y^2)\end{aligned}$$

$$= [2i\pi^{3/2}/\{\Gamma(\alpha)\Gamma(\tfrac12-\alpha)\}](\tfrac14 y^2)^{-\alpha}(1-2/y)F(\alpha,\alpha+\tfrac12;\alpha;4/y^2)\,. \tag{2.75}$$

It turns out that this final result is valid even if $\alpha = 1,2,3,\cdots$, though to obtain it in these cases one must use formulas other than Eq. (2.74) to perform the analytic continuation. So Eq. (2.75) is valid for all $\alpha > 0$ (and, of course, $y > 2$). It follows that for all values of α other than $\frac12,\frac32,\frac52,\cdots$, $h(\sigma;\alpha)$ certainly has a branch point at $\sigma = 2i$, with a nonzero discontinuity across the cut, so no solution exists for the moment problem (2.69). Thus, in the para-Bose representation $\mathscr{D}_\alpha$ with $\alpha \neq \frac12,\frac32,\frac52,\cdots$, there is no diagonal coherent state resolution of the identity.

If $\alpha = m+\frac12$ with $m = 0,1,2,\cdots$, the expression (2.75) vanishes, so $h(\sigma;m+\frac12)$ has no branch point in the upper half σ plane. We must now check whether it has a pole at $\sigma = 2i$. As mentioned earlier, it definitely has no singularities anywhere else in the upper half σ plane. It turns out that we are able to express h in quite elementary form for the set of values of α being considered. One has, in fact, the results

$$F(m+\tfrac12,1;\tfrac12;u) = -\tfrac12[\Gamma(m+1)/\{\Gamma(m+\tfrac12)(1-u)^{m+1}\}]\sum_{n=0}^{m}\frac{\Gamma(n-\frac12)(1-u)^n}{n!},$$

$$F(m+\tfrac32,1;\tfrac12;u) = \frac{\Gamma(m+1)}{2\Gamma(m+3/2)(1-u)^{m+1}}\sum_{n=0}^{m}\frac{\Gamma(n+\frac12)(1-u)^n}{n!},\quad m=0,1,2,\cdots. \tag{2.76}$$

[See, for example, Ref. 5, p. 110, formula (14)]. This leads to the following explicit expression for h:

$$h(\sigma;m+\tfrac12) = -\tfrac12[m!/\Gamma(m+\tfrac12)](1+\tfrac14\sigma^2)^{-m-1}\sum_{n=0}^{m}\Gamma(n-\tfrac12)[1+i(\tfrac12-n)\sigma](1+\tfrac14\sigma^2)^n/n!,\quad m=0,1,2,\cdots. \tag{2.77}$$

For the normal Bose case $m = 0$, the potential pole at $\sigma = 2i$ due to the factor standing ahead of the sum is killed by the sum (actually just one term) and

$$h(\sigma;\tfrac12) = (1-\tfrac12 i\sigma)^{-1}, \tag{2.78}$$

so the moment problem has a solution and we get $\mathscr{K}$ inverting the Fourier transform in (2.72):

$$\mathscr{K}(\rho;\tfrac12) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\sigma\exp(-i\rho\sigma)h(\sigma;\tfrac12) = \begin{cases}2e^{-2\rho}, & \rho>0,\\ 0, & \rho<0.\end{cases} \tag{2.79}$$

This, when used in Eq. (2.68), leads to the known diagonal resolution of the identity for the normal Bose case:

$$\hat{1}_{1/2} = \frac{1}{\pi}\int d^2z|z;\tfrac12\rangle\langle z;\tfrac12|\,. \tag{2.80}$$

However, for $\alpha = \frac32,\frac52,\cdots$, i.e., $m = 1,2,\cdots$, $h(\sigma;\alpha)$ always has a pole, of order m, at $\sigma = 2i$. Thus, except for the very special case $\alpha = \frac12$, $h(\sigma;\alpha)$ always has a singularity at $\sigma = 2i$ in the upper half-plane, this being either a branch point ($\alpha \neq \frac32,\frac52,\cdots$) or a pole ($\alpha = \frac32,\frac52,\cdots$). We conclude that the moment problem (2.69) has no solution if $\alpha \neq \frac12$.

A quick way to reach this conclusion avoiding an analysis of singularities in the complex σ plane is to note that the Fourier inverse transform of $h(\sigma;\alpha)$ is explicitly calculable [Ref. 4, p. 853 formulas (7.531.1), (7.531.2)]:

$$\mathscr{K}'(\rho;\alpha) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\rho\sigma)h(\sigma;\alpha)d\sigma$$
$$= 2|\rho|^\alpha[\Gamma(\alpha)]^{-1}\{K_{\alpha-1}(2|\rho|)+\epsilon(\rho)K_\alpha(2|\rho|)\},\quad -\infty<\rho<\infty\,. \tag{2.81}$$

Thus, if Eq. (2.69) or (2.72) has a solution, it must be given by Eq. (2.81), which should automatically vanish for $\rho < 0$. This happens only when $\alpha = \frac12$.

Hence, we find that a diagonal coherent state resolution (2.66) *of the identity operator does not exist, even if we allow* $\mathscr{K}$ *to be a distribution* (*as usually defined*) *in any para-Bose representation* $\mathscr{D}_\alpha$ *except* $\mathscr{D}_{1/2}$.

3. PARA-BOSE OPERATOR DESCRIPTION

We now consider some aspects of the description operators acting on a space carrying the para-Bose representations $\mathscr{D}_\alpha$. We first recall the situation in the familiar $\alpha = \frac12$ case and then consider the problem of generalizing those results.

