Separability in relativistic Hamiltonian particle dynamics

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The problem of separability in recent models of classical relativistic interacting particles is examined. This physical requirement is shown to be more subtle than naive separability of all the constraints defining the system: it is adequate to be able to canonically transform the time-fixing constraints from an unseparated to a separated form when clusters emerge. Viewing separability in this way, and within a specific framework, we are led to a new no-interaction theorem which states the incompatibility of nontrivial interaction with relativistic invariance, separability, and invariant world lines for more than two particles.

I. INTRODUCTION

Classical physics contains the point particle and aggregates of point particles as essential ingredients. Newtonian physics with its clear formulation of the equations of motion as second-order differential equations suggested that position and momentum were required to specify the kinematic state of the particle. Later developments of Lagrangian generalized coordinates allowed the treatment of holonomic constraints, and the passage to the Hamiltonian form. The development of constraint dynamics, initiated in the work of Dirac1 and completed in the work of several others makes the next grand generalization to nonholonomic constraints. The constraint formalism in its turn has led to significant new work in relativistic classical dynamics.

The fundamental principles underlying the Hamiltonian description of classical relativistic systems were first clearly enunciated by Dirac.2 The basic idea involved is to construct a canonical realization of the Poincaré group on the phase space appropriate to the given dynamical system. However, as Dirac pointed out, one has to make a choice from several possible forms of such descriptions. These are the instant, front, and point forms which correspond to different kinematic choices of Cauchy surfaces in space-time. It would be desirable to strengthen the Lie algebra formalism of Dirac to furnish a representation of the Poincaré group by canonical transformations to incorporate the idea of relativistic invariance. Within the above general framework, for a system of classical point particles, it seems desirable to incorporate the following features: (1) a representation of the Poincaré group by canonical transformations acting on particle variables, (2) the presence of nontrivial interactions, (3) the existence of invariant particle world lines, and (4) the property of separability.

Within the instant form, Thomas and Bakamjan and Thomas3 succeeded in constructing nontrivial models which, however, did not possess the preceding features (3) and (4). While the world-line condition (WLC) was explicitly abandoned by these authors, it was Foldy4 who pointed out that separability was also missing in these models.

Even if the requirement of separability, it was shown by Currie, Jordan, and Sudarshans5 that the features (1), (2), and (3) above were already incompatible within the instant form of dynamics. This is the well-known no-interaction theorem. An important contribution of this work was the formulation of the WLC in the language of Poisson brackets and appropriate to the instant form.

Recent work by several authors6 has shown ways to avoid the no-interaction theorem and to construct models possessing features (1), (2), and (3) by using the constraint Hamiltonian formalism elaborated by Dirac.1 A careful analysis7 of this work has shown that these models involve an enlargement of the original Dirac framework: it is necessary to consider dynamical choices of Cauchy surfaces (rather than the kinematic ones offered by Dirac). The expression for the WLC in a form appropriate to this enlarged framework has also been accomplished.8 Clearly, what remains to be done is the incorporation of separability in this new framework.

In nonrelativistic particle mechanics with short-range potentials, when groups of particles are far
removed from each other, each group follows its
own dynamics unaffected by the presence of other
groups. Such a property is what is qualitatively
meant by separability.

The intuitive basis of such a property is evident.
At a technical level, in scattering theory, this prop-
erty plays an important role.

In this paper, we want to discuss the implica-
tions of separability for relativistic Hamiltonian
dynamics. Within the constraint formal-
ism one recognizes that the full specification of the
dynamics involves the specification of all the con-
straints. Here several new features arise since be-
sides the separability of equations of motion and
initial conditions, we have also to face the problem
of the separability of all the constraints and Pois-
son brackets in a suitable sense. The conceptual
framework is further complicated by the WLC
which has to be fulfilled whether or not the par-
ticles are far apart. If the WLC is ignored, separa-
bility (and relativistic invariance) can be readily
achieved; this we shall show. On the other hand,
as we mentioned earlier, it is possible to fulfill the
WLC for interacting systems if we ignore the ques-
tion of separability. It is thus the combination of
these two requirements of separability and the
WLC that seems difficult to achieve in relativistic
particle dynamics with interactions. Some recent
work\textsuperscript{9} suggests that the problem of separability has
been solved, but our analysis shows that the impli-
cations of both separability and the WLC had not
been fully examined in that work; our conclusions
naturally differ.\textsuperscript{10,11}

