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Separability in relativistic Hamiltonian particle dynamics

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The problem of separability in recent models of classical relativistic interacting particles is examined. This physical requirement is shown to be more subtle than naive separability of all the constraints defining the system: it is adequate to be able to canonically transform the time-fixing constraints from an unseparated to a separated form when clusters emerge. Viewing separability in this way, and within a specific framework, we are led to a new no-interaction theorem which states the incompatibility of nontrivial interaction with relativistic invariance, separability, and invariant world lines for more than two particles.

I. INTRODUCTION

Classical physics contains the point particle and aggregates of point particles as essential ingredients. Newtonian physics with its clear formulation of the equations of motion as second-order differential equations suggested that position and momentum were required to specify the kinematic state of the particle. Later developments of Lagrangian generalized coordinates allowed the treatment of holonomic constraints, and the passage to the Hamiltonian form. The development of constraint dynamics, initiated in the work of Dirac[1] and completed in the work of several others makes the next grand generalization to nonholonomic constraints. The constraint formalism in its turn has led to significant new work in relativistic classical dynamics.

The fundamental principles underlying the Hamiltonian description of classical relativistic systems were first clearly enunciated by Dirac.[2] The basic idea involved is to construct a canonical realization of the Poincarè group on the phase space appropriate to the given dynamical system. However, as Dirac pointed out, one has to make a choice from several possible forms of such descriptions. These are the instant, front, and point forms which correspond to different kinematic choices of Cauchy surfaces in space-time. It would be desirable to strengthen the Lie algebra formalism of Dirac to furnish a representation of the Poincarè group by canonical transformations to incorporate the idea of relativistic invariance.

Within the above general framework, for a system of classical point particles, it seems desirable to incorporate the following features: (1) a representation of the Poincarè group by canonical transformations acting on particle variables, (2) the presence of nontrivial interactions, (3) the existence of invariant particle world lines, and (4) the property of separability.

Within the instant form, Thomas and Bakamjian and Thomas[3] succeeded in constructing nontrivial models which, however, did not possess the preceding features (3) and (4). While the world-line condition (WLC) was explicitly abandoned by these authors, it was Foldy[4] who pointed out that separability was also missing in these models.

Even without the requirement of separability, it was shown by Currie, Jordan, and Sudarshan[5] that the features (1), (2), and (3) above were already incompatible within the instant form of dynamics. This is the well-known no-interaction theorem. An important contribution of this work was the formulation of the WLC in the language of Poisson brackets and appropriate to the instant form.

Recent work by several authors[6] has shown ways to avoid the no-interaction theorem and to construct models possessing features (1), (2), and (3) by using the constraint Hamiltonian formalism elaborated by Dirac.[1] A careful analysis[7] of this work has shown that these models involve an enlargement of the original Dirac framework: it is necessary to consider *dynamical* choices of Cauchy surfaces (rather than the *kinematic* ones offered by Dirac[2]). The expression for the WLC in a form appropriate to this enlarged framework has also been accomplished.[8] Clearly, what remains to be done is the incorporation of separability in this new framework.

In nonrelativistic particle mechanics with short-range potentials, when groups of particles are far

removed from each other, each group follows its own dynamics unaffected by the presence of other groups. Such a property is what is qualitatively meant by separability.

The intuitive basis of such a property is evident. At a technical level, in scattering theory, this property plays an important role.

In this paper, we want to discuss the implications of separability for relativistic Hamiltonian particle dynamics. Within the constraint formalism one recognizes that the full specification of the dynamics involves the specification of all the constraints. Here several new features arise since besides the separability of equations of motion and initial conditions, we have also to face the problem of the separability of all the constraints and Poisson brackets in a suitable sense. The conceptual framework is further complicated by the WLC which has to be fulfilled whether or not the particles are far apart. If the WLC is ignored, separability (and relativistic invariance) can be readily achieved; this we shall show. On the other hand, as we mentioned earlier, it is possible to fulfill the WLC for interacting systems if we ignore the question of separability. It is thus the combination of these two requirements of separability and the WLC that seems difficult to achieve in relativistic particle dynamics with interactions. Some recent work[9] suggests that the problem of separability has been solved, but our analysis shows that the implications of both separability and the WLC had not been fully examined in that work; our conclusions naturally differ.[10,11]