In the $\mathscr{D}_{1/2}$ space we can write the coherent state $|z;\frac12\rangle$ in the form

$$|z;\tfrac12\rangle = e^{z\hat{a}^\dagger - z^*\hat{a}}|0;\tfrac12\rangle$$

$$= e^{-(1/2)|z|^2}e^{z\hat{a}^\dagger}|0;\tfrac12\rangle\,. \tag{3.1}$$

A fairly large and important class of operators can be described by the Weyl representation

$$\hat{A} = \int d^2z\,F(z)e^{z\hat{a}^\dagger - z^*\hat{a}}, \tag{3.2}$$

which is analogous to a Fourier representation with a c-number weight function $F(z)$. We also have the diagonal coherent state representation valid for a certain class of operators

$$\hat{A} = \int \phi(z)|z;\tfrac{1}{2}\rangle\langle z;\tfrac{1}{2}|d^2z\,. \tag{3.3}$$

There is close relationship between a particular ordered form of $\hat{A}$ and the various representations of $\hat{A}$ such as $F(z)$ or $\phi(z)$.[6] From the commutation relations of $\hat{a}$ and $\hat{a}^\dagger$ one obtains the operator relations

$$e^{z\hat{a}^\dagger + z'\hat{a}} = e^{z\hat{a}^\dagger}e^{z'\hat{a}}e^{zz'/2} = e^{z'\hat{a}}e^{z\hat{a}^\dagger}e^{-zz'/2}\,, \tag{3.4}$$

with z,z' any two complex numbers. Such relations when used in Eq. (3.2) allow us to express $\hat{A}$ in normal ordered form with dependences on $\hat{a}^\dagger$ standing to the left of the dependences on $\hat{a}$, or in the antinormal ordered form with the dependences on $\hat{a}$ to the left of the dependences on $\hat{a}^\dagger$. Putting $z' = -z^*$ in Eq. (3.4) we find that the normal (antinormal) ordered description involves better (worse) behaved weight function relative to that of Weyl representation. Thus, for example, we obtain the following normal and antinormal ordered forms of $\hat{A}$ from Eq. (3.2):

$$\hat{A} = \int F(z)e^{-(1/2)|z|^2}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}d^2z\,, \tag{3.5}$$

$$= \int F(z)e^{(1/2)|z|^2}e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}d^2z\,. \tag{3.6}$$

Using the resolution of the identity (2.60), one can immediately obtain the diagonal coherent state representation $\hat{A}$ from its antinormal ordered form

$$\hat{A} = \frac{1}{\pi}\int F(z')e^{(1/2)|z'|^2}e^{-z'^*\hat{a}}|z;\tfrac{1}{2}\rangle\langle z;\tfrac{1}{2}|e^{z'\hat{a}^\dagger}d^2z\,d^2z'$$
$$= \int d^2z\int d^2z'F(z')\exp\{\tfrac{1}{2}|z'|^2 - z'^*z + z'z^*\}|z;\tfrac{1}{2}\rangle\langle z;\tfrac{1}{2}|\,. \tag{3.7}$$

Thus, we find

$$\phi(z) = \frac{1}{\pi}\int d^2z'F(z')e^{(1/2)|z'|^2}\exp(z^*z' - zz'^*)\,. \tag{3.8}$$

Similarly, the normal ordered form of $\hat{A}$ is related to the diagonal matrix elements of $\hat{A}$ in the coherent state:

$$\langle z|\hat{A}|z\rangle = \int d^2z'F(z')e^{-(1/2)|z'|^2}\exp(z^*z' - zz'^*)\,. \tag{3.9}$$

It is unfortunately not easy to obtain generalizations of these results in the para-Bose representation $\mathscr{D}_\alpha$ for $\alpha \neq \frac{1}{2}$. For example, we have seen in Sec. 2 that even the identity operator does not have a diagonal coherent state representation for $\alpha \neq \frac{1}{2}$. Most of the statements we can make about operator descriptions in $\mathscr{D}_\alpha$ rest on general arguments and not on any explicit calculations.

From Eqs. (I2.20) and (2.34) we obtain the following generalizations of Eq. (3.1):

$$|z;\alpha\rangle = \mathscr{N}_\alpha(|z|)\Gamma(\alpha)2^{\alpha-1}\mathscr{F}_\alpha(z\hat{a}^\dagger)|0;\alpha\rangle\,, \tag{3.10}$$

where $\mathscr{N}_\alpha$ and $\mathscr{F}_\alpha$ are given by Eqs. (2.30) and (2.31), respectively.

We now consider whether we can express an operator in normal or antinormal ordered forms for general α. An operator $\hat{A}$ is clearly determined by the values of its coherent state matrix elements

$$\langle z';\alpha|\hat{A}|z;\alpha\rangle \tag{3.11}$$

in the sense that if this matrix element vanishes for all z and z', then $\hat{A}$ must vanish. Now that bra and ket vectors involved above depend on z and z' such that the function

$$\langle z';\alpha|\hat{A}|z;\alpha\rangle/\langle z';\alpha|z;\alpha\rangle \tag{3.12}$$

is analytic in z'^* and z. This is evident from the relation (2.34). We may thus define an analytic function $f(z'^*,z)$, which does not depend on z' and z^*, as

$$f(z'^*,z) = \langle z';\alpha|\hat{A}|z;\alpha\rangle/\langle z';\alpha|z;\alpha\rangle\,. \tag{3.13}$$

If in f we set $\hat{a}^\dagger$ in place of z'^* and $\hat{a}$ in place of z, and always keep the former to the left of latter, we obtain an operator $:f(\hat{a}^\dagger|\hat{a}):$ in a normal ordered form all of whose coherent state matrix elements coincide with those of $\hat{A}$. The two operators must then be equal

$$\hat{A} = :f(\hat{a}^\dagger|\hat{a}):\,. \tag{3.14}$$

We have used a bar rather than a coma to separate the arguments of f to stress the normal ordered nature of this operator. Thus, in principle, every operator is expressible as some normal ordered function of $\hat{a}^\dagger$ and $\hat{a}$. It is perhaps important to stress that this argument rests on the properties of coherent states and not on a recipe for moving $\hat{a}^\dagger$ and $\hat{a}$ past each other.[7]