We have made a precise definition of separabil-
ity but we have not been able to treat the problem
in complete generality. Rather, we base our dis-
cussion on a specific definition of the meaning of
separability and a specific choice of the con-
straints. Our choice of constraints, though specif-
ic, is reasonable and common, while our require-
ment of separability is satisfied by noninteracting
systems and only by them. (This last property is
to be contrasted with a recent formulation due to
Rohrlich.\textsuperscript{9})

We find that with our hypotheses it is not possi-
bile to construct a model possessing all the four
features. Our analysis suggests a new no-interac-
tion theorem stating the general incompatibility of
these four eminently reasonable requirements (see
Ref. 10 and Sec. V in this context). While such
indecomposability was already pointed out for a
quantum system by Einstein, Podolsky, and
Rosen\textsuperscript{12} we believe it is the first time that such a
feature has been discerned for classical systems.

The plan of the paper is as follows. In Sec. II,
we adapt the new techniques based on constraint
dynamics to solve the separability problem in the
instant form. This may be viewed as a completion
of the Bakamjian-Thomas work\textsuperscript{1} and of course
does not maintain invariant world lines. In Sec.
III we state the complete set of constraints work-
ing in the Droz-Vincent-Komar-Todorov (DVKT)
framework,\textsuperscript{6} and derive the separability require-
ments on these constraints. We also verify that
these requirements are fulfilled in the free-particle
case. In Sec. IV we examine these requirements
along with the WLC and prove explicitly that the
noninteracting system is the unique solution meet-
ing all our conditions. In the concluding Sec. V
we offer some comments comparing our approach
to other existing ones.

II. SEPARABILITY IN THE INSTANT FORM

Following the DVKT approach, for an \(N\)-
particle system, we start with an 8\(N\)-dimensional
phase space \(\Gamma\) spanned by 2\(N\) four-vectors \(q_a^\mu, p_a^\mu\)
\((a = 1,2,\ldots,N)\) under the Poincaré group. They
fulfill the Poisson-bracket (PB) relations

\[
\{q_a^\mu, q_b^\nu\} = \{p_a^\mu, p_b^\nu\} = 0, \quad \{q_a^\mu, p_b^\nu\} = \delta_{ab} g^{\mu\nu}.
\]

(2.1)

In this space we impose the \(N\) constraints

\[
K_a = p_a^2 - m_a^2 - V_a(q,p) = 0
\]

(2.2)

which are required to be Poincaré invariant and
first class:

\[
[K_a, K_b] = 0.
\]

(2.3)

We take from Sazdjian\textsuperscript{11} the following result:
separable solutions \(V_a\) to this system of equations
exist. That is to say, if the \(N\) particles are divided
into two clusters \(S_1\) and \(S_2\) of \(N_1\) and \(N_2\) particles,
respectively, and all difference four-vectors
\(q_{ab} \equiv q_a - q_b, a \in S_1, b \in S_2\) are large and spacelike,
then the "potentials" \(V_a, a \in S_1,\) depend only upon
the \(q\)'s and \(p\)'s of that cluster (and correspondingly
for \(S_2\)). Note that this result does not use the
WLC and is a property of the partial differential
equations (2.3).

To describe a true physical system of \(N\) particles
with the correct number, i.e., \(6N\), of degrees of
freedom, it is essential to supplement the \(K\) con-
straints by an additional set of \(N\) constraints \(X_a,\)
such that the \(2N\) constraints \(X, X\) form a second-
class set. To get the instant form of dynamics, we
choose the $X_a$ to be
$$ X_a = q^0_a - \tau, \quad (2.4) $$

where $\tau$, the evolution parameter, is the coordinate time. (An evolution parameter $\tau$ must occur in any choice of the set $X_a$.) With this choice, we have of course abandoned all hope of invariant world lines. But we recognize that the fixation constraints (2.4) are separable: the constraint on any one particle does not depend on the other particles.

