Proc. R. Soc. Lond. A **379**, 103–107 (1982)
Printed in Great Britain

Dirac's new relativistic wave equation in interaction with an electromagnetic field

BY N. MUKUNDA,† C. C. CHIANG‡ AND E. C. G. SUDARSHAN
Center for Particle Theory, University of Texas, Austin, Texas 78712, *U.S.A.*

(*Communicated by P.A.M. Dirac, F.R.S. – Received* 18 *March* 1981)

Dirac had discovered a relativistic wave equation for a spinless massive particle with only positive energies. It has a defect: it became inconsistent when electromagnetic interaction was included. In this paper this problem is surmounted simply.

1. THE WAVE EQUATION OF DIRAC

The relativistic wave equations most often considered for the relativistic description of quantum mechanical particles have both positive and negative energy solutions. Ten years ago Dirac (1971, 1972) proposed a new system of relativistic wave equations that had a formal similarity with the Dirac (1928*a*, *b*) relativistic equation for the electron, but the new equation had only positive energy solutions with a unique mass. The new wave equation was for a one-component function but seems to describe some internal structure with four corresponding operators $Q_a (a = 1, 2, 3, 4)$. Nevertheless the particle described by the new equation has a unique mass. The quantities Q_a satisfy the commutation relations

$$[Q_a, Q_b] = Q_a Q_b - Q_b Q_a = i\beta_{ab}, \tag{1.1}$$

where the matrix

$$\beta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{1.2}$$

We use units such that $\hbar = c = 1$. Note that β so defined is anti-Hermitian and satisfies $\beta^2 = -1$. Dirac's new equation is

$$(\partial/\partial x_0 + \alpha_r \partial/\partial x_r + \beta) Q\psi = 0. \tag{1.3}$$

† Permanent address: Indian Institute of Science, Bangalore 560012, India.
‡ On leave from the Institute of Physics, National Taiwan Normal University, Taipei, Taiwan. Permanent address after 1 September 1981: Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan.

The variable Q is treated as a four-component column vector. The quantities α_r, β are Dirac (1928 *a*, *b*) matrices but are all chosen to be real. A particular choice is

$$\alpha_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.4}$$

Note that

$$\left.\begin{aligned} \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = -\beta^2 = 1,& \\ \alpha_1\alpha_2+\alpha_2\alpha_1 = 0, \quad \alpha_2\alpha_3+\alpha_3\alpha_2 = 0,& \\ \alpha_3\alpha_1+\alpha_1\alpha_3 = 0, \quad \beta\alpha_1+\alpha_1\beta = 0,& \\ \beta\alpha_2+\alpha_2\beta = 0, \quad \beta\alpha_3+\alpha_3\beta = 0.& \end{aligned}\right\} \tag{1.5}$$

It is straightforward to verify that, by virtue of equation (1.3), ψ satisfies the equation (Dirac 1971)

$$(\partial_\mu \partial^\mu + 1)\psi = 0, \tag{1.6}$$

where $\partial^\mu = \partial/\partial x_\mu$. If we put $\alpha_0 = 1$ we can write the equation (1.3) in the form

$$(\alpha_\mu \partial^\mu + \beta)_{ab}\, Q_b\, \psi = P_a \psi = 0. \tag{1.7}$$

For equations (1.7) to be consistent it must be true that

$$[P_a, P_b]\psi = 0. \tag{1.8}$$

In turn this leads to equation (1.6).

If we take plane wave solutions of the form

$$\psi(x) = \exp\{-\mathrm{i}(p_0 x_0 - \boldsymbol{p}\cdot\boldsymbol{x})\}\psi(0), \tag{1.9}$$

equation (1.6) implies $\quad p_0^2 = 1+p_1^2+p_2^2+p_3^2.$

The action of Q_a on ψ is especially simple when displayed on such plane wave states. Take a representation with Q_1 and Q_3 diagonal:

$$Q_2 = -\mathrm{i}\partial/\partial Q_1; \quad Q_4 = -\mathrm{i}\partial/\partial Q_3. \tag{1.10}$$

For the state with $p_0 = 1, p_1 = p_2 = p_3 = 0$, we obtain the specially simple relations

$$(Q_1+\partial/\partial Q_1)\psi = 0; \quad (Q_3+\partial/\partial Q_3)\psi = 0;$$

so that

$$\psi \sim \exp\{-\tfrac{1}{2}(Q_1^2+Q_3^2)\}. \tag{1.11}$$

But for negative energy $p_0 = -1, p_1 = p_2 = p_3 = 0$, the solution

$$\psi \sim \exp\{+\tfrac{1}{2}(Q_1^2+Q_3^2)\}$$

is not normalizable and does not correspond to a quantum state. By relativistic invariance there are no negative energy solutions. Since the zero momentum solution is unique and invariant under rotations it follows that the particle described by Dirac's new relativistic equations has no spin.