One can show that any normal ordered operator can be rewritten as the sum of two parts in the form

$$:f(\hat{a}^\dagger|\hat{a}): = \text{“}g(\hat{a}|\hat{a}^\dagger)\text{”} + \hat{R}_1\text{“}h(\hat{a}|\hat{a}^\dagger)\text{,”} \tag{3.15}$$

where $\hat{R}_1$ is defined from the relation

$$[\hat{a},\hat{a}^\dagger] = 1 + (2\alpha - 1)\hat{R}_1\,, \tag{3.16}$$

and g and h are both antinormal normal ordered and uniquely determined by f. One reaches this conclusion by working with simple monomials and obtaining results such as

$$\hat{a}^\dagger\hat{a}^{2l} = \hat{a}^{2l}\hat{a}^\dagger - 2l\hat{a}^{2l-1}\,, \tag{3.17a}$$

$$\hat{a}^\dagger\hat{a}^{2l+1} = \hat{a}^{2l+1}\hat{a}^\dagger - (2l+1)\hat{a}^{2l} - (2\alpha - 1)\hat{R}_1\hat{a}^{2l}\,, \tag{3.17b}$$

$$\hat{a}^{\dagger 2l}\hat{a} = \hat{a}\hat{a}^{\dagger 2l} - 2l\hat{a}^{\dagger 2l-1}\,, \tag{3.17c}$$

$$\hat{a}^{\dagger 2l+1}\hat{a} = \hat{a}\hat{a}^{\dagger 2l+1} - (2l+1)\hat{a}^{\dagger 2l} - (2\alpha - 1)\hat{R}_1\hat{a}^{\dagger 2l}\,. \tag{3.17d}$$

These relations can be established by induction. Using these relations, one may verify that the general expression

$$\hat{a}^{+m}\hat{a}^n$$

can be systematically transformed to finally assume the form of the right-hand side of Eq. (3.15). Unfortunately, no simple analytical expression can be worked out for general m and n; if it were, one could try generalizing Eq. (3.4) for $\alpha \neq \frac{1}{2}$ (but with $\hat{R}_1$ present in the expressions). Thus, we see that any operator has, in addition to the normal ordered form (3.14), the possibility of being expressed in the form (3.15), which way be called a quasi-antinormal ordered form. This result of course neither confirms nor denies the possibility of achieving a true antinormal ordered form, which is possible if $\hat{R}_1$ itself is expressible in antinormal form.

Using the structure (3.15) and inserting the resolution of identity (2.59) in between $\hat{a}$ and $\hat{a}^\dagger$ and also using the obvious result

$$\hat{R}_1|z;\alpha\rangle = |-z;\alpha\rangle\,, \tag{3.18}$$

we readily see that every operator $\hat{A}$ posseses, in principle, a representation

$$\hat{A} = :f(\hat{a}^\dagger|\hat{a}): = \int \phi_1(z,z^*)|z;\alpha\rangle\langle z;\alpha|d^2z + \int \phi_2(z,z^*)|z;\alpha\rangle\langle -z;\alpha|d^2z\,. \tag{3.19}$$

Here

$$\phi_1(z,z^*) = F_1(|z|^2)g(z,z^*) + F_2(|z|^2)h(-z,z^*)\,, \tag{3.20a}$$

$$\phi_2(z,z^*) = F_2(|z|^2)g(z,-z^*) + F_1(|z|^2)h(-z,-z^*)\,, \tag{3.20b}$$

and

$$F_1(x) = \frac{1}{2\pi}x^\alpha \mathscr{F}_\alpha(x)\{K_{\alpha-1}(x) + K_\alpha(x)\}\,, \tag{3.21a}$$

$$F_2(x) = \frac{1}{2\pi}x^\alpha \mathscr{F}_\alpha(x)\{K_{\alpha-1}(x) - K_\alpha(x)\}\,. \tag{3.21b}$$

Thus, we find that if we are given g and h we obtain ϕ_1 and ϕ_2. Conversely, knowing a representation of the type (3.19), i.e., ϕ_1 and ϕ_2, we may determine g and h. For this, we regard ϕ_1, ϕ_2, g, and h as functions of two independent complex variables. We rewrite Eq. (3.20b) by replacing z^* with $-z^*$:

$$\phi_2(z,-z^*) = F_2(-|z|^2)\,g(z,z^*) + F_1(-|z|^2)h(-z,z^*)\,. \tag{3.22}$$

We then obtain, from Eqs. (3.20a) and (3.22) the following expressions for g and h:

$$g(z,z^*) = \frac{F_1(-|z|^2)\phi_1(z,z^*) - F_2(|z|^2)\phi_2(z,-z^*)}{F_1(|z|^2)F_1(-|z|^2) - F_2(|z|^2)F_2(-|z|^2)}\,, \tag{3.23}$$

$$h(-z,z^*) = \frac{F_2(-|z|^2)\phi_1(z,z^*) - F_1(|z|^2)\phi_2(z,-z^*)}{F_2(|z|^2)F_2(-|z|^2) - F_1(|z|^2)F_1(-|z|^2)}\,. \tag{3.24}$$

Lastly, we discuss the existence of diagonal and discrete diagonal coherent state approximations (not representations) to operators. For definiteness let us restrict ourselves to the family of Hilbert–Schmidt (H–S) operators $\hat{A}$ for which

$$\mathrm{Tr}(\hat{A}^\dagger\hat{A}) < \infty\,. \tag{3.25}$$

The expression

$$(A,B) = \mathrm{Tr}(\hat{A}^\dagger\hat{B}) \tag{3.26}$$

serves as an inner product among such operators, making them elements of a Hilbert space. Condition (3.25) can be expressed in the basis $|n;\alpha\rangle$ as

$$\sum_m\sum_n |\langle m;\alpha|\hat{A}\,|n;\alpha\rangle|^2 < \infty\,, \tag{3.27}$$

