THE TIME SCALE FOR THE QUANTUM ZENO PARADOX AND PROTON DECAY

C.B. CHIU
Center for Particle Theory, The University of Texas at Austin, Austin, TX 78712, USA

B. MISRA
Institut Solvay, Université Libre de Bruxelles, Brussels, Belgium

and

E.C.G. SUDARSHAN [1]
Institute of Theoretical Physics, S-412 96 Göteborg, Sweden

Received 17 June 1982

At very small times the decay law departs from the exponential one. We examine the possibility that this Zeno effect could suppress proton decay. We conclude that it is very unlikely, contrary to L.A. Khalfin who has recently suggested such a possibility.

1. Introduction. In recent years, there has been considerable effort devoted to understanding the time evolution of unstable quantum systems [1–4]. By now it is well known that this time evolution can be characterized in general terms of three time regions divided by the time scales t_1 and t_2. In the region $t_1 < t < t_2$ the usual exponential decay law: $\exp(-\Gamma t/2)$ is essentially satisfied. On the other hand, in the small-t region, i.e. $0 < t < t_1$, there is a significant departure from the exponential law. One can readily see this in the following way. Due to the assumed time reversal invariance of the system in question, the time derivative of the survival probability must be the same with respect to the forward development and the backward development in time. This implies that the derivative of the survival probability at $t = 0$ should either be zero or infinity. One can easily check that for a quantum system with finite energies, the correct choice for the derivative is zero. This implies that there must be deviations from the exponential decay law for sufficiently small values of t.

[1] On leave of absence from Center for Particle Theory, The University of Texas at Austin, Austin, TX 78712, USA.

There is also a departure from the exponential behavior in the large-t region where $t \gg 1/\Gamma$. We recall that the survival amplitude is given by the Fourier transform of the energy spectrum of the quantum system [see eq. (2.1) below]. In the region $t \gg 1/\Gamma$, one finds that the contribution to the survival amplitude is not dominated by the resonance pole contribution, instead the main contributions come from the region near the threshold. The survival amplitude here is governed by the phase space of the decay products from threshold up to $E \sim 1/t$. In turn, in the large-t region, it is the geometric property of the system which governs the time evolution of the system.

At $t = 0$, the very vanishing of the time derivative of the survival probability can lead to the Zeno paradox. In particular, consider successive observations on a system of unstable particles. With the stipulated time dependence of the survival probability, successive observations separated by small time intervals may lead to the prolongation of the life-time of the unstable particles. Pushing to its extreme, if the observation interval can be reduced to arbitrarily small values, then the unstable particles may end up not decaying at all. This is the Zeno paradox [3,4].

Recently, there is great interest in the possibility of proton decay. The proton life-time estimates based on the grand unified theory is about 10^{31} yr [5,6]. Now if the Zeno interval for proton decay is longer than the experimental time resolution, e.g.

$$\Delta t \sim 10^{-12}\ \mathrm{s}\,, \tag{1.1}$$

then the Zeno effects could lead to a rather nontrivial consequence. In particular, the experimental setup of detecting proton decay amounts to performing successive observations on the initial proton system, with a time interval Δt. This according to the Zeno paradox alluded above, should prolong the original proton life-time. Now if the Zeno time scale is sufficiently long as compared to 10^{-12} s, then the very experimental setup would paradoxically prevent the detection of proton decay.

Recently Khalfin considered another circumstance in which the deviation from exponential decay law may lead to an effective suppression of proton decay. *in the present epoch of the universe* [7]. This would be the case if the Zeno interval for the proton is as long as 10^{10} yr, i.e., the estimated age of the universe.

The possibility of an effective suppression of proton decay implied by the Zeno effect and the possibility considered by Khalfin are both consequences of the deviation from the exponential decay law at the "small time" region. They are, however, physically distinct. The Zeno effect is the combined result of the deviation from the exponential decay law and the quantum-mechanical effect of repeated observations. It is independent of cosmological theories or the age of the universe. The possibility envisaged by Khalfin, on the other hand, does not make use of the effect of repeated measurements, but invokes the currently accepted "big bang" cosmological theory and depends on the age of the universe.