By translation invariance, the $V_a$ depend on the $q$'s only through the differences, $q_{ab}$. On the hypersurface $\Sigma_\tau$ in $\Gamma$ defined by $K_a \approx X_a \approx 0$, we can thus imagine solving (2.2) for $p^a_\mu$ in the form
$$ p^0_a = h_a(\bar{q}_a, \bar{p}) . \quad (2.5) $$
The surface $\Sigma_\tau$ is obviously spanned by the $2N$ three-vectors $\bar{q}_a, \bar{p}_a$, and their Dirac brackets (DB) are
$$ \{q_{aj}, q_{bk}\}^* = [p_{aj}, p_{bk}]^* = 0, \quad (2.6) $$
$$ \{q_{aj}, p_{bk}\}^* = \delta_{ab} \delta_{jk} . $$

Starting from the Poincaré generators in $\Gamma$,
$$ P_\mu = \sum_a p_{a\mu}, \quad (2.7) $$
$$ M_{\mu\nu} = \sum_a (q_a \wedge p_a)_{\mu\nu} $$

we get their forms for use in $\Sigma_\tau$ to be
$$ \bar{p} = \sum_a \bar{p}_a, \quad \bar{q} = \sum_a \bar{q}_a \times \bar{p}_a , $$
$$ H = p^0 = \sum_a h_a(\bar{q}_a, \bar{p}) , $$
$$ \bar{K} = \sum_a \bar{q}_a h_a(\bar{q}_a, \bar{p}) - \tau \bar{p} . \quad (2.8) $$

From general theory we know that these are guaranteed to generate the Poincaré algebra under Dirac brackets.

To verify the separability of the system, we have to check this property for kinematics (i.e., the DB's) and for dynamics, i.e., the Poincaré generators (2.8). The basic DB's (2.6) are already in separated form. As for the generators, we have from Sazdjian the separability of the $V_a$; this obviously implies the separability of $H$, and of all the other generators as well. These generalizations go beyond the Bakamjian-Thomas scheme of constructing an invariant-mass-squared function in the center-of-mass frame.

Thus with the help of the constraint formalism developed by DVT/K, we are able to solve the separability problem which had been a difficulty for the Bakamjian-Thomas model.

### III. SEPARABILITY AND THE WLC

The choice of the $X$ constraints in the last section was incompatible with the WLC. A set of $X$'s known from previous work to be compatible with invariant world lines is
$$ X_a = P \cdot q_a - \tau. \quad (3.1) $$

In this section, we discuss separability in terms of this choice. At first sight, in view of the occurrence of the total four-momentum $P$ in $X_a$, one might hastily conclude that all hope of separability must be abandoned. However, a more careful analysis is necessary.

For any choice of $X$'s these constraints play two roles. On the one hand, their forms at $\tau = 0$ combined with the $K$ constraints give a complete set of restrictions on initial conditions in $\Gamma$. On the other hand, through the requirement that they be maintained for all $\tau$, they determine a definite dynamics, i.e., a definite linear combination $v_a K_a$ which generates $\tau$ evolution through the PB's:

$$ \frac{dX_a}{d\tau} = \frac{\partial X_a}{\partial \tau} + [X_a, v_b K_b] \approx 0, $$
$$ v_a = -C_{ab} \frac{\partial X_b}{\partial \tau} , \quad (3.2) $$
$$ C_{ab} [X_b, K_r] \equiv \delta_{ar} . $$

For the choice (3.1) of $X$'s, we have

$$ \frac{\partial X_a}{\partial \tau} = -1, $$
$$ [X_a, K_b] = P_\mu M_{ab}^\mu, \quad (3.3) $$
$$ M_{ab}^\mu \equiv [q^0_a, K_b] . $$

From the separability of the "potentials" $V_a$, it is clear that if the total system separates into clusters $S_1$ and $S_2$, the matrix $M_{ab}^\mu$ and hence the matrix $[X_a, K_b]$ becomes block diagonal. Thus the equations of motion for the $q$'s and $p$'s in $S_i$ involve only $K_a$ for $a \in S_i$. Moreover, the only dependence of these equations on the variables of $S_j$ ($j \neq i$) is through the total four-momentum of $S_j$. For complete separability of the equations of motion and the constraints, it is thus sufficient to remove this dependence in some way.