so that these matrix elements of $\hat{A}$ are surely bounded, and in fact go to zero for large values of m and n. Using the argument used in Ref. 8, it now follows that the diagonal coherent matrix elements of $\hat{A}$ are separately analytic, in fact entire, in the real and imaginary parts of z, the eigenvalue of $\hat{a}$. Thus, using Eq. (2.34),

$$\begin{aligned}\langle z;\alpha|\hat{A}\,|z;\alpha\rangle = \mathscr{N}_\alpha^2\Gamma(\alpha)\sum_{l,m=0}^{\infty}2^{-m-l}&\times[m!l!\Gamma(\alpha+m)\Gamma(\alpha+l)]^{-1/2}\\ &\times\Big\{(x-iy)^{2m}(x+iy)^{2l}A_{2m,2l}\\ &+\frac{(x-iy)^{2m}(x+iy)^{2l+1}}{(2\alpha+2l)^{1/2}}A_{2m,2l+1}\\ &+\frac{(x-iy)^{2m+1}(x+iy)^{2l}}{(2\alpha+2m)^{1/2}}A_{2m+1,2l}\\ &+\frac{(x-iy)^{2m+1}(x+iy)^{2l+1}}{2[(\alpha+m)(\alpha+l)]^{1/2}}A_{2m+1;2l+1}\Big\}\end{aligned} \tag{3.28}$$

is the boundary value, for real x and y, of an entire analytic function in two complex variable ξ and η, say, defined by the replacement $x\to\xi$ and $y\to\eta$ on the right hand side of Eq. (3.28). This allows us to assert that an (H–S) operator $\hat{A}$ is fully determined by its diagonal coherent state matrix elements since by the principle of uniqueness of analytic continuation

$$\langle z;\alpha|\hat{A}\,|z;\alpha\rangle = 0 \text{ for all } z\Rightarrow\hat{A} = 0\,. \tag{3.29}$$

Using the inner product notation (3.26) we could state this result as

$$(|z;\alpha\rangle\langle z;\alpha|,\hat{A}) = 0,\quad \text{all } z \Rightarrow\hat{A} = 0\,. \tag{3.30}$$

However, this has the interpretation that in the Hilbert space of all H–S operators, "linear combinations" of the (continuous) family of elements $|z;\alpha\rangle\langle z;\alpha|$ form a dense set, so that any operator $\hat{A}$ can be approximated arbitrarily closely by such linear combinations. We can therefore assert that the diagonal coherent state approximation to a given operator $\hat{A}$:

$$\hat{A}\sim\int d^2z\varphi(z)|z;\alpha\rangle\langle z;\alpha| \tag{3.31}$$

can be found to arbitrary accurary.

Actually, it is possible to replace Eq. (3.29) by a much more economical one. It is well known that an entire function vanishes identically if it vanishes on a suitably chosen infinite sequence of points in the complex plane. For example, a sequence with a finite limit point has this property. In general, a set of points in the complex plane with the property that the only entire function (out of the class of entire functions under discussion) that vanishes on this set is the zero function is called a characteristic set. For entire functions in two variables, characteristic sets are defined in the product of the complex plane by itself. We can now replace Eq. (3.29) by the following one: choose two characteristic sets $\{x_j\}$, $\{y_k\}$ in the complex plane, both restricted to the real axis. Define the set of points z_{jk} by

$$z_{jk} = x_j + iy_k\,. \tag{3.32}$$

Then

$$\langle z_{jk};\alpha|\hat{A}\,|z_{jk};\alpha\rangle = 0,\ \text{all } j \text{ and } k\Rightarrow\hat{A} = 0\,. \tag{3.33}$$

Alternatively,

$$(|z_{jk};\alpha\rangle\langle z_{jk};\alpha|,\hat{A}) = 0,\quad \text{all } j \text{ and } k \Rightarrow\hat{A} = 0\,. \tag{3.34}$$

Thus, such a denumerable sequence of coherent state projection operators already yields via its linear combinations a dense set in the Hilbert space of H–S operators, leading to the existence of discrete diagonal coherent state approximations to a given $\hat{A}$:

$$\hat{A} \sim \sum_{j,k} \varphi_{ik} |z_{jk};\alpha\rangle\langle z_{jk};\alpha| \tag{3.35}$$

to any desired accuracy.

4. POSITION–MOMENTUM UNCERTAINTY PRODUCT

In this section we consider the problem of constructing states with a minimum value of the product of the uncertainties in position and momentum variables. It is well known that if

$$[\hat{A},\hat{B}] = i\hat{C}, \tag{4.1}$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are Hermitian operators, then one has the inequality

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geqslant \tfrac{1}{4}|\langle\hat{C}\rangle|^2, \tag{4.2}$$

where

$$\langle(\Delta\hat{A})^2\rangle = \langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2, \tag{4.3}$$

and the sharp brackets denote the quantum expectation values in the given state. Relation (4.2) reduces to an equality if and only if the given state is an eigenstate of $(\hat{A} + i\lambda\hat{B})$, where λ is some real number. Let us now identify $\hat{A}$ and $\hat{B}$ as the position and momentum variables respectively of the para-Bose system

$$\hat{A} = \hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \hat{B} = \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{(\sqrt{2})i}, \tag{4.4}$$

so that

$$[\hat{A},\hat{B}] = i[\hat{a},\hat{a}^\dagger] = i\{1 + (2\alpha - 1)\hat{R}_1\}, \tag{4.5}$$

where [cf. Eq. (I2.22)]

$$\hat{R}_1|n;\alpha\rangle = (-1)^n|n;\alpha\rangle. \tag{4.6}$$

We then find that

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \tfrac{1}{4}\langle[\hat{a},\hat{a}^\dagger]\rangle^2. \tag{4.7}$$

We may readily verify that relation (4.7) is an equality for the para-Bose coherent states (being the eigenstates of the operator $\hat{q} + i\hat{p}$). However, since $[\hat{a},\hat{a}^\dagger]$ is in general not a c-number, the right-hand side of (4.7) itself depends on the given state. Hence, the para-Bose coherent states do not minimize the product of the uncertainties in $\hat{q}$ and $\hat{p}$ in the absolute sense, except for the ordinary Bose case where $\alpha = \frac{1}{2}$. Relation (4.7) gives the minimum value of the product of the uncertainties only in a restricted sense. Consider all those states for which $\frac{1}{4}\langle[\hat{a},\hat{a}^\dagger]\rangle^2$ is a given definite number. The uncertainty product in any of these states is greater than or equal to this number. There is no guarantee that such states would include any coherent state.