In the present note, we present some general considerations to estimate the Zeno effect time scale t_1. This is presented in section 2. In section 3, we come back to address the question of proton decay. Here we disagree with the conclusion of ref. [7], we find that it is very unlikely that the Zeno effect and a fortiori, the possibility considered by Khalfin, can be operative in the case of proton decay. However, in section 4 we will mention other situations where the Zeno effect does play a nonnegligible role in connection with the understanding of the time evolution of the corresponding quantum systems.

2. General discussion on the Zeno time scale.

(a) A general requirement on the energy weight function. The survival amplitude for an unstable quantum state $|M\rangle$ is defined by

$$a(t) = \langle M|M(t)\rangle = \langle M|\exp(-\mathrm{i}\lambda t)|M\rangle$$

$$= \int_0^\infty \mathrm{d}\lambda \langle M|\lambda\rangle \exp(-\mathrm{i}\lambda t)\langle\lambda|M\rangle$$

$$= \int_0^\infty \mathrm{d}\lambda\, \exp(-\mathrm{i}\lambda t)\rho(\lambda)\,. \tag{2.1}$$

In other words, the survival amplitude is given by the Fourier transform of the weight function:

$$\rho(\lambda) = |\langle M|\lambda\rangle|^2\,. \tag{2.2}$$

Expanding the integrand, we get

$$a(t) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\langle\lambda^n\rangle\,, \tag{2.3}$$

with the nth moment defined by [8]:

$$\langle\lambda^n\rangle = \int_0^\infty \mathrm{d}\lambda\, \lambda^n \rho(\lambda)\,. \tag{2.4}$$

For the Zeno paradox to occur, a general requirement is that the radius of convergence, τ, in the series expansion of eq. (2.3) should be nonzero. This implies that all the moments should exist. Specifically,

$$\langle\lambda^n\rangle \leqslant n! L^n\,, \tag{2.5}$$

where L is some constant. Of course, the inequality of eq. (2.5) covers a wide range of possibilities for τ. The equality in eq. (2.5) implies

$$\tau \sim 1/L\,, \tag{2.6}$$

while for $\langle\lambda^n\rangle \sim L^n$,

$$\tau \sim \infty\,. \tag{2.7}$$

In terms of the spectrum $\rho(\lambda)$, the requirement for the existence of all moments necessarily implies that for large λ, $\rho(\lambda)$ must fall off faster than any inverse power of λ.

(b) Statements on the relationship between the spectrum and the survival amplitude. We find that it is instructive at this point to analyze the t-dependence of the survival amplitude contributed by a Breit–

Wigner weight function with various cutoffs. We first consider the cutoff spectrum:

$$\rho(\lambda) = \pi^{-1}\,(\Gamma/2)/[(\lambda - m)^2 + (\Gamma/2)^2]\ ,$$
$$m - c \leqslant \lambda \leqslant m + c\ ; \tag{2.8}$$

$$\rho(\lambda) = 0, \quad \text{otherwise}\ . \tag{2.9}$$

with $c \gg \Gamma/2$. Later we comment on the case where eq. (2.9) is replaced by suitable smooth background contributions.

For now we write,

$$a(t) = \int_{-c}^{c} \mathrm{d}\lambda\, \rho(\lambda) \exp(-\mathrm{i}\lambda t)$$
$$\equiv a_1(t) + a_2(t) + a_3(t)\ , \tag{2.10}$$

with the three terms on the right-hand side of eq. (2.10) being contributions of the spectrum in the ranges: $[m - b, m + b]$, $[m - c, m - b]$ and $[m + b, m + c]$, respectively. The parameter b is chosen in the interval: $\Gamma/2 \ll b \ll c$. We confine our attention here to the time interval: $1/c \ll t \ll 1/b$.