Now we make the following remark. In the
models which evade the no-interaction theorem, the Cauchy surfaces are determined dynamically and autonomously; this means in particular that for each cluster that separates from the total system, its Cauchy surface should not depend on the rest of the system. As things stand, this condition is not fulfilled. When the total system splits into two clusters \( S_1 \) and \( S_2 \), it is easy to see from the equations of motion that the total momentum \( P_\mu \) also splits into the sum \( P_\mu^{(1)} + P_\mu^{(2)} \) of two individually conserved momenta; \( P_\mu^{(1)} \) is just the total momentum of cluster \( S_1 \). For the subsequent description of \( S_1 \), the parent system would lead to Cauchy surfaces given by \( K_a = 0 \) and

\[
\chi_a = (P^{(1)} + P^{(2)}) \cdot q_a - \tau = 0 , \tag{3.4}
\]

while the constraints determined by \( S_1 \) alone ought to read \( K_a = 0 \) and

\[
\chi_a^{(1)} = P^{(1)} \cdot q_a - \tau = 0 . \tag{3.5}
\]

(Now and hereafter, \( a,b, \ldots \) run over cluster \( S_1 \).)

In space-time, the passage from \( \chi_a \) to \( \chi_a^{(1)} \) corresponds to tilting the family of hyperplanes normal to \( P^{(1)} + P^{(2)} \) to the family normal to \( P^{(1)} \). Complete separability of description thus evidently means that in the phase space of \( S_1 \), we should be able to transform the description based on the Cauchy surfaces given by \( K_a = 0, \chi_a = 0 \) to the ones given by \( K_a = 0, \chi_a^{(1)} = 0 \), \textit{without however altering the world lines of the particles in} \( S_1 \).

How is this tilting of the Cauchy surface to be achieved? We shall assume that this change of description, if at all possible, should be the result of a succession of infinitesimal canonical transformations generated by a linear combination \( u_a K_a \) of the \( K_a \)'s. In partial justification of this assumption, we may remark that such transformations preserve the \( K \) constraints in view of their first-class character.

Such \( u_a \)'s which can perform the tilting are subject to two more equations. The first comes from the requirement that the world lines are preserved. Thus the PB of \( u_b K_b \) \( \delta \sigma \) with any \( q_a \) should be equal to the PB of \( \delta q_a \tau \) with \( q_a \) where the infinitesimal quantity \( \delta q_a \tau \) can depend on \( \delta \sigma \) (and other variables of the problem). The second condition comes from the requirement that \( \chi_a^{(1)} \) is preserved by \( \tau \) evolution.

We shall now formalize these three requirements on \( \chi_a^{(1)} \).

Consider the family of constraints

\[
\chi_a^{(\sigma)} = [P^{(1)} + (1 - \sigma)P^{(2)}] \cdot q_a - \tau , \quad 0 \leq \sigma \leq 1 \tag{3.6}
\]

which interpolates between \( \chi_a \) and \( \chi_a^{(1)} \). If there exists

\[
P^{(\sigma)} = u_a^{(\sigma)} K_a \tag{3.7}
\]

such that

\[
\frac{\partial \chi_a^{(\sigma)}}{\partial \sigma} = \{ \chi_a^{(\sigma)}, P^{(\sigma)} \} , \tag{3.8}
\]

then we can deform \( \chi_a \) to \( \chi_a^{(1)} \) by canonical transformations.

There seems to be no loss of generality in the choice of the interpolation (3.6). For if there is an interpolation for a given value of \( P^{(2)} \), there should also be an interpolation when \( P^{(2)} \) is changed to \( P^{(2)} + \delta P^{(2)} \). That is, we expect that there are canonical transformations which change \( P^{(2)} \) alone in \( \chi_a \). The form (3.6) is consistent with this expectation.