A surprising and unwelcome feature of Dirac's new equations is that a gauge invariant electromagnetic interaction obtained by the replacement

$$p_\mu = -\mathrm{i}\partial/\partial x_\mu \rightarrow p_\mu - \mathrm{i}eA_\mu(x) \tag{1.12}$$

makes the equations inconsistent since the integrability conditions are violated. This is a serious shortcoming since all particles that appear to have internal structure are either electrically charged or have electrically charged partners.

We have found that this shortcoming of Dirac's new wave equations can be remedied by a suitable modification of the internal structure associated with the operators Q_a (Sudarshan 1981).

2. Generalized Dirac equation

It is convenient to start with the new Dirac equation in the form

$$(\gamma_\mu \partial^\mu + m)\,\xi\psi(x) = 0, \tag{2.1}$$

where the covariant Dirac matrices γ_μ satisfy the anticommutation relations

$$\left.\begin{aligned} \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= -2g_{\mu\nu}, \\ g_{\mu\nu} &= \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu \neq 0, \\ 0, & \mu \neq \nu. \end{cases} \end{aligned}\right\} \tag{2.2}$$

The four quantities ξ_a satisfy the trilinear relations

$$\tfrac{1}{2}[\xi_a\xi_b + \xi_b\xi_a, \xi_c] = \delta_{ac}\xi_b + \delta_{bc}\xi_a \tag{2.3}$$

first given by Green (1953). A particular choice of the operators ξ_a is given by

$$\xi_a = Q_a + Q'_a, \tag{2.4}$$

with Q_a and Q'_a satisfying the relations given by equation (1.1) among themselves but anticommuting between them (Green 1953):

$$\left.\begin{aligned} Q_aQ'_b + Q'_bQ_a &= 0; \\ [Q_a, Q_b] = [Q'_a, Q'_b] &= \mathrm{i}\beta_{ab}. \end{aligned}\right\} \tag{2.5}$$

We make special choices for the γ matrices:

$$\left.\begin{aligned} \gamma_0 = \beta = \mathrm{i}\rho_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_1 = \rho_3\sigma_3 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \\ \gamma_2 = -\rho_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_3 = -\rho_3\sigma_1 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}. \end{aligned}\right\} \tag{2.6}$$

All the γ-matrices are real. With this choice the proof of relativistic invariance given by Dirac (1971, 1972) can be extended to the present generalization. The essential point is that the generators of rotations and pure Lorentz transformations are given by the six quantities

$$S_{\mu\nu} = \tfrac{1}{8}\xi^{\mathrm{T}}\beta[\gamma_\mu, \gamma_\nu]\,\xi, \tag{2.7}$$

which satisfy

$$[S_{\mu\nu}, \xi] = \tfrac{1}{4}\mathrm{i}[\gamma_\mu, \gamma_\nu]\,\xi. \tag{2.8}$$

If the wave function $\psi(x)$ changes according to

$$\psi'(x') = U(\Lambda)\,\psi(x), \tag{2.9}$$

where $U(\Lambda)$ is the unitary representation of the Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu} \tag{2.10}$$

(generated by $S_{\mu\nu}$), then Dirac's new equation is relativistically invariant.

The plane wave solution given by

$$\psi(x) = u(p)\exp\{-\mathrm{i}(p_0 x_0 - p_1 x_1 - p_2 x_2 - p_3 x_3)\} \tag{2.11}$$

has an amplitude function $u(p)$ in the rest frame $(p_1 = p_2 = p_3 = 0)$ satisfying

$$\left.\begin{aligned} (\mathrm{i}p_0\,\xi_3 + m\xi_1)\,u(p) = 0, \quad & (\mathrm{i}p_0\,\xi_4 + m\xi_2)\,u(p) = 0, \\ (-\mathrm{i}p_0\,\xi_1 + m\xi_3)\,u(p) = 0, \quad & (-\mathrm{i}p_0\,\xi_2 + m\xi_4)\,u(p) = 0. \end{aligned}\right\} \tag{2.12}$$

Consequently $p_0^2 = m^2$, so $p_0 = \pm m$. For $p_0 = m$ there is a normalizable solution proportional to $\exp\{-\tfrac{1}{2}(Q_1^2 + Q_1'^2 + Q_3^2 + Q_3'^2)\}$; there is no such solution for p_0 negative. This system is therefore very similar to the system discovered by Dirac. It differs from it by the wave function being essentially a function of *four* variables Q_1, Q_3, Q_1', Q_3' in addition to space–time variables.