We have made a precise definition of separability but we have not been able to treat the problem in complete generality. Rather, we base our discussion on a specific definition of the meaning of separability and a specific choice of the constraints. Our choice of constraints, though specific, is reasonable and common, while our requirement of separability is satisfied by noninteracting systems and only by them. (This last property is to be contrasted with a recent formulation due to Rohrlich.[9])

We find that with our hypotheses it is not possible to construct a model possessing all the four features. Our analysis *suggests* a new no-interaction theorem stating the general incompatibility of these four eminently reasonable requirements (see Ref. 10 and Sec. V in this context). While such indecomposibility was already pointed out for a *quantum* system by Einstein, Podolsky, and Rosen[12] we believe it is the first time that such a feature has been discerned for classical systems.

The plan of the paper is as follows. In Sec. II, we adapt the new techniques based on constraint dynamics to solve the separability problem in the instant form. This may be viewed as a completion of the Bakamjian-Thomas work[3] and of course does not maintain invariant world lines. In Sec. III we state the complete set of constraints working in the Droz-Vincent-Komar-Todorov (DVKT) framework,[6] and derive the separability requirements on these constraints. We also verify that these requirements are fulfilled in the free-particle case. In Sec. IV we examine these requirements along with the WLC and prove explicitly that the noninteracting system is the unique solution meeting all our conditions. In the concluding Sec. V we offer some comments comparing our approach to other existing ones.

II. SEPARABILITY IN THE INSTANT FORM

Following the DVKT approach, for an N-particle system, we start with an $8N$-dimensional phase space Γ spanned by $2N$ four-vectors q_a^μ, p_a^μ $(a=1,2,\ldots,N)$ under the Poincaré group. They fulfill the Poisson-bracket (PB) relations

$$\{q_a^\mu,q_b^\nu\}=\{p_a^\mu,p_b^\nu\}=0\ ,\quad \{q_a^\mu,p_b^\nu\}=\delta_{ab}g^{\mu\nu}\ . \tag{2.1}$$

In this space we impose the N constraints

$$K_a=p_a{}^2-m_a{}^2-V_a(q,p)\approx 0 \tag{2.2}$$

which are required to be Poincaré invariant and first class:

$$\{K_a,K_b\}\approx 0\ . \tag{2.3}$$

We take from Sazdjian[11] the following result: *separable solutions V_a to this system of equations exist.* That is to say, if the N particles are divided into two clusters S_1 and S_2 of N_1 and N_2 particles, respectively, and all difference four-vectors $q_{ab}\equiv q_a-q_b$, $a\in S_1$, $b\in S_2$ are large and spacelike, then the "potentials" V_a, $a\in S_1$, depend only upon the q's and p's of that cluster (and correspondingly for S_2). Note that this result does not use the WLC and is a property of the partial differential equations (2.3).

To describe a true physical system of N particles with the correct number, i.e., $6N$, of degrees of freedom, it is essential to supplement the K constraints by an additional set of N constraints χ_a, such that the $2N$ constraints K, χ form a second-class set. To get the instant form of dynamics, we

choose the χ_a to be

$$\chi_a = q_a^0 - \tau \ , \tag{2.4}$$

where τ, the evolution parameter, is the coordinate time. (An evolution parameter τ must occur in any choice of the set χ.) With this choice, we have of course abandoned all hope of invariant world lines. But we recognize that the fixation constraints (2.4) are separable: the constraint on any one particle does not depend on the other particles.