We can simplify (3.8) to

\[
-P^{(2)} \cdot q_a \approx u_a^{(\sigma)} \{ \chi_a^{(\sigma)}, K_b \}
= [P^{(1)} + (1 - \sigma)P^{(2)}] \mu M_{ab}^{\mu} u_b^{(\sigma)} .
\tag{3.9}
\]

Here we have used the fact that \( K_a \)'s separate.

Translational invariance then implies that

\[
\{ P^{(1)}, K_b \} = 0, \; b \in S_1 \quad \text{while} \quad \{ P^{(2)}, K_b \} = 0, \; b \in S_2
\]

since \( K_b \) does not depend on \( q_a, \; a \in S_2 \).

The condition for the preservation of world lines is

\[
\{ q_a^{(\sigma)}, F^{(\sigma)} \} \delta \sigma \cdot q_a^{(\sigma)} \cdot u_a^{(\sigma)} K_b \delta \sigma \tau
= 0 , \tag{3.10}
\]

or

\[
M_{ab}^{\mu} u_b^{(\sigma)} \delta \sigma \cdot M_{ab}^{\mu} u_b^{(\sigma)} \delta \sigma \tau \quad \text{(no a sum)} .
\]

Here we have used \( u_a^{(\sigma)} K_a \) for the generator of \( \tau \) evolution.

The final equation involving \( \chi_a^{(\sigma)} \) is

\[
\frac{\partial \chi_a^{(\sigma)}}{\partial \tau} + \{ \chi_a^{(\sigma)}, u_a^{(\sigma)} K_b \} \approx 0
\]

or

\[
[P^{(1)} + (1 - \sigma)P^{(2)}] \mu M_{ab}^{\mu} u_b^{(\sigma)} \approx 1 . \tag{3.11}
\]

This serves to determine \( u_a^{(\sigma)} \).

It is easy to verify that the three equations (3.9), (3.10), and (3.11) can be fulfilled by the free-particle system. In this case \( M^\mu \) is diagonal:

\[
M_{ab}^{\mu} = 2 \delta_{ab} p_a^\mu \quad \text{(no a sum)} . \tag{3.12}
\]

Therefore each of the equations can be algebraical-
ly solved for \( u^{(\sigma)}_a, \delta_\sigma \tau, \) and \( u^{(\sigma)}_q \).

Thus despite the initial impression that the \( \chi \)'s do not separate, we are able to achieve separability for the free system by the tilting operation. This shows that the latter operation is not without content.

**IV. PROOF OF THE MAIN RESULT**

We shall now prove that the only system compatible with (3.9), (3.10), and (3.11) is the free system.

It is convenient to introduce the notation

\[
M_{ab} = [P^{(1)}+(1-\sigma)P^{(2)}]_\mu M^\mu_{ab}.
\]  

In terms of \( M \) the solution for \( v^{(\sigma)} \) is

\[
v^{(\sigma)}_a = \sum_b M^{-1}_{ab} \delta a^\sigma b
\]  

while the solution for \( u^{(\sigma)} \) is

\[
u^{(\sigma)}_a = -M^{-1}_{ab} P^{(2)} q_b.
\]

Substitution of these solutions in (3.10) leads to

\[
\delta \sigma M^\mu_{ac} M^{-1}_{ce} P^{(2)} q_b = \delta a^\sigma b M^{-1}_{ce} (no \ a \ \text{sum})
\]

Multiplication by \([P^{(1)}+(1-\sigma)P^{(2)}]_\mu\) shows that \( \delta_\sigma \tau \), if they exist at all, are given by

\[
\delta_\sigma \tau = -P^{(2)} q_a \delta \sigma
\]

which may be put back in (4.4) to find

\[
E^{(2)} \cdot P^{(2)} q_b = P^{(2)} q_a M^\mu_{ac} \sum_b M^{-1}_{ce}.
\]  

(4.6)