In the Appendix, we show that

$$\exp(\mathrm{i}mt)a_1(t) = 1 - 2f(t) + \mathrm{O}[\Gamma^2 t^2, \Gamma b^3 t^4]\ ; \tag{2.11}$$

$$\exp(\mathrm{i}mt)a_2(t)$$
$$= -\tfrac{1}{4}\Gamma t + f(t) - \mathrm{i}g(t) + \mathrm{O}(\Gamma t/(ct)^2)\ ; \tag{2.12}$$

$$\exp(\mathrm{i}mt)a_3(t)$$
$$= -\tfrac{1}{4}\Gamma t + f(t) + \mathrm{i}g(t) + \mathrm{O}(\Gamma t/(ct)^2)\ ; \tag{2.13}$$

where

$$f(t) = \tfrac{1}{2}(\Gamma/\pi b + \Gamma b t^2/2\pi)\ ;$$
$$g(t) = (\Gamma t/2\pi)(\gamma - 1 + \ln bt) \tag{2.14}$$

with the Euler constant $\gamma \simeq 0.58$. Eqs. (2.11) to (2.13) lead to

$$\exp(\mathrm{i}mt)a(t) \simeq 1 - \Gamma t/2 + \mathrm{O}(\Gamma t/(ct)^2)\ . \tag{2.15}$$

Consider now the implication of eq. (2.11). Here $t \ll 1/b \ll 1/\Gamma$.

$$\exp(\mathrm{i}mt)a_1(t) \simeq 1 - \Gamma/\pi b - \Gamma b t^2/2\pi\ . \tag{2.16}$$

If the spectrum from $\lambda = m - b$ to $m + b$ is the entire spectrum, it will be necessary to properly normalize the spectrum, giving the corresponding normalized amplitude:

$$\exp(\mathrm{i}mt)\,\tilde{a}(t) \simeq 1 - \Gamma b t^2/2\pi\ . \tag{2.17}$$

For this case, for $t < 1/b$, the Zeno effect is important. The Zeno time scale is: $t_1 \sim 1/b$. This leads to our first statement in this section.

Statement 1: If a simple resonance spectrum is truncated at $\lambda = m - b$ and $m + b$, the Zeno time scale is $t_1 \sim 1/b$.

Eq. (2.15) corresponds to the case where the spectrum is cut off at $\lambda = m - c$ and at $\lambda = m + c$. The time interval considered is in the range: $t \ll 1/c \ll 1/\Gamma$. From eq. (2.15), we see that the exponential decay law $\exp(-\Gamma t/2) \approx 1 - \Gamma t/2$ dominates the t-dependence here. In the Appendix we show that this exponential decay law persists even in the presence of a suitably smooth background outside of the resonance region. This then leads to our second statement.

Statement 2: If the Breit–Wigner spectrum is approximately valid within the energy interval $m - c$ to $m + c$, then an exponential behavior is expected for the survival amplitude in the time interval $1/c \ll t \ll 1/\Gamma$. The exponential law persists even in the presence of some smooth background contributions outside of the resonance region considered.

Next consider the asymmetric cutoffs of the resonance peak. Inspection of eqs. (2.11)–(2.13) lead to following modified conclusions:

Statement 1′: Statement 1 can be generalized to include the case where the spectrum is taken to be $[m - b', m + b]$, where $t \ll 1/b', \ll 1/\Gamma$. For this case,, the Zeno scale is given by $t_1 \sim \min(1/b, 1/b')$.

Statement 2′: Statement 2 can be generalized to include the case where the resonance spectrum is approximately valid within the range $[m - c', m + c]$ for $1/c' \ll t \ll 1/\Gamma$. For this case the exponential term dominates in the time interval: $\max(1/c, 1/c') \ll t \ll 1/\Gamma$.

A separate discussion is needed for the truncated

Breit–Wigner spectrum within the energy interval $[m - c, m + c]$. For this case, adding the contributions on the right-hand side of eq. (2.11) and that of eq. (2.12) we arrive at

$$\exp(\mathrm{i}mt)\, a(t) \approx 1 - \Gamma/2\pi b - \tfrac{1}{2}\Gamma t(1 + 2bt/\pi) - \mathrm{i}g(t)\,, \qquad (2.18)$$

where $g(t) = \frac{1}{2}\Gamma t(\gamma - 1 + \ln bt)$. Keeping to the lowest order in Γ, the properly normalized survival amplitude is given by

$$\exp(\mathrm{i}mt)\, a(t) \approx \exp(\mathrm{i}g) \exp[-\tfrac{1}{2}\Gamma t(1 + 2bt/\pi)]$$
$$\simeq \exp(\mathrm{i}g) \exp(-\Gamma t/2)\,. \qquad (2.19)$$

This leads to our third statement.