In order to determine the minimum value of the uncertainty product, we consider the cases $\alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\alpha > \frac{1}{2}$ separately.

Case 1: $\alpha < \frac{1}{2}$. The commutator $[\hat{a},\hat{a}^\dagger]$ can be expressed in the form [cf. Eq. (4.5)]

$$[\hat{a},\hat{a}^\dagger] = 2\alpha\hat{P}_e + 2(1-\alpha)\hat{P}_o, \tag{4.8}$$

where $\hat{P}_e$ and $\hat{P}_o$ are the projection operators on the even and odd number states respectively:

$$\hat{P}_e = \sum_{n=0}^{\infty} |2n;\alpha\rangle\langle 2n;\alpha|, \tag{4.9a}$$

$$\hat{P}_o = \sum_{n=0}^{\infty} |2n+1;\alpha\rangle\langle 2n+1;\alpha|. \tag{4.9b}$$

Since $\hat{P}_e + \hat{P}_o = \hat{1}$, we readily find from Eq. (4.8) that

$$2(1-\alpha) - 2(1-2\alpha)\,\hat{P}_e = [\hat{a},\hat{a}^\dagger] = 2\alpha + 2(1-2\alpha)\,\hat{P}_o. \tag{4.10}$$

Further, $\alpha < \frac{1}{2}$ and both $\hat{P}_e$ and $\hat{P}_o$ are positive definite operators; we have

$$2(1-\alpha) \geqslant \langle[\hat{a},\hat{a}^\dagger]\rangle \geqslant 2\alpha. \tag{4.11}$$

From the equalities (4.7) and (4.11) we the obtain

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \alpha^2, \tag{4.12}$$

giving us a lower bound to the product of the uncertainties in $\hat{q}$ and $\hat{p}$. In order to search for the states for which this lower bound is actually reached, we observe that for such states we must have [cf. relations (4.7), (4.11), and (4.12)]

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}\langle[\hat{a},\hat{a}^\dagger]\rangle^2 \tag{4.13}$$

and

$$\langle[\hat{a},\hat{a}^\dagger]\rangle = 2\alpha \tag{4.14}$$

separately. For Eq. (4.13) to hold, the given state must be an eigenstate of $\hat{q} + i\lambda\hat{p}$ for some real λ (in fact $\lambda \geqslant 0$, since there are no eigenstates of $\hat{q} + i\lambda\hat{p}$ for $\lambda < 0$). Also, from Eqs. (4.10) and (4.14) we find that such a state could contain only the even number states and hence

$$\langle\hat{q} + i\lambda\hat{p}\rangle = 0, \tag{4.15}$$

which follows from the fact that $\hat{q}$ and $\hat{p}$ have nonzero matrix elements between the neighboring number states only [cf. Eq. (I2.20)].

We thus conclude that for $\alpha < \frac{1}{2}$, the minimum value of the product of the uncertainties $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$ is α^2, and that this is achieved for the states which are the eigenstates of $\hat{q} + i\lambda\hat{p}$ with eigenvalue zero. One may readily see that this state is given by

$$|\psi\rangle = \left\{\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!}\left(\frac{\lambda-1}{\lambda+1}\right)^{2n}\right\}^{-1/2} \times \sum_{n=0}^{\infty}\left\{\left(\frac{\Gamma(\alpha+n)}{n!}\right)^{1/2}\left(\frac{\lambda-1}{\lambda+1}\right)^{n}|2n;\alpha\rangle\right\}, \tag{4.16}$$

which in fact contains only the even number states.

For such a state, we find that[9]

$$\langle(\Delta\hat{q})^2\rangle = \lambda\alpha, \tag{4.17a}$$

$$\langle(\Delta\hat{p})^2\rangle = \alpha/\lambda. \tag{4.17b}$$

We also have, therefore,

$$\langle(\Delta\hat{q})^2\rangle + \langle(\Delta\hat{p})^2\rangle \geqslant 2\alpha, \tag{4.18}$$

with equality holding only if $\lambda = 1$, i.e., for the ground state $|0;\alpha\rangle$.

It is interesting to note that the minimum uncertainty state (4.16) may also be written directly as

$$|\psi\rangle = \exp\{\tfrac{1}{4}\log\lambda\,(\hat{a}^{\dagger 2} - \hat{a}^2)\}|0;\alpha\rangle, \tag{4.19}$$

which follows from the fact that

$$e^{(1/4)\log\lambda\,(\hat{a}^{\dagger 2} - \hat{a}^2)}\hat{a}e^{-(1/4)\log\lambda\,(\hat{a}^{\dagger 2} - \hat{a}^2)} = \hat{q} + i\lambda\hat{p}, \quad \lambda > 0, \tag{4.20}$$

and that $|0;\alpha\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue 0.

Case 2: $\alpha = \frac{1}{2}$. This is the familiar case of the ordinary Bose oscillator. In this case the commutator $[\hat{a},\hat{a}^\dagger]$ is a c-number, and the situation is very simple and in fact well known. The minimum uncertainty product $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$ is now $\frac{1}{4}$ and this is achieved for the eigenstates of the operator $\hat{q} + i\lambda\hat{p}$, $\lambda > 0$. Again, here also, we find that $\langle(\Delta\hat{q})^2\rangle + \langle(\Delta\hat{p})^2\rangle \geqslant 1$ and the minimum value 1 is reached for the coherent states, i.e., the eigenstates of $\hat{q} + i\hat{p}$.