Defining the matrix \( \psi^\mu \)

\[
\psi^\mu_{ab} = M^\mu_{ac} M^{-1}_{cd} - \delta_{ab} M^\mu_{ac} \sum_d M^{-1}_{ed}
\]

we see that \( P^{(2)} q_1, \ldots, P^{(2)} q_N \) is a null eigenvector of \( \psi^\mu \):

\[
\psi^\mu_{ab} P^{(2)} q_b = 0.
\]

(4.8)

But \( 1, \ldots, 1 \) is also a null eigenvector of \( \psi^\mu \):

\[
\sum_b \psi^\mu_{ab} = 0.
\]  

(4.9)

Now consider the case where the cluster \( S \) has two particles. Then the 2 x 2 matrix \( \psi^\mu \) has two linearly independent null eigenvectors and therefore is identically zero:

\[
M^\mu_{ac} M^{-1}_{ce} = \delta_{ab} M^\mu_{ac} \sum_b M^{-1}_{ce}.
\]  

(4.10)

For this two-particle cluster, we can assume the canonical form

\[
V_a = V(q_1^2),
\]

\[
q_{1\mu} = q_\mu = \frac{P^{(1)} q_1}{P^{(1)}},
\]

\[
q = q_1 - q_2
\]

and explicitly evaluate \( M^\mu \) and \( M^{-1} \):

\[
M^\mu = 2 \begin{bmatrix}
0 & 0 \\
p_1^\mu & 0
\end{bmatrix} + V' \begin{bmatrix}
0 & 1 \\
p_1^\mu & 0
\end{bmatrix}
\]

\[
M^{-1} = \frac{2}{\det M} \begin{bmatrix}
0 & 0 \\
p_1 & \varphi
\end{bmatrix} + V' \begin{bmatrix}
p_1 & 0 \\
0 & \varphi
\end{bmatrix}
\]

\[
\varphi = P^{(1)}+(1-\sigma)P^{(2)}, \quad V' = \frac{dV}{dq_1^2}.
\]

The \( a = 1, b = 2 \) element of the left-hand side of (4.10) is proportional to

\[
V' (\varphi^\mu q_1 - \varphi^\mu q_2)
\]

which will not vanish for interacting systems. This completes our proof for those cases where the total number of particles is at least three.

If the system had consisted only of two particles which separate in the course of their motion, they must thereafter move freely. The analysis of the previous section then shows that with the help of the tilting operation, complete separability of description can be achieved for such a two-particle system.

**V. CONCLUDING REMARKS**

The work described in this paper is in essence an attempt at the completion of the researches into
the Hamiltonian description of interacting relativistic particles. In the spirit of Newtonian physics of motion of material particles one starts with a phase space and introduces a parameter of evolution. There are two aspects of relativistic invariance: the existence of a canonical realization of the Poincaré group acting on the phase space on the one hand, and the requirement that there be objective world lines, the so-called world-line condition, on the other. Within the context of these requirements the (first) no-interaction theorem states that if \( \tau \) is chosen to be the laboratory time there can be no interaction.

Constraint dynamics provides a richer variety of evolution parameters chosen through the specification of the fixation constraints. It then turns out that the world-line condition can be made compatible with interaction provided the evolution parameter is chosen to be a Lorentz invariant, and provided there is essentially only one constraint which is not Poincaré invariant. A large number of interacting systems may be so described.\(^7\)

A new constraint on the system of interacting particles is that of separability. When we have a collection of particles that separate into two clusters we would expect that, provided the interactions fall off sufficiently fast, the two systems have autonomous descriptions with one subsystem having no reference to the variables of the other subsystem. Such a property is a natural requirement on an interacting system since otherwise the description for a relativistic system of a finite number of particles in a nonempty universe would become meaningless. We want to study physics in this universe. No external directions can enter the description. It is difficult to see how this requirement can be ignored, since every system may be thought of as part of a larger system.

What our work suggests is that if all these requirements are imposed on a system with three or more particles within a definite framework, there can be no interaction. This has been a surprise to us since there is no such problem with a two-particle system: interacting two-particle systems obeying the world-line condition exist for which separability imposes no additional requirement beyond the fast falling off of the interaction with the separation between the two particles. In view of its importance we call the result of the present work the second no-interaction theorem.