Statement 3: Consider the contribution of the pole spectrum within the energy interval $\lambda = m - c$ to $\lambda = m + b$. For the time interval $1/c \ll t \ll 1/b$, there is an approximate exponential decay law for the survival amplitude given by $\exp(\mathrm{i}g) \exp(-\Gamma t/2)$.

3. Applications. The main lesson which we have learned from the last section is that the Zeno time scale depends on the suppression of the spectrum in the proximity of the pole position. More specifically, if the "pole spectrum" is approximately valid for the interval from $\lambda = m - c$ to $\lambda = m + c$, the exponential law begins to dominate beyond $t \sim 1/c$. This then imposes an upper limit for the Zeno time scale, i.e. $t_1 \sim 1/c$. On the other hand, if the pole spectrum is suppressed outside the region $\lambda = m - b$ to $\lambda = m + b$, then there is a Zeno region for $t < 1/b$. So the Zeno scale here is $\sim 1/b$. Notice that as $b \to 0$, the survival amplitude will approach that for a stable particle, which is given by the phase factor $\exp(-\mathrm{i}mt)$.

The weight function in a system with resonance contributions can in general be parameterized in terms of the discontinuity of the inverse of a certain denominator function, $d(\lambda)$, i.e.

$$\delta(\lambda) = [1/d(\lambda - \mathrm{i}t) - 1/d(\lambda + \mathrm{i}t)]/2\mathrm{i} \qquad (3.1)$$
$$= |f(\lambda)|^2/[d(\lambda - \mathrm{i}t)\, d(\lambda + \mathrm{i}t)]\,,$$

where

$$|f(\lambda)|^2 = [d(\lambda + \mathrm{i}t) - d(\lambda - \mathrm{i}t)]/2\mathrm{i}\,, \qquad (3.2)$$

with $f(\lambda)$ being the usual numerator function [4].

Barring some judicious concoction, we expect both the $f(\lambda)$ and $d(\lambda)$ functions to be smooth function of λ. For $\lambda < m$, $|f(x)|^2$ is proportional to the phase space factor. Beyond $\lambda = m$, and eventually as λ increases, $f(\lambda)$ being the form factor, is expected to fall off. However, the energy scale beyond which the fall-off occurs depends on the dynamics. For instance for proton decay, within the framework of grand unified theory, $f(\lambda)$ falls of at $\sim 10^{15}$ GeV. The d-function is expected to satisfy a dispersion relation.

Taking into account the facts enumerated, the energy scale which characterizes the variation of the numerator function and the denominator functions should be of the order of the Q value involved, with $Q = m - \lambda_0$ and λ_0 being the threshold energy. The important point here is that this scale is insensitive to the width of the resonance or to the coupling involved. To put it differently, given a dynamical model in which there is a resonance as the coupling decreases the width of the resonance will decrease. Notice that as long as the width of the resonance is finite the Zeno time scale is always going to be controlled by the energy scale measured from the position of the peak to the point beyond which there is a strong suppression giving rise to a significant departure from the pole form. Hence the energy scale governing such a deviation is insensitive to the width of the resonance.

Now let us take proton decay as a numerical example. Using the numbers quoted in section 1, to get $t_1 \sim 10^{-12}$ s, it is necessary for the suppression to occur at $b \sim 10^{-3}$ eV. Is such a suppression expected? We do not think so. Take for example the process $p \to \pi^0 e^+$. Our discussion above implies that the deviation from the simple pole form should become appreciable at energies which differ by about 10^2 MeV but not 10^{-3} eV.

We mentioned earlier, Khalfin recently conjectured that $t_1 \sim 10^{10}$ yr [7]. This would require the suppression to occur at an energy interval $b \sim 10^{-33}$ eV away from the peak position. This should be regarded as rather unrealistic!