Case 3: $\alpha > \frac{1}{2}$. Consider first the case when $\frac{1}{2} \leqslant \alpha \leqslant 1$. From Eq. (4.10), we find in this case that

$$\langle[\hat{a},\hat{a}^\dagger]\rangle \geqslant 2(1-\alpha)$$

and hence

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant (1-\alpha)^2 \,. \tag{4.21}$$

It may readily be seen that equality in (4.21) can never hold. For if it does, the given state must be an eigenstate of $\hat{q} + i\lambda\hat{p}$ and in addition we must have

$$\langle[\hat{a},\hat{a}^\dagger]\rangle = 2(1-\alpha)\,.$$

Equation (4.10) then shows that the minimum uncertainty state should contain only the odd number states and such a state can never be an eigenstate of $\hat{q} + i\lambda\hat{p}$ for any real positive λ.

For the case when $\alpha \geqslant 1$, we can have states for which $\langle[\hat{a},\hat{a}^\dagger]\rangle = 0$; however, such a state can never be an eigenstate of $\hat{q} + i\lambda\hat{p}$ so that the product of the uncertainties in $\hat{q}$ and $\hat{p}$ can never be made to vanish.

It is believed that when $\alpha > \frac{1}{2}$, the lower bound of $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$ is $\frac{1}{4}$, the same as in the ordinary Bose case. This conjecture is based on the following observations:

(1) The coherent states are the eigenstates of $\hat{q} + i\lambda\hat{p}$ with $\lambda = 1$. For such states we have

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}|\langle z;\alpha|[\hat{a},\hat{a}^\dagger]|z;\alpha\rangle|^2\,. \tag{4.22}$$

Further, we find from Eqs. (4.5), (2.64), and (2.35) that[2]

$$\langle z;\alpha|[\hat{a},\hat{a}^\dagger]|z;\alpha\rangle = 1 + (2\alpha - 1)\,\frac{I_{\alpha-1}(|z|^2) - I_\alpha(|z|^2)}{I_{\alpha-1}(|z|^2) + I_\alpha(|z|^2)}\,, \tag{4.23}$$

where I_α is the modified Bessel functions. Hence, for coherent states we obtain

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}\Big\{1 + (2\alpha-1) \times \frac{I_{\alpha-1}(|z|^2) - I_\alpha(|z|^2)}{I_{\alpha-1}(|z|^2) + I_\alpha(|z|^2)}\Big\}^2. \tag{4.24}$$

We show in Appendix B that

$$I_{\alpha-1}(|z|^2) > I_\alpha(|z|^2), \quad \alpha \geqslant \tfrac{1}{2}\,, \tag{4.25}$$

so that from Eq. (4.24) we find for $\alpha \geqslant \frac{1}{2}$ the inequality

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle > \tfrac{1}{4}\,. \tag{4.26}$$

It may be observed that there is no coherent state for which the uncertainty product actually takes the value $\frac{1}{4}$. However, by letting z be very large we may approach this value as close as we like.

(2) Consider any stationary state represented by a density operator $\hat{\rho}$. This implies that $\hat{\rho}$ commutes with the Hamiltonian, i.e., $\hat{\rho}$ is diagonal in the number representation

$$\hat{\rho} = \sum_{n=0}^{\infty} \rho_n |n;\alpha\rangle\langle n;\alpha|\,. \tag{4.27}$$

For such a state we find that $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$ and

$$\langle q^2\rangle = \langle\hat{p}^2\rangle = \sum_{n=0}^{\infty} \rho_n(n+\alpha) \geqslant \alpha, \tag{4.28}$$

with equality holding only for the ground state ($\rho_n = \delta_{n,0}$). Hence

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \alpha^2. \tag{4.29}$$

(3) From Eq. (4.10) we find that

$$\langle[\hat{a},\hat{a}^\dagger]\rangle = 2(1-\alpha) + 2(2\alpha-1)\langle\,\hat{P}_e\rangle\,. \tag{4.30}$$

Thus, if $\langle\hat{P}_e\rangle \geqslant \frac{1}{2}$ (and $\alpha \geqslant \frac{1}{2}$),

$$\langle[\hat{a},\hat{a}^\dagger]\rangle \geqslant 1\,. \tag{4.31}$$

Hence, for all those states for which $\langle\hat{P}_e\rangle \geqslant \frac{1}{2}$ we find

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \tfrac{1}{4}\,. \tag{4.32}$$

Also, if $\langle\hat{P}_e\rangle \leqslant [\alpha - (3/2)]/(2\alpha - 1)$, we have

$$\langle[\hat{a},\hat{a}^\dagger]\rangle \leqslant -1\,,$$

and again we find that

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \tfrac{1}{4}. \tag{4.33}$$

Of course, it is not necessary that the lower bound is actually obtained in these cases. Thus, we find that the product of the uncertainties cannot be less than $\frac{1}{4}$ if either $\langle\hat{P}_e\rangle \geqslant \frac{1}{2}$ or $\langle\hat{P}_e\rangle \leqslant [\alpha - (3/2)]/(2\alpha - 1)$. For α very large, it includes almost the whole range.

In Table I below, we summarize the results of this section.

It is interesting to observe that the ground state $|0;\alpha\rangle$ is actually an extremum state for the uncertainty product $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$ for all α in the sense that it satisfies

$$\delta\{\langle\psi|(\hat{q} - \langle\hat{q}\rangle)^2|\psi\rangle\langle\psi|(\hat{p} - \langle\hat{p}\rangle)^2|\psi\rangle\} = 0\,, \tag{4.34}$$

subject to the condition $\langle\psi/\psi\rangle = 1$. For $\alpha < \frac{1}{2}$, the ground state $|0;\alpha\rangle$ is the minimum uncertainty product state as discussed earlier. For $\alpha > \frac{1}{2}$ it turns out that $|0;\alpha\rangle$ is actually the maximum uncertainty product state. However, this maximum is only a local maximum, i.e., if we look into neighborhood of the ground state, we get $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$, a maximum for the ground state. The inequality (4.29) is not a contradiction, since any stationary state other than the ground state is not in the neighborhood of it. It appears that the general solution of Eq. (4.34) for $|\psi\rangle$ is the eigenstate of $\hat{q} + i\lambda\hat{p}$ with eigenvalue 0 [except for the normal Bose case $\alpha = \frac{1}{2}$, for which the general solution of Eq. (4.34) is any eigenstate of $\hat{q} + i\lambda\hat{p}$].

