STRONG COUPLING TRANSMUTATION OF YUKAWA THEORY ☆

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In the strong coupling limit, it is shown that the Yukawa-type theory can be made to undergo a transmutation into an attractive separable potential theory, provided a single state is removed from the spectrum in the lowest nontrivial sector and the states at infinity which include a continuum in the next sector. If these states are not removed, the two theories are distinct. It is suggested that the full equivalence and the renormalization of four-fermion theories need further examination.

Since the work of Houard and Jouvet [1] and in particular the work of Nambu and Jona-Lasinio [2] two decades ago, the possibility of constructing theories with Yukawa-type interactions from four-fermion interactions has been investigated by many authors [3–7]. Different attempts have been made to obtain gauge theories from four-fermion interaction theories, where gauge bosons are not the fundamental fields. This attempt was carried out in the context of QED in the early sixties [4]. More recently, this program has been extended to other theories like the Yang–Mills theory, the standard SU(2) × U(1) model and the SU(3) × SU(2) × U(1) model [5].

We first recall briefly the usual argument for the demonstration of the equivalence of the trilinear Yukawa-type and the four-fermion interaction theories. Take the lagrangian,

$$\mathcal{L}_1 = \bar{\psi}(\not\partial - m_0)\psi + \tfrac{1}{2} g_0^2 (\bar{\psi}\psi)^2 \,. \tag{1}$$

With the introduction of the auxiliary boson field ϕ, this lagrangian can be rewritten in the equivalent form:

$$\mathcal{L}_2 = \bar{\psi}(\not\partial - m_0)\psi - g_0 \bar{\psi}\psi\phi - \tfrac{1}{2}\phi^2 \,. \tag{2}$$

Notice that the variation with respect to ϕ gives the constraint $\phi = -g_0\bar{\psi}\psi$. The bona fide Yukawa-type theory, which one wishes to compare with $\mathcal{L}_1$, or $\mathcal{L}_2$, has the kinetic term, i.e.,

$$\mathcal{L}_3 = \mathcal{L}_2 + \tfrac{1}{2}(\partial_\mu \phi)^2 \,. \tag{3}$$

For the purpose of the present discussion, it suffices to consider explicitly the effect of renormalization on the boson field ϕ. If we denote the wavefunction renormalization constant by Z, the renormalized field by ϕ_R and the corresponding renormalized coupling constant by g_R, we have $\phi = \sqrt{Z}\phi_R$ and $g_R = \sqrt{Z} g_0$. So

$$\mathcal{L}_3 = \mathcal{L}_2 + \tfrac{1}{2} Z (\partial_\mu \phi_R)^2 \,. \tag{4}$$

When $Z = 0$, $\mathcal{L}_2 = \mathcal{L}_3$. This has led many to believe that the $\mathcal{L}_3$-type theory in the limit of vanishing renormalization constant is equivalent to the $\mathcal{L}_2$-type theory, and hence the four-fermion interaction.

We recall that even in the sixties, it was recognized [6] that the direct comparison of the forms of the two

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lagrangians, and the one-to-one match between Feynman graphs in the two theories are not completely satisfactory ways to demonstrate the equivalence. Recently, Rajaraman [7] has questioned the validity of some of the equivalence proofs. He argued that the nonequivalence of the two theories for a quantum mechanical system with only a finite number of degrees of freedom is obvious, since the analog of the renormalization constant there does not vanish. Furthermore, he asserted that the $Z = 0$ case is not an interesting case; with g_0 fixed, $Z = 0$ would lead to $g_R = \sqrt{Z} g_0 = 0$, which would suggest that the equivalence of the two theories is realized for the trivial case of free-field theories only.

In this letter, we would like to come back to the question of equivalence once again. We base our investigation on two soluble models, the Lee model and the separable potential model. We will be comparing the limiting theory of the Lee model where $g_0 \to \infty$ and $Z \to 0$, and the separable potential model. We shall display in detail the nonequivalent aspect of the two models.