By translation invariance, the V_a depend on the q's only through the differences, q_{ab}. On the hypersurface Σ_τ in Γ defined by $K_a \approx \chi_a \approx 0$, we can thus imagine solving (2.2) for p_a^0 in the form

$$p_a^0 = h_a(\vec{q},\vec{p}) \ . \tag{2.5}$$

The surface Σ_τ is obviously spanned by the $2N$ three-vectors $\vec{q}_a$, $\vec{p}_a$, and their Dirac brackets (DB) are

$$\begin{aligned} &\{q_{aj},q_{bk}\}^* = \{p_{aj},p_{bk}\}^* = 0 \ , \\ &\{q_{aj},p_{bk}\}^* = \delta_{ab}\delta_{jk} \ . \end{aligned} \tag{2.6}$$

Starting from the Poincaré generators in Γ,

$$\begin{aligned} &P_\mu = \sum_a p_{a\mu} \ , \\ &M_{\mu\nu} = \sum_a (q_a \wedge p_a)_{\mu\nu} \end{aligned} \tag{2.7}$$

we get their forms for use in Σ_τ to be

$$\begin{aligned} &\vec{\mathrm{P}} = \sum_a \vec{\mathrm{p}}_a, \quad \vec{\mathrm{J}} = \sum_a \vec{\mathrm{q}}_a \times \vec{\mathrm{p}}_a \ , \\ &H = P^0 = \sum_a h_a(\vec{\mathrm{q}},\vec{\mathrm{p}}) \ , \\ &\vec{\mathrm{K}} = \sum_a \vec{\mathrm{q}}_a h_a(\vec{\mathrm{q}},\vec{\mathrm{p}}) - \tau\vec{\mathrm{P}} \ . \end{aligned} \tag{2.8}$$

From general theory we know that these are guaranteed to generate the Poincaré algebra under Dirac brackets.

To verify the separability of the system, we have to check this property for kinematics (i.e., the DB's) and for dynamics, i.e., the Poincaré generators (2.8). The basic DB's (2.6) are already in separated form. As for the generators, we have from Sazdjian the separability of the V_a; this obviously implies the separability of H, and of all the other generators as well. These generalizations go beyond the Bakamjian-Thomas scheme[3] of constructing an invariant-mass-squared function in the center-of-mass frame.

Thus with the help of the constraint formalism developed by DVKT, we are able to solve the separability problem which had been a difficulty for the Bakamjian-Thomas model.

III. SEPARABILITY AND THE WLC

The choice of the χ constraints in the last section was incompatible with the WLC. A set of χ's known from previous work to be compatible with invariant world lines is

$$\chi_a = P \cdot q_a - \tau \ . \tag{3.1}$$

In this section, we discuss separability in terms of this choice. At first sight, in view of the occurrence of the *total* four-momentum P in χ_a, one might hastily conclude that all hope of separability must be abandoned. However, a more careful analysis is necessary.

For any choice of χ's these constraints play two roles. On the one hand, their forms at $\tau=0$ combined with the K constraints give a complete set of restrictions on initial conditions in Γ. On the other hand, through the requirement that they be maintained for all τ, they determine a definite dynamics, i.e., a definite linear combination $v_a K_a$ which generates τ evolution through the PB's:

$$\begin{aligned} &\frac{d\chi_a}{d\tau} = \frac{\partial \chi_a}{\partial \tau} + \{\chi_a, v_b K_b\} \approx 0 \ , \\ &v_a \approx -C_{ab}\frac{\partial \chi_b}{\partial t} \ , \\ &C_{ab}\{\chi_b, K_c\} \equiv \delta_{ac} \ . \end{aligned} \tag{3.2}$$

For the choice (3.1) of χ's, we have

$$\begin{aligned} &\frac{\partial \chi_a}{\partial \tau} = -1 \ , \\ &\{\chi_a, K_b\} = P_\mu M_{ab}^\mu \ , \\ &M_{ab}^\mu \equiv \{q_a^\mu, K_b\} \ . \end{aligned} \tag{3.3}$$

From the separability of the "potentials" V_a, it is clear that if the total system separates into clusters S_1 and S_2, the matrix M_{ab}^μ and hence the matrix $(\{\chi_a, K_b\})$ becomes block diagonal. Thus the equations of motion for the q's and p's in S_i involve only K_a for $a \in S_i$. Moreover, the only dependence of these equations on the variables of S_j $(j \neq i)$ is through the total four-momentum of S_j. For complete separability of the equations of motion *and* the constraints, it is thus sufficient to remove this dependence in some way.