Starting with the definition (2.1), we can express the survival probability in the form

$$P(t) = \int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\mu \exp[\mathrm{i}(\lambda - \mu)t]\, \rho(\lambda)\rho(\mu)$$
$$= \int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\mu \cos((\lambda - \mu)t)\rho(\lambda)\rho(\mu)\,,$$

so that

$$1 - P(t) = \int_0^\infty d\lambda \int_0^\infty d\mu \, [1 - \cos((\lambda - \mu)t)] \rho(\lambda)\rho(\mu) \,. \quad (3.3)$$

Using the inequality

$$1 - \cos x \geqslant \tfrac{1}{4}(1 - \cos 2x) \,, \quad (3.4)$$

we get the inequality

$$1 - P(t) \geqslant \tfrac{1}{4}[1 - P(2t)] \,. \quad (3.5)$$

Khalfin has suggested using this to estimate a Zeno time scale. But to do so it is necessary to invoke the equality in eq. (3.5). But an inspection of eq. (3.4) reveals that the equality relation can be satisfied only for the case $x = 0$. For $t \neq 0$, this corresponds to setting $\lambda = \mu$ in (3.3). The last relation can occur only for the uninteresting case when the density function $\rho(\lambda)$ corresponds to one discrete state. For this case the "Zeno time" is infinite. So no useful information on the value of Zeno time scale t_1 can be derived solely based on the equality relation for eq. (3.5).

4. Discussions and outlook. We have studied the possibility of making use of the Zeno effect to suppress the observation of proton decay. Based on our analysis, we do not envision such a possibility. This is because of the fact that to arrive at some acceptable time scale for the Zeno effect to be important, it is necessary to suppress the pole spectrum very close to the position of the pole. At the present time there is no dynamical model which can naturally predict such a suppression. Of course, we have not completely ruled out the possibility of having Zeno suppression in proton decay; our point is that to demonstrate the Zeno effect as a viable possibility, it is necessary to look for dynamical models which can give rise to cutoffs in the pole spectrum at some fraction of an electron-volt away from the position of the pole.

While it is not likely for the Zeno effect to play an important role in proton decay, such effect does manifest itself in other physical situations. In the past we have explored the Zeno effect in connection with hadron–nucleus collisions [9,10] and the equation of state appropriate for neutron stars [10]. For these two cases, scattering centers are closely packed together, in turn the Zeno time scale can be of the order of $t_1 \sim 10^{-23}$ s $\sim 1/m_\pi$. In the case of strong interactions, the form factor cutoff scale here is of the order of $b \sim m_\pi$, so the product bt_1 is of the order of unity and the Zeno effect is non-negligible.

In the future we plan to further explore other physical situations where the Zeno effect is important and also to explore new dynamical situations where strong suppression near the resonance pole could indeed occur as applied to the case of proton decay.

This work was supported by the US Department of Energy.

Appendix: Details of calculations. In this Appendix we provide the detailed calculations for the model presented in section 2.

The cutoff Breit–Wigner spectrum is defined by:

$$\rho(\lambda) = \pi^{-1}(\Gamma/2)/[(\lambda - m)^2 + (\Gamma/2)^2] \,, \quad m - c \leqslant \lambda < m + c; \quad (2.8)$$

and

$$\rho(\lambda) = 0 \,, \quad \text{otherwise.} \quad (2.9)$$

with $c \geqslant \Gamma/2$. The survival amplitude is written as

$$a(t) = \int_{-c}^{c} d\lambda \, \rho(\lambda) \exp(i\lambda t) = a_1(t) + a_2(t) + a_3(t) \,, \quad (2.10)$$

with the three terms on the right-hand side contributed by the spectrum in the ranges: $[m - b, m + b]$, $[m - c, m - b]$ and $[m + b, m + c]$, respectively.

(1) Evaluate $a_1(t)$:

$$\begin{aligned} e(imt)\, a_1(t) &= \frac{\Gamma}{2\pi} \int_{-b}^{b} dx \exp(-ixt) [x^2 + (\Gamma/2)^2]^{-1} \\ &= \frac{\Gamma}{2\pi} \int_{-b}^{b} dx \, [1 - ixt + (-ixt)^2/2! + ...] \times [x^2 + (\Gamma/2)^2]^{-1} \quad (A.1) \\ &\simeq 1 - \frac{\Gamma}{2\pi} \left(\int_{-\infty}^{b} dx/x^2 + \int_{b}^{\infty} dx/x^2 \right) \\ &\quad - \frac{\Gamma}{2\pi} \frac{t^2}{2} \int_{-b}^{b} dx \, x^2 [x^2 + (\Gamma/2)^2]^{-1} + \ldots \,. \end{aligned}$$

so

$$\exp(imt)\, a_1(t) = 1 - \Gamma/\pi b - (\Gamma b/2\pi)t^2 + O[(\Gamma t)^2, (\Gamma b^3 t^4)] \,. \tag{A.2}$$

(2) Evaluate a_2, a_3 and a.