For the separable potential model, the system consists of two fields associated with N and particle θ. The hamiltonian is given by

$$H^{\mathrm{s}} = \int k a^+(\boldsymbol{k}) a(\boldsymbol{k})\, \mathrm{d}^3 k - \int h(k) N^+ a^+(\boldsymbol{k})\, \mathrm{d}^3 k \int h(k') N a(\boldsymbol{k}')\, \mathrm{d}^3 k' \,. \tag{5}$$

The theory, as is well known, decomposes into a countable set of noncombining sectors. The matrix element of the hamiltonian between the one-N–one-θ sector states is given by

$$\langle N\theta_k | H^{\mathrm{s}} | N\theta_{k'} \rangle = k\delta(k - k') - h(k)h(k') \,. \tag{6}$$

Notice that the interaction gives rise to an attractive potential.

The hamiltonian for the Lee model involves a third field V and is given by

$$H = m_0 V^+ V + \int \mathrm{d}^3 k\, k a^+(\boldsymbol{k}) a(\boldsymbol{k}) + \int \mathrm{d}^3 k\, g_0 f(k) [V^+ N a(\boldsymbol{k}) + V N^+ a^+(\boldsymbol{k})] \,. \tag{7}$$

This theory also has a countable set of noncombining sectors. The matrix of the hamiltonian with respect to the bare states $|V\rangle$ and $|N\theta_k\rangle$ is given by

$$\langle x | H | x' \rangle = \begin{pmatrix} m_0 & g_0 f(k') \\ g_0 f(k) & k\delta(k - k') \end{pmatrix} . \tag{8}$$

The corresponding eigenvectors are standard. We confine ourselves to the case where there is a bound state with "physical mass" M. The corresponding renormalization constant is given by

$$Z = \left(1 + \int \mathrm{d}^3 l\, g_0^2 |f(l)|^2 / (M - l)\right)^{-1} . \tag{9}$$

We first concentrate on the Lee model. We consider the strong coupling (SC) limit with care: we define it by first introducing the cutoff L for the momentum integral, take the limit of $g_0 \to \infty$ first. From (9), in the SC limit $Z_1 \to g_0^{-2}$. We shall see that our final conclusion is L-independent.

The bound state in the Lee model occurs when $\alpha(z) = 0$, with

$$\alpha(z) = z - m_0 - g_0^2 \int_0^L \mathrm{d}^3 k' \, |f(k')|^2 / (z - k') \,. \tag{10}$$

This bound state is again assumed to be at finite energy $z = M_1$ which is independent of g_0. In the SC limit, at $z = L$, $\alpha(z)$ is negative. From the fact that for $z \gg L$, the slope of $\mathrm{d}\alpha/\mathrm{d}z \sim 1$, $\alpha(z)$ increases indefinitely. Hence the presence of a bound state beyond the cutoff, say at $z = M_2$. In the strong coupling limit, it can be shown that as $g_0 \to \infty, M_2 \propto g_0^2 \to \infty$. The corresponding wavefunctions for the bound state M_1, the continuum and the bound state M_2 are given, respectively, by

$$\psi_{M_1} = \sqrt{Z_1}\begin{pmatrix} 1 \\ \dfrac{g_0 f(k)}{M_1 - k} \end{pmatrix}, \quad \psi_\lambda = \begin{pmatrix} g_\lambda^* \\ \delta(\lambda - k) + \dfrac{g_\lambda^* g_0 f(k)}{\lambda - k + i\epsilon} \end{pmatrix}, \quad \psi_{M_2} = \sqrt{Z_2}\begin{pmatrix} 1 \\ \dfrac{g_0 f(k)}{M_2 - k} \end{pmatrix}, \tag{11}$$

where

$$Z_i = \left(1 + \int_0^L d^3 l\, g_0 |f_l|^2/(M_i - l)\right)^{-1}, \quad g_\lambda = g_0 f_\lambda/\alpha(\lambda + i\epsilon).$$

In the above strong coupling limit: $Z_2 \to 1, M_2 \to m_0$, and the g_0 dependence of the quantities are: $Z \propto 1/g_0^2$, and $m_0 \propto g_0^2$.