Now we make the following remark. In the

models which evade the no-interaction theorem, the Cauchy surfaces are determined *dynamically* and *autonomously;* this means in particular that for each cluster that separates from the total system, its Cauchy surface should not depend on the rest of the system.[13] As things stand, this condition is *not* fulfilled. When the total system splits into two clusters S_1 and S_2, it is easy to see from the equations of motion that the total momentum P_μ also splits into the sum $P_\mu^{(1)}+P_\mu^{(2)}$ of two individually conserved momenta; $P_\mu^{(1)}$ is just the total momentum of cluster S_1. For the subsequent description of S_1, the parent system would lead to Cauchy surfaces given by $K_a=0$ and

$$\chi_a=(P^{(1)}+P^{(2)})\cdot q_a-\tau=0\ , \tag{3.4}$$

while the constraints determined by S_1 alone ought to read $K_a=0$ and

$$\chi_a^{(1)}=P^{(1)}\cdot q_a-\tau=0\ . \tag{3.5}$$

(Now and hereafter, $a,b,\ldots$ run over cluster S_1.) In space-time, the passage from χ_a to $\chi_a^{(1)}$ corresponds to tilting the family of hyperplanes normal to $P^{(1)}+P^{(2)}$ to the family normal to $P^{(1)}$. Complete separability of description thus evidently means that in the phase space of S_1, we should be able to transform the description based on the Cauchy surfaces given by $K_a=0$, $\chi_a=0$ to the ones given by $K_a=0$, $\chi_a^{(1)}=0$, *without however altering the world lines of the particles in* S_1.

How is this tilting of the Cauchy surface to be achieved? We shall assume that this change of description, if at all possible, should be the result of a succession of infinitesimal canonical transformations generated by a linear combination u_aK_a of the K's. In partial justification of this assumption, we may remark that such transformations preserve the K constraints in view of their first-class character.

Such u_a's which can perform the tilting are subject to two more equations. The first comes from the requirement that the world lines are preserved. Thus the PB of $u_bK_b\ \delta\sigma$ with any q_a should be equal to the PB of $\delta_a\tau v_bK_b$ with q_a where the infinitesimal quantity $\delta_a\tau$ can depend on $\delta\sigma$ (and other variables of the problem). The second condition comes from the requirement that $\chi_a^{(\sigma)}$ are preserved by τ evolution.

We shall now formalize these three requirements on $\chi_a^{(\sigma)}$.

Consider the family of constraints

$$\chi_a^{(\sigma)}=[P^{(1)}+(1-\sigma)P^{(2)}]\cdot q_a-\tau\ ,\quad 0\le\sigma\le 1 \tag{3.6}$$

which interpolates between χ_a and $\chi_a^{(1)}$. If there exists

$$F^{(\sigma)}=u_a^{(\sigma)}K_a \tag{3.7}$$

such that

$$\frac{\partial\chi_a^{(\sigma)}}{\partial\sigma}=\{\chi_a^{(\sigma)},F^{(\sigma)}\}\ , \tag{3.8}$$

then we can deform χ_a to $\chi_a^{(1)}$ by canonical transformations.

There seems to be no loss of generality in the choice of the interpolation (3.6). For if there is an interpolation for a given value of $P^{(2)}$, there should also be an interpolation when $P^{(2)}$ is changed to $P^{(2)}+\delta P^{(2)}$. That is, we expect that there are canonical transformations which change $P^{(2)}$ alone in χ_a. The form (3.6) is consistent with this expectation.