5. CONCLUDING REMARKS

We have considered energy, position, and momentum eigenstates and the Bargmann description of the para-Bose system with one degree of freedom. Using the coherent states, a resolution of the identity operator containing the diagonal and pseudodiagonal term has been obtained. Normal and antinormal ordering of para-Bose operators has been discussed. We also discussed the minimum value of the product of the uncertainties in position and momentum var-

TABLE I. Lower bound of the uncertainty product $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$.

α	Restriction on the state	Lower bound of $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle$	Remarks
$\alpha<\frac{1}{2}$	All states	α^2	Lower bound is obtained for the eigenstate of $\hat{q}+i\lambda\hat{p}$, with eigenvalue 0 ($\lambda\geqslant 0$).
$\alpha=\frac{1}{2}$	All states	$\frac{1}{4}$	Lower bound is obtained for any eigenstate of $\hat{q}+i\lambda\hat{p}$.
$\alpha>\frac{1}{2}$	Coherent states	$>\frac{1}{4}$	By taking z large, we can approach the value $\frac{1}{4}$ as close as we like.
	$\langle\hat{P}_e\rangle\geqslant\frac{1}{2}$, $\langle\hat{P}_e\rangle\leqslant\dfrac{\alpha-3/2}{2\alpha-1}$	$>\frac{1}{4}$	
All α	Stationary states $[\hat{\rho},\hat{H}]=0$	α^2	Lower bound is obtained for the ground state $\lvert 0;\alpha\rangle$.

iables. We have tried to generalize the several known results for the normal Bose case ($\alpha=\frac{1}{2}$) to the para-Bose case (general α). We have found significantly different results in the general case.

It is obvious that the coherent states form an overcomplete set. For the case $\alpha=\frac{1}{2}$, it has been shown[10] that the existence of a diagonal coherent state representation is analogous to the existence of an expansion of a given state in terms of coherent states with imaginary eigenvalues, i.e., in terms of the states of the form $\lvert ix\rangle$, with x real. This is demonstrated by introducing "super operators" whose action on the space consisting of ordinary operators is suitably defined. It will be of interest to see if such a formalism could be generalized for the para-Bose system. It is necessary, for this purpose, to consider first a para-Bose system with more than one degree of freedom. Unfortunately, the algebra for systems with more than one degree of freedom becomes much complicated[11] especially because even the operators belonging to different modes need not commute. One may then naturally ask for the minimum value of the uncertainty product $\langle(\Delta\hat{q}_1)^2\rangle\langle(\Delta\hat{p}_2)^2\rangle$ for the position and momentum variables in different modes. It will also be of interest to study the Weyl representation for para-Bose systems with one or more degrees of freedom.

APPENDIX A: A MOMENT PROBLEM

In this Appendix we consider the moment problem

$$\int_0^\infty dx\, K(x,\alpha)x^{2n+1}=n!\Gamma(\alpha+n)/(2\pi\Gamma(\alpha)), \tag{A1}$$

and show that if we restrict $K(x,\alpha)$ to be positive, it has a unique solution. We further show that this leads to unique resolutions of the identity operator in the spaces of the representations D_β [Eq. (2.23)] and $\mathscr{D}_\alpha$ [Eq. (2.61)] as long as the corresponding weight functions $F(\omega)$ or $F_1(z)\pm F_2(z)$ are restricted to be positive definite.

We set

$$x^2=t, \tag{A2}$$

$$\pi\Gamma(\alpha)K(x,\alpha)=\phi(t), \tag{A3}$$

and write Eq. (A1) in the form

$$\int_0^\infty \phi(t)t^n dt=n!\Gamma(\alpha+n). \tag{A4}$$

A solution of Eq. (A4) is given by

$$\phi(t)=2t^{(\alpha-1)/2}K_{\alpha-1}(2t^{1/2}), \tag{A5}$$

where $K_{\alpha-1}$ is the modified Bessel function of the second kind.[12]

Shohat and Tamarkin give a sufficient condition under which the moment problem

$$\int_0^\infty \varphi(t)t^n dt=\mu_n \tag{A6}$$

is determined [i.e., $\varphi(t)$ is unique as long as $\varphi(t)$ is restricted to be positive]. This condition is (Ref. 13, theorem 1.11, p. 20) that the series

$$\sum_{n=1}^{\infty}\mu_n^{-1/(2n)} \tag{A7}$$

is divergent. In the present case the moment problem (A4) is determined, i.e., has a unique positive definite solution (A5), if the series

$$\sum_{n=1}^{\infty}[n!\Gamma(\alpha+n)]^{-1/(2n)} \tag{A8}$$

is divergent. It is readily seen that the nth term of this series for large n behaves as n^{-1} and hence the series is in fact divergent. This establishes the required result.