Through the closure relations, we rewrite the hamiltonian matrix explicitly in terms of summing over the contributions of the entire spectra. More explicitly,

$$\langle x|H|x'\rangle = \sum_\lambda \lambda \psi_\lambda^+(x)\, \psi_\lambda(x'). \tag{12}$$

We divide the contributions of the spectrum of states of H into two parts: (1) the finite energy contribution and (2) the contribution which in the SC limit retreats to infinity ($\lambda = M_2$ for the present case). In the SC limit, we find,

$$H_1 = \begin{pmatrix} m_0 - M_2 & 0 \\ 0 & k\delta(k - k') - h'(k)h'(k') \end{pmatrix}, \quad H_2 = \begin{pmatrix} M_2 & g_0 f(k') \\ g_0 f(k) & h'(k)h'(k') \end{pmatrix}, \tag{13}$$

where $h'(k) = g_0 f(k)/\sqrt{M_2}$ was used. As expected, the sum over the complete spectrum reproduces the original expression, eq. (8).

The SC limit of the Lee model is to be compared with the separable potential case where a cutoff L is also applied to the function of $H(k)$ of eq. (6). The cutoff separable potential has no additional discrete states beyond L. We find that at finite energies, the part of the Lee model hamiltonian, H_1 of eq. (13) is identical to that of the separable potential if we identify $h'(k) = h(k)$. Furthermore, one can explicitly show that the wavefunctions and the S-matrices of the two theories are also the same at finite energies. In fact in the SC limit, the Lee model and the separable potential model coincide except for the contribution of the discrete state at infinity, which is present in the Lee model but absent in the separable potential model. Hence, through a one-point compactification, one may go from the separable potential model to the Lee model.

We have also carried out a lengthy calculation for the next sector to see whether the agreement of the two theories in the lowest sector was fortuitous. Making use of some of the techniques of ref. [8], we found the wave functions for the bare states $|V\theta_k\rangle$ and $|N\theta_k\theta_l\rangle$ to be

$$\psi_\nu(k) = \sqrt{Z_1}\phi_\nu^1(l) + \sqrt{Z_2}\phi_\nu^2(l) + \int_0^L d^3k\, g^*(k)\phi_\nu(k,l),$$

$$\psi_\nu(k,l) = \frac{1}{2}\left(\phi_\nu^1(k)F_1(l) + \phi_\nu^2(k)F_2(l) + \phi_\nu(k,l) + \int_0^L d^3l' \frac{g_0 g^*(k) f(l')}{k - l' + i\epsilon}\, \phi_\nu(k,l) + (k \leftrightarrow l)\right), \tag{14}$$

where $F_i(l) = \sqrt{Z_i} g_0 f(l)/(M_i - l)$ and the eigenstate labels $\nu = \{\Lambda; \xi_1; \xi_2; \xi, \xi'\}$, which are detailed below.

For a discrete state at $\lambda = \Lambda$, the auxiliary functions

$$\phi_\Lambda^i(l) = g_0 f(l)\chi_\Lambda^i/(\Lambda - M_i - l), \quad \phi_\Lambda(k,l) = g_0 f(l)\chi_\Lambda(k)/(\Lambda - k - l), \tag{15}$$

with

$$\chi_\Lambda^2 = [\alpha(\lambda - M_1)/\alpha(\lambda - M_2)](Z_2/Z_1)^{1/2}\chi_\Lambda^1, \quad \chi_\Lambda(k) = [g(k)\alpha(\Lambda - M_1)/\sqrt{Z_1}\alpha(\Lambda - k)]\chi_\Lambda^1.$$

The quantity χ^{l}_{Λ} is a normalization constant. There may be one or more discrete states depending on the values of g_0.