We can simplify (3.8) to

$$\begin{aligned}-P^{(2)}\cdot q_a&\approx u_b^{(\sigma)}\{\chi_a^{(\sigma)},K_b\}\\&=[P^{(1)}+(1-\sigma)P^{(2)}]_\mu M_{ab}^\mu u_b^{(\sigma)}\ .\end{aligned} \tag{3.9}$$

Here we have used the fact that K_b's separate. Translational invariance then implies that $\{P^{(1)},K_b\}=0$, $b\in S_1$ while $\{P^{(2)},K_b\}=0$, $b\in S_1$ since K_b does not depend on q_a, $a\in S_2$.

The condition for the preservation of world lines is

$$\{q_a^\mu,F^{(\sigma)}\}\delta\sigma\approx\{q_a^\mu,v_b^{(\sigma)}K_b\}\delta_a\tau$$

or (3.10)

$$M_{ab}^\mu u_b^{(\sigma)}\delta\sigma\approx M_{ab}^\mu v_b^{(\sigma)}\delta_a\tau\ \ (\text{no } a \text{ sum})\ .$$

Here we have used $v_a^{(\sigma)}K_a$ for the generator of τ evolution.

The final equation involving $\chi_a^{(\sigma)}$ is

$$\frac{\partial\chi_a^{(\sigma)}}{\partial\tau}+\{\chi_a^{(\sigma)},v_b^{(\sigma)}K_b\}\approx 0$$

or

$$[P^{(1)}+(1-\sigma)P^{(2)}]_\mu M_{ab}^\mu v_b^{(\sigma)}\approx 1\ . \tag{3.11}$$

This serves to determine $v_a^{(\sigma)}$.

It is easy to verify that the three equations (3.9), (3.10), and (3.11) can be fulfilled by the free-particle system. In this case M^μ is diagonal:

$$M_{ab}^\mu=2\delta_{ab}p_a^\mu\ \ (\text{no } a \text{ sum})\ . \tag{3.12}$$

Therefore each of the equations can be algebraical-

ly solved for $u_a^{(\sigma)}$, $\delta_a\tau$, and $v_a^{(\sigma)}$.

Thus despite the initial impression that the χ's do not separate, we are able to achieve separability for the free system by the tilting operation. This shows that the latter operation is not without content.

IV. PROOF OF THE MAIN RESULT

We shall now prove that the only system compatible with (3.9), (3.10), and (3.11) is the free system.

It is convenient to introduce the notation

$$M_{ab}=[P^{(1)}+(1-\sigma)P^{(2)}]_\mu M^\mu_{ab} \ . \tag{4.1}$$

In terms of M the solution for $v^{(\sigma)}$ is

$$v_a^{(\sigma)}=\sum_b M^{-1}_{ab} \tag{4.2}$$

while the solution for $u^{(\sigma)}$ is

$$u_a^{(\sigma)}=-M^{-1}{}_{ab}P^{(2)}\cdot q_b \ . \tag{4.3}$$

Substitution of these solutions in (3.10) leads to

$$\delta\sigma M^\mu_{ac}M^{-1}{}_{cb}P^{(2)}\cdot q_b = -\delta_a\tau M^\mu_{ac}\sum_b M^{-1}{}_{cb} \quad (\text{no } a \text{ sum}) \ . \tag{4.4}$$

Multiplication by $[P^{(1)}+(1-\sigma)P^{(2)}]_\mu$ shows that $\delta_a\tau$, if they exist at all, are given by

$$\delta_a\tau=-P^{(2)}\cdot q_a\delta\sigma \tag{4.5}$$

which may be put back in (4.4) to find

$$M^\mu_{ac}M^{-1}{}_{cb}P^{(2)}\cdot q_b=P^{(2)}\cdot q_a M^\mu_{ac}\sum_b M^{-1}{}_{cb} \ . \tag{4.6}$$