To facilitate our calculations below, we first quote several mathematical identities [11]:

$$\int \mathrm{d}w \exp(\mathrm{i}w)\,(w^2 + a^2)^{-2} = (\mathrm{i}/2a)[\exp(-a)\, E_1(-a - \mathrm{i}w) - \exp(a)\, E_1(a - \mathrm{i}w)] + \text{const.}, \tag{A.3}$$

where $w > 0$ and $E_1(z)$ is the exponential integral function. Its small argument expansion is given by

$$E_1(Z) \simeq -\gamma - \ln Z + Z - Z^2/4 + O(Z^3) \,. \tag{A.4}$$

The corresponding large argument expansion is given by

$$E_1(Z) \simeq [\exp(-Z)\, Z^{-1}]\,[1 + O(1/Z)] \,. \tag{A.5}$$

For the evaluation of a_2 and a_3 we write:

$$\exp(imt)\, a(t) = \frac{\Gamma}{2\pi} \int_{-c}^{c} \mathrm{d}x \exp(-\mathrm{i}xt)\,[x^2 + (\Gamma/2)^2]^{-1} \,. \tag{A.6}$$

Denote

$$F = \int_{-c}^{-b} \mathrm{d}x \exp(-\mathrm{i}xt)\,[x^2 + (\Gamma/2)^2]^{-1} = \int_{b}^{c} \mathrm{d}x \exp(\mathrm{i}xt)\,[x^2 + (\Gamma/2)^2]^{-1} \,. \tag{A.7}$$

Then from (A.6) and (A.7)

$$\exp(imt)\, a_2(t) = (\Gamma/2\pi)F \,, \tag{A.8}$$

$$\exp(imt)\, a_3(t) = (\Gamma/2\pi)F^* \,. \tag{A.9}$$

We identify $w = xt$ and $a = \Gamma t/2$,

$$F = t \int_{bt}^{ct} \mathrm{d}w \exp(\mathrm{i}w)\,(w^2 + a^2)^{-1} \equiv t[G(ct) - G(bt)] \,. \tag{A.10}$$

For $bt \ll 1$ and $a \ll 1$, we get

$$-2a\mathrm{i}G(w) \simeq (1 - a)[-\gamma - \ln Z + Z - Z^2/4]_{Z=-a-\mathrm{i}w} - (1 + a)\,[-\gamma - \ln Z + Z - Z^2/4]_{Z=a-\mathrm{i}w} \tag{A.11}$$

where the identities (A.3) and (A.4) were used. Since only the difference of G's appears in (A.10), the constant term in (A.3) has been dropped. Now for $z = -a - \mathrm{i}w$,

$$\ln Z = \ln[-\mathrm{i}(w - \mathrm{i}a)] \simeq -\mathrm{i}\pi/2 + \ln w - \mathrm{i}a/w \tag{A.12}$$

and

$$Z - Z^2/4 = (-a - \mathrm{i}w) - (a + \mathrm{i}w)^2/4 \simeq -a - \mathrm{i}w - \mathrm{i}aw/2 \,. \tag{A.13}$$

So

$$-2a\mathrm{i}G \simeq (1 - a)(A + aB) - (1 + a)(A - aB) \tag{A.14}$$

where

$$A = -\gamma - \ln w + \mathrm{i}(\pi/2 - w) \,,$$

$$B = -1 + \mathrm{i}(1/w - w/2) \,.$$

In turn

$$G(bt) = \mathrm{i}(\gamma - 1 + \ln bt) + (\pi/2 - bt/2 - 1/bt) \,, \tag{A.15}$$

or

$$(\Gamma t/2\pi)G(bt) = (\Gamma t/4 - \Gamma bt^2/4\pi - \Gamma/2\pi b) + (\mathrm{i}\Gamma t/2\pi)(\gamma - 1 + \ln bt) \,, \quad bt \ll 1 \,. \tag{A.16}$$