For the $V_1\theta$ and $V_2\theta$ scattering states, introduce $\xi_1 = \lambda - M_1$, $\xi_2 = \lambda - M_2$ with the range $0 \leqslant \xi_1, \xi_2 \leqslant L$. The auxiliary functions are

$$\phi^i_\nu(l) = \delta(\xi_i - l) + g_0 f(l)\chi^i_\nu/(\xi_i - l + \mathrm{i}\epsilon)\ , \qquad \phi_\nu(k,l) = g_0 f(l)\chi_\nu(k)/(\lambda - l - k + \mathrm{i}\epsilon)\ , \tag{16}$$

with

$$\chi^i_\nu = g^*(\xi_i) - [2\sqrt{Z_i}/\alpha^+(\xi_i)]\,[\sqrt{Z_1}\,g^*(\xi_1) + \sqrt{Z_2}\,g^*(\xi_2)]/\eta^+_\lambda, \qquad \eta_\lambda = Z_1/\alpha(\xi_1) + Z_2/\alpha(\xi_2) + K_\lambda,$$

$$K_\lambda = \int_0^L \mathrm{d}^3k\ |g(k)|^2/\alpha(\lambda - k)\ ,$$

$$\chi_\nu(k) = \sqrt{Z_1}\,\delta(\xi_1 - k) + \sqrt{Z_2}\,\delta(\xi_2 - k) + [g(k)/\alpha^+(\lambda - k)]\,[\alpha^+(\xi_i)\chi^i_\lambda - g_0 f(\xi_i)]/\sqrt{Z_i}.$$

For the $N\theta_\xi\theta_{\xi'}$ scattering states introduce ξ, ξ' with $\lambda = \xi + \xi'$ and $0 \leqslant \xi, \xi' \leqslant 1$. The auxiliary functions are

$$\phi^i_\nu(l) = g_0 f(l)\chi^i_\nu/(\xi_i - l + \mathrm{i}\epsilon)\ ,$$

$$\phi_\nu(k,l) = \delta(\xi - l)\delta(\xi' - k) + \delta(\xi - k)\delta(\xi' - l) + g_0 f(l)\chi_\nu(k)/(\lambda - l - k + \mathrm{i}\epsilon)\ , \tag{17}$$

with

$$\chi_\nu(k) = g^*(\xi)\delta(\xi' - k) + g^*(\xi')\delta(\xi - k) - [2g(k)/\alpha^+(\lambda - k)]\,g^*(\xi')g^*(\xi)/\eta^+_\lambda,$$

$$\chi^i_\nu = -[(2\sqrt{Z_i}/\alpha^+(\xi_i)]\,g^*(\xi')g(\xi)/\eta^+_\lambda.$$

In the strong coupling limit, the hamiltonian is found to be

$$H = H_1 + H_2\ ,$$

with

$$H_1 = \begin{pmatrix} 0 & 0 \\ 0 & (k+l)\,[\delta(k-k')\delta(l-l') + \delta(k-l')\delta(l-k')] - A \end{pmatrix},$$

$$H_2 = \begin{pmatrix} (m_0 + l)\,\delta(l - l') & g_0\,[f(k)\delta(l-l') + f(l)\delta(k-l')] \\ g_0\,[f(l')\delta(k'-l) + f(k')\delta(l-l')] & A \end{pmatrix} \tag{18}$$

$$A = h(k)h(k')\delta(l-l') + (k \leftrightarrow l) + (k' \leftrightarrow l') + (k \leftrightarrow l)(k' \leftrightarrow l')\ ,$$

where the contribution (1) is again from the spectrum in the finite energy region and (2) corresponds to the singularities at infinity, which include the $M_2\theta$ continuum. As before, the form of H_1 here coincides with the corresponding expression for the separable potential in the same sector here.

To conclude: We observe that some authors in the past have argued the renormalizability for the four-fermion interaction theory based on the assumption of its equivalence with Yukawa-type theory. We have demonstrated that the equivalence of the two theories may be achieved in the strong coupling limit provided we proceed in a certain fashion. In particular, the identity of the two theories can be made only after we remove the spectral contributions at infinity. Once this is done the Yukawa theory undergoes a transmutation into the attractive separable potential model. Without this, as far as the full spectrum of the theory is concerned, the Lee model is distinctly different from the separable potential model even in the strong coupling limit. It is suggested that the full equivalence and the renormalization of four-fermion theories need further examination.

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