Defining the matrix ψ^μ by

$$\psi^\mu_{ab}=M^\mu_{ac}M^{-1}{}_{cb}-\delta_{ab}M^\mu_{ac}\sum_d M^{-1}{}_{cd} \tag{4.7}$$

we see that $(P^{(2)}\cdot q_1,\ldots,P^{(2)}\cdot q_N)$ is a null eigenvector of ψ^μ:

$$\psi^\mu_{ab}P^{(2)}\cdot q_b=0 \ . \tag{4.8}$$

But $(1,\ldots,1)$ is also a null eigenvector of ψ^μ:

$$\sum_b \psi^\mu_{ab}=0 \ . \tag{4.9}$$

Now consider the case where the cluster S_1 has two particles. Then the 2×2 matrix ψ^μ has two linearly independent null eigenvectors and therefore is identically zero:

$$M^\mu_{ac}M^{-1}{}_{cb}=\delta_{ab}M^\mu_{ac}\sum_b M^{-1}{}_{cb} \ . \tag{4.10}$$

For this two-particle cluster, we can assume the canonical form

$$V_a=V(q_\perp{}^2) \ , \qquad q_{\perp\mu}=q_\mu-\frac{P^{(1)}\cdot q}{P^{(1)2}}P^{(1)}_\mu \ , \qquad q=q_1-q_2 \tag{4.11}$$

and explicitly evaluate M^μ and M^{-1}:

$$\begin{aligned} M^\mu &= 2\left[\begin{bmatrix} p_1^\mu & 0 \\ 0 & p_2^\mu \end{bmatrix}+V'\frac{P^{(1)}\cdot q}{P^{(1)2}}q_\perp^\mu\begin{bmatrix}1&1\\1&1\end{bmatrix}\right], \\ M^{-1} &= \frac{2}{\det M}\left[\begin{bmatrix} \mathscr{P}\cdot p_1 & 0 \\ 0 & \mathscr{P}\cdot p_2 \end{bmatrix}+V'\frac{P^{(1)}\cdot q}{P^{(1)2}}\mathscr{P}\cdot q_\perp\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\right], \\ \mathscr{P} &= P^{(1)}+(1-\sigma)P^{(2)}, \quad V'=\frac{dV}{dq_\perp{}^2} \ . \end{aligned} \tag{4.12}$$

The $a=1$, $b=2$ element of the left-hand side of (4.10) is proportional to

$$V'(\mathscr{P}\cdot q_\perp p_1^\mu-\mathscr{P}\cdot p_2 q_\perp^\mu) \tag{4.13}$$

which will not vanish for interacting systems. This completes our proof for those cases where the total number of particles is at least three.

If the system had consisted only of two particles which separate in the course of their motion, they must thereafter move freely. The analysis of the previous section then shows that with the help of the tilting operation, complete separability of description can be achieved for such a two-particle system.

V. CONCLUDING REMARKS

The work described in this paper is in essence an attempt at the completion of the researches into

the Hamiltonian description of interacting relativistic particles. In the spirit of Newtonian physics of motion of material particles one starts with a phase space and introduces a parameter of evolution. There are two aspects of relativistic invariance: the existence of a canonical realization of the Poincaré group acting on the phase space on the one hand, and the requirement that there be objective world lines, the so-called world-line condition, on the other. Within the context of these requirements the (first) no-interaction theorem states that if τ is chosen to be the laboratory time there can be no interaction.

Constraint dynamics provides a richer variety of evolution parameters chosen through the specification of the fixation constraints. It then turns out that the world-line condition can be made compatible with interaction provided the evolution parameter is chosen to be a Lorentz invariant, and provided there is essentially only one constraint which is not Poincaré invariant. A large number of interacting systems may be so described.[7]

A new constraint on the system of interacting particles is that of separability. When we have a collection of particles that separate into two clusters we would expect that, provided the interactions fall off sufficiently fast, the two systems have autonomous descriptions with one subsystem having no reference to the variables of the other subsystem. Such a property is a natural requirement on an interacting system since otherwise the description for a relativistic system of a finite number of particles in a nonempty universe would become meaningless. We want to study physics in *this* universe. No external directions can enter the description. It is difficult to see how this requirement can be ignored, since every system may be thought of as part of a larger system.

What our work suggests is that if all these requirements are imposed on a system with three or more particles within a definite framework, there can be no interaction. This has been a surprise to us since there is no such problem with a two-particle system: interacting two-particle systems obeying the world-line condition exist for which separability imposes no additional requirement beyond the fast falling off of the interaction with the separation between the two particles. In view of its importance we call the result of the present work the second no-interaction theorem.