3. Interaction with the electromagnetic field

The gauge invariant minimal interaction with the electromagnetic field associated with the four-potential A_μ is obtained by the replacement of ∂_μ by $\mathrm{D}_\mu = \partial_\mu - \mathrm{i}eA_\mu$. The equations of motion become

$$(\gamma_\mu \mathrm{D}^\mu + m)\,\xi\psi(x) = 0. \tag{3.1}$$

We may rewrite this in the form

$$\left.\begin{aligned} \tau_a\,\psi(x) &= 0, \\ \tau_a &= (\gamma_\mu \mathrm{D}^\mu + m)_{ab}\,\xi_b. \end{aligned}\right\} \tag{3.2}$$

Consistency of these equations requires that

$$[\tau_a, \tau_b]\,\psi(x) = 0.$$

Straightforward computation gives

$$\{(\mathrm{D}^2 + m^2)\,\xi^{\mathrm{T}}\beta\xi + \tfrac{1}{2}\mathrm{i}eF^{\mu\nu}S_{\mu\nu}\}\,\psi = 0, \tag{3.3}$$

$$\{(\mathrm{D}^2 + m^2)\,\xi^{\mathrm{T}}\beta\gamma_5\,\xi + 2m\mathrm{D}^\mu\xi^{\mathrm{T}}\beta\gamma_5\gamma_\mu\,\xi + \tfrac{1}{4}\mathrm{i}e\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}\}\,\psi = 0, \tag{3.4}$$

$$\{2m\mathrm{D}^\lambda\xi^{\mathrm{T}}\beta\gamma_5\,\xi + (\mathrm{D}^\lambda\mathrm{D}^\rho + \mathrm{D}^\rho\mathrm{D}^\lambda + (m^2 - \mathrm{D}^\sigma\mathrm{D}_\sigma)\,g^{\lambda\rho})\,\xi^{\mathrm{T}}\beta\gamma_5\gamma_\rho\,\xi + \tfrac{1}{8}\mathrm{i}e\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}\,\xi^{\mathrm{T}}\beta\gamma_\rho\,\xi\}\,\psi = 0. \tag{3.5}$$

Further calculations and simplifications of these relations together with equation (3.1) yield

$$\{(D^2+m^2)\,\xi^T\beta\xi-\tfrac{1}{2}ieF^{\mu\nu}S_{\mu\nu}\}\,\psi=0, \tag{3.6}$$

$$\{(D^2+m^2)\,\xi^T\beta\gamma_5\,\xi-\tfrac{1}{2}ie\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}\}\,\psi=0, \tag{3.7}$$

$$\{(D^2+m^2)\,\xi^T\beta\gamma_5\,\gamma^\lambda\xi-ieF^{\lambda\mu}\xi^T\beta\gamma_5\,\gamma_\mu\,\xi-\tfrac{1}{2}e\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}\,\xi^T\beta\gamma_\rho\,\xi\}\,\psi=0. \tag{3.8}$$

By virtue of the fact that ξ contains the sum of two anticommuting quantities Q, Q', (3.6)–(3.8) are all nontrivial equations of motion: this obtains since the quantities $\xi^T\beta\xi$, $\xi^T\beta\gamma_5\,\xi$, $\xi^T\beta\gamma_5\,\gamma_\lambda\,\xi$ are all non-zero. The electromagnetic interaction therefore changes equations of motion from the free equations of motion. It does not introduce any inconsistencies as in the equations discovered by Dirac. For the simpler Dirac equations,

$$\xi^T\beta\xi\to Q^T\beta Q=2i,$$

$$\xi^T\beta\gamma_5\,\xi\to Q^T\beta\gamma_5\,Q=0,$$

$$\xi^T\beta\gamma_5\,\gamma_\lambda\,\xi\to Q^T\beta\gamma_5\,\gamma_\lambda\,Q=0.$$

Consequently we obtain the constraint equations

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,Q^T\beta[\gamma_\rho,\gamma_\sigma]\,Q\psi=0, \tag{3.9}$$

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,Q^T\beta\gamma_\rho Q\psi=0 \tag{3.10}$$

first obtained by Biedenharn *et al.* (1973) and Biedenharn & van Dam (1974).

4. Conclusion

The present generalization of the work of Dirac provides a relativistic wave equation for a spinless particle of finite mass capable of interacting with the electromagnetic field. The inconsistency of Dirac's equation is avoided by having a more complex structure for the particle. It would be interesting to study in detail the electromagnetic form factor for such a particle.

This work was supported by the U.S. Department of Energy.

References

Biedenharn, L. C., Han, M.-Y. & van Dam, H. 1973 *Phys. Rev.* D **8**, 1735.
Biedenharn, L. C. & van Dam, H. 1974 *Phys. Rev.* D **9**, 471.
Dirac, P. A. M. 1928*a* *Proc. R. Soc. Lond.* A **117**, 610.
Dirac, P. A. M. 1928*b* *Proc. R. Soc. Lond.* A **118**, 341.
Dirac, P. A. M. 1971 *Proc. R. Soc. Lond.* A **322**, 435.
Dirac, P. A. M. 1972 *Proc. R. Soc. Lond.* A **328**, 1.
Green, H. S. 1953 *Phys. Rev.* **90**, 270.
Sudarshan, E. C. G., Mukunda, N. & Chiang, C. C. 1981 *Phys. Rev.* (In the press.)