Consider the resolution (2.23) of the identity operator in the space of the representation D_β:

$$\hat{1}_\beta=\int F(\omega)\lvert\omega;\beta\rangle\langle\omega;\beta\rvert d^2\omega. \tag{A9}$$

One may readily show using Eq. (2.19) and comparing Eq. (A9) with the relation

$$\sum_{n=0}^{\infty} |n;\beta\rangle\langle n;\beta| = \hat{1}_\beta). \tag{A10}$$

That $F(\omega)$ must be a function of $|\omega|$ only and that it must satisfy the moment condition

$$\int_0^\infty \beta\,[F(|\omega|)\{I_{2\beta-1}(2|\omega|)\}^{-1}|\omega|^{2\beta-1}]|\omega|^{2n+1}\,d|\omega| = \frac{n!\Gamma(\alpha+n)}{2\pi}. \tag{A11}$$

Since $I_{2\beta-1} > 0$ for $\beta > 0$, we conclude that as long as $F(\omega)$ is positive, it is given by

$$F(|\omega|) = \frac{2}{\pi} I_{2\beta-1}(2|\omega|)K_{2\beta-1}(2|\omega|). \tag{A12}$$

Finally, we consider the resolution of the identity operator in the space of the representation $\mathscr{D}_\alpha$:

$$\hat{1}_\alpha = \int \{F_1(z)|z\rangle\langle z| + F_2(z)|z\rangle\langle -z|\}d^2z. \tag{A13}$$

Again using Eq. (2.34) and comparing Eq. (A13) with the relation

$$\hat{1}_\alpha = \sum_{n=0}^{\infty} |n;\alpha\rangle\langle n;\alpha|, \tag{A14}$$

one may readily show that both $F_1 \pm F_2$ must be functions of $|z|^2$ only. This implies F_1 and F_2 must also separately be functions of $|z|^2$ only. Further, one finds that

$$\int_0^\infty \{F_1(|z|^2) + F_2(|z|^2)\}\{\mathscr{F}_\alpha(|z|^2)\}^{-1}|z|^{4n+1}d|z| = \pi^{-1}2^{2n+\alpha-2}n!\Gamma(n+\alpha) \tag{A15}$$

and

$$\int_0^\infty \{F_1(|z|^2) - F_2(|z|^2)\}\{\mathscr{F}_\alpha(|z|^2)\}^{-1}|z|^{4n+3}d|z| = \pi^{-1}2^{2n+\alpha-1}n!\Gamma(n+\alpha+1). \tag{A16}$$

Setting $|z|^2 = 2x$, Eqs. (A15) and (A16) may be written in the form

$$\int_0^\infty \{F_1(2x) + F_2(2x)\}\{x\mathscr{F}_\alpha(2x)\}^{-1}x^{2n+1}dx = \pi^{-1}2^{\alpha-2}n!\Gamma(n+\alpha), \tag{A17}$$

$$\int_0^\infty \{F_1(2x) - F_2(2x)\}\{\mathscr{F}_\alpha(2x)\}^{-1}x^{2n+1}dx = \pi^{-1}2^{\alpha-2}n!\Gamma(n+\alpha+1). \tag{A18}$$

Since $\mathscr{F}_\alpha(2x)$ is always positive for $\alpha > 0$, we conclude that as long as $F_1 \pm F_2$ are positive they are uniquely determined. The resolution of the identity operator (2.59) is unique under this restriction.

APPENDIX B: PROOF OF THE INEQUALITY $I_{\alpha-1}(z) > I_\alpha(z)$

In this Appendix we show that for $\alpha \geqslant \frac{1}{2}$ and $z \geqslant 0$, we have the inequality

$$I_{\alpha-1}(z) > I_\alpha(z), \tag{B1}$$

where $I_\alpha(z)$ is the modified Bessel function

$$I_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(n+\alpha+1)}. \tag{B2}$$

Define a function $f(z,\alpha)$ as

$$f(z,\alpha) = I_{\alpha-1}(z) - I_\alpha(z). \tag{B3}$$

From Eqs. (B2) and (B3) we find that

$$f(z,\alpha) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\alpha-1}}{n!\Gamma(n+\alpha+1)}[n+\alpha-z/2]. \tag{B4}$$

Each term on the right-hand side of Eq. (B4) is positive for $\alpha > z/2$. Hence, it follows that

$$f(z,\alpha) > 0, \quad \text{for } 0 \leqslant z \leqslant 2\alpha. \tag{B5}$$

Further, for large values of z, we know the asymptotic nature of $I_\alpha(z)$:

$$I_\alpha(z) \sim (2\pi z)^{-1/2}e^z\left[1 - \frac{1}{2z}(\alpha^2 - \tfrac{1}{4})\right] \tag{B6}$$

so that

$$f(z,\alpha) \sim (8\pi z^3)^{-1/2}e^z(2\alpha - 1). \tag{B7}$$

Hence, it also follows that for some large and positive number M, and $2\alpha > 1$,

$$f(z,\alpha) > 0, \quad z > M. \tag{B8}$$

Further, we also know that $f(z,\alpha)$ is an analytic function of z for $z > 0$. Hence, from Eqs. (B5) and (B8), it follows that if $f(z,\alpha)$ was negative for some z, such that $z\alpha < z < M$, then it must have a minimum at some point where its value is negative. Thus, $f(z,\alpha)$ can be negative for $2\alpha < z < M$ only if for some $z = z_0$, we have the following three conditions satisfied:

$$f(z_0,\alpha) < 0, \quad \frac{d}{dz}f(z_0,\alpha) = 0, \quad \text{and} \quad \frac{d^2}{dz^2}f(z_0,\alpha) > 0. \tag{B9}$$

Now $I_\alpha(z)$ satisfies the differential equation[12]

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - (z^2+\alpha^2)\right]I_\alpha(z) = 0,$$

from which it follows that for all z,

$$z^2\frac{d^2}{dz^2}f(z,\alpha) + z\frac{d}{dz}f(z,\alpha) - (z^2+\alpha^2)f(z,\alpha) + (2\alpha-1)I_{\alpha-1}(z) = 0. \tag{B10}$$

Since[14] $I_{\alpha-1}(z) > 0$ for $z > 0$ and $\alpha > \frac{1}{2}$, Eq. (B10) at $z = z_0$ is obviously in contradiction with Eq. (B9). Hence, $f(z,\alpha)$ cannot take any negative value. This established the inequality (4.25) of the text, viz.,

$$I_{\alpha-1}(z) > I_\alpha(z), \quad z \geqslant 0, \alpha \geqslant \tfrac{1}{2}. \tag{B11}$$

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