This conclusion is at variance with the conclusions of some other authors who believe that they have found, at least in principle, such interactions. We believe that these authors have not carried out their analysis far enough to make sure that the dynamical descriptions of the two subsystems are really independent. In particular if the K constraints are separable that alone does *not* guarantee the separability of the *dynamics.*

One could relax the world-line condition and perhaps altogether give it up. As we have seen in Sec. III we can then find infinitely many interactions with Poincaré invariance and separability but without the world-line condition. One could argue that it need be obeyed only for the asymptotic segments of the world lines when the particles are practically free; in the corresponding quantum theory the scattering amplitudes would be manifestly invariant, but not the wave functions.

A somewhat novel approach to this problem is due to Sazdjian.[11] He considers the coordinates x_a^μ to be distinct from q_a^μ, the canonical coordinate; x_a^μ is defined as a dynamical variable of the form

$$x_a^\mu = A_{ab}(q,p)q_b^\mu + B_{ab}(q,p)p_b^\mu$$

with the Lorentz-invariant functions A,B satisfying the relations

$$\sum_b A_{ab} = 1 \ ,$$

$$\sum_c \frac{\partial}{\partial q_c} A_{ab} = \sum_c \frac{\partial}{\partial q_c} B_{ab} = 0 \ .$$

The world-line condition is now considered to be applied to the coordinates x_a^μ rather than q_a^μ. It leads therefore to partial differential equations to determine these coordinates. As we understand it, Sazdjian's program set in a multitime formalism has not been carried out fully to the extent of demonstrating that *curved* world lines can exist.

We now recognize that one could have started the Sazdjian program in terms of the variables x_a^μ and p_a^μ which transform in the usual manner with respect to the Poincaré group. For sufficiently well-behaved potentials they have the proper separable behavior with respect to the coordinate x_a^μ considered as independent arguments of the potentials. Seen thus the Sazdjian program may be viewed as the search for a way out of the second no-interaction theorem by taking the primitive bracket relation between the phase-space variables x_a^μ, p_a^μ to be a generalized Poisson bracket. This interaction-dependent generalized bracket relation is then an essential ingredient of the theory. It is even possible that one can use the free-particle forms for the K and χ constraints but alter the

primitive bracket relations to obtain interaction. Clearly this possibility merits further investigation.

In our work we have chosen a specific form (3.1) for the χ constraints and proved our main result from it. We note that this particular choice has much merit: it has the τ parameter as the time in an inertial system and its form is stable with respect to the addition or subtraction of particles. It is also an explicit linear relationship between the canonical coordinate variables and the temporal parameter. We believe it to be the most natural choice and to be the one most commonly chosen.

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[10]In formulating the WLC, we insist that of the two canonically conjugate four-vectors q_a^μ, p_a^μ $(a=1,2,\ldots,N)$ which span the primitive phase space Γ (cf. Sec. II), q_a^μ is the position four-vector of the ath particle. This is in agreement with the work of Dirac[2] and seems natural to us. Note however that by abandoning this requirement and allowing the particle positions x_a^μ to differ from the canonical q_a^μ, Sazdjian[11] seems able to fulfill all the four requirements. See, however, the remarks in Sec. V.

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[13]In their work [Phys. Rev. D 23, 2218 (1981)] Sudarshan, Mukunda, and Goldberg have suggested a model of constraints which may be seen as an exception to this. The model involves $N+1$ particles, the $(N+1)$th particle being free and not interacting with any of the others. The χ constraints are chosen to be

$$\chi_a = p_{N+1}\cdot q_a - m_{N+1}\tau \ , \ a=1,2,\ldots,N+1 \ .$$

The $(N+1)$th particle thus acts as a timekeeper. When the N interacting particles split into two groups of N_1 and $N_2 = N - N_1$ particles we can keep the same timekeeper for both of them. These constraints are obviously separable as they stand.