This conclusion is at variance with the conclusions of some other authors who believe that they have found, at least in principle, such interactions. We believe that these authors have not carried out their analysis far enough to make sure that the dynamical descriptions of the two subsystems are really independent. In particular if the \( K \) constraints are separable that alone does not guarantee the separability of the dynamics.

One could relax the world-line condition and perhaps altogether give it up. As we have seen in Sec. III we can then find infinitely many interactions with Poincaré invariance and separability but without the world-line condition. One could argue that it need be obeyed only for the asymptotic segments of the world lines when the particles are practically free; in the corresponding quantum theory the scattering amplitudes would be manifestly invariant, but not the wave functions.

A somewhat novel approach to this problem is due to Szajdian.\(^11\) He considers the coordinates \( x^\mu \) to be distinct from \( q^\mu \), the canonical coordinate; \( x^\mu \) is defined as a dynamical variable of the form

\[
x^\mu = A_{\mu a}(q,p)q^a + B_{\mu a}(q,p)p^a
\]

with the Lorentz-invariant functions \( A, B \) satisfying the relations

\[
\sum_b A_{\mu b} = 1, \quad \sum_c \frac{\partial}{\partial q_c} A_{\mu b} = \sum_c \frac{\partial}{\partial q_c} B_{\mu b} = 0.
\]

The world-line condition is now considered to be applied to the coordinates \( x^\mu \) rather than \( q^\mu \). It leads therefore to partial differential equations to determine these coordinates. As we understand it, Szajdian’s program set in a multitime formalism has not been carried out fully to the extent of demonstrating that curved world lines can exist.

We now recognize that one could have started the Szajdian program in terms of the variables \( x^\mu_a \) and \( p^\mu_a \) which transform in the usual manner with respect to the Poincaré group. For sufficiently well-behaved potentials they have the proper separable behavior with respect to the coordinate \( x^\mu_a \) considered as independent arguments of the potentials. Seen thus the Szajdian program may be viewed as the search for a way out of the second no-interaction theorem by taking the primitive bracket relation between the phase-space variables \( x^\mu_a, p^\mu_a \) to be a generalized Poisson bracket. This interaction-dependent generalized bracket relation is then an essential ingredient of the theory. It is even possible that one can use the free-particle forms for the \( K \) and \( X \) constraints but alter the
primitive bracket relations to obtain interaction. Clearly this possibility merits further investigation.

In our work we have chosen a specific form (3.1) for the \( \chi \) constraints and proved our main result from it. We note that this particular choice has much merit: it has the \( \tau \) parameter as the time in an inertial system and its form is stable with respect to the addition or subtraction of particles. It is also an explicit linear relationship between the canonical coordinate variables and the temporal parameter. We believe it to be the most natural choice and to be the one most commonly chosen.

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2P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).
10In formulating the WLC, we insist that of the two canonically conjugate four-vectors \( q_a^u, p_a^u \) \( (a = 1, 2, \ldots, N) \) which span the primitive phase space \( \Gamma \) (cf. Sec. II), \( q_a^u \) is the position four-vector of the \( a \)th particle. This is in agreement with the work of Dirac\dagger and seems natural to us. Note however that by abandoning this requirement and allowing the particle positions \( x_a^u \) to differ from the canonical \( q_a^u \), Sazdjian\‡ seems able to fulfill all the four requirements. See, however, the remarks in Sec. V.
13In their work [Phys. Rev. D 23, 2218 (1981)] Sudarshan, Mukunda, and Goldberg have suggested a model of constraints which may be seen as an exception to this. The model involves \( N + 1 \) particles, the \( (N + 1) \) th particle being free and not interacting with any of the others. The \( \chi \) constraints are chosen to be

\[ x_a = p_{N+1} q_a - m_{N+1} \tau, \quad a = 1, 2, \ldots, N + 1. \]

The \( (N + 1) \) th particle thus acts as a timekeeper. When the \( N \) interacting particles split into two groups of \( N_1 \) and \( N_2 = N - N_1 \) particles we can keep the same timekeeper for both of them. These constraints are obviously separable as they stand.