For $ct \gg 1$,

$$G(ct) \simeq (\mathrm{i}/2a)\,[\exp(-a)\exp(a + \mathrm{i}ct)\,(-a - \mathrm{i}ct)^{-1} - \exp(a)\exp(-a + \mathrm{i}ct)\,(a - \mathrm{i}ct)^{-1}] = -\mathrm{i}\exp(\mathrm{i}ct)\,[(\Gamma t/2)^2 + (ct)^2]^{-1} = -\mathrm{i}\exp(\mathrm{i}ct)\,(ct)^{-2} \,. \tag{A.17}$$

From (A.8), (A.10), (A.16) and (A.17).

$$\exp(imt)\, a_2(t) = [-(\Gamma t/4) + (\Gamma bt^2/4\pi) + (\Gamma/2\pi b)] - \mathrm{i}(\Gamma t/2\pi)(\gamma - 1 + \ln bt) + O(\Gamma t/(ct)^2) \,. \tag{A.18}$$

From (A.9),

$$\exp(imt)\, a_3(t) = [-(\Gamma t/4) + (\Gamma b t^2/4\pi) + (\Gamma/2\pi b)] + i(\Gamma t/2\pi)(\gamma - 1 + \ln bt) + O(\Gamma t/(ct)^2)\,. \quad (A.19)$$

Finally from (A.18) and (A.19), we get

$$\exp(imt)\, a(t) \simeq 1 - (\Gamma t/2) + O[(\Gamma t)^2, \Gamma t/(ct)^2]\,. \quad (A.20)$$

(3) Estimate the effect of background contributions.
Let us now return to eqs. (2.8) and (2.9) at the beginning of this Appendix. We replace $\rho(\lambda) = 0$ by some smooth background contribution which is assumed to be smoothly joined on to the pole contributions at $\lambda = m - c$ and at $\lambda = m + c$. We write the full survival amplitude

$$a'(t) = a(t) + a_{bk}(t) \quad (A.21)$$

with the background contribution defined by

$$a_{bk}(t) = \int_0^{m-c} d\lambda\, \rho_{bk}(\lambda) \exp(-i\lambda t) + \int_{m+c}^{\infty} d\lambda\, \rho_{bk}(\lambda) \exp(-i\lambda t) = a'_{bk}(t) + a''_{bk}(t)\,. \quad (A.22)$$

The second term is given by:

$$a''_{bk}(t) = \exp[-i(m+c)t] \times \int_{m+c}^{\infty} d\lambda\, \rho_{bk}(\lambda) \exp[-x(\lambda - m - c)t] \simeq t^{-1} \cdot \exp[-i(m+c)t]\; \rho_{bk}(m+c)\,. \quad (A.23)$$

An order of magnitude estimate gives:

$$\rho_{bk}(m+c) \simeq \rho(m+c) = \Gamma/2\pi c^2\,. \quad (A.24)$$

Thus

$$a''_{bk}(t) \sim \exp[-i(m+c)t]\; (\Gamma t/2\pi)/(ct)^2\,. \quad (A.25)$$

In order to evaluate a'_{bk}, we need to consider the threshold behavior of $\rho(\lambda)$. Take for example proton decay, the fundamental process is $qq \to \bar{q}\ell$, with final state particle masses being m_1 and m_2. The corresponding threshold behavior is given by $\Gamma \propto k^{2l+1}/\lambda$. For a vector coupling both $l = 0$ and $l = 1$ states contribute. Near threshold, the $l = 0$ contribution should dominate There we write

$$\Gamma \sim \tilde{\Gamma}[(\lambda - m_1 - m_2)/m]^{1/2}\,. \quad (A.26)$$

Thus

$$a'_{bk}(t) \sim \tilde{\Gamma} m^{-5/2} \int_0^{1/t} dx\, x^{1/2} \sim \tilde{\Gamma} t (mt)^{-5/2}\,. \quad (A.27)$$

For the case of present interest, $m \gg c$. So in the time interval $1/\Gamma \gg t \gg 1/c \gg 1/m$ from (A.25) and (A.27), one sees that the background contribution is negligible compared to the quantity $-\Gamma t/2$ in (A.20), which characterizes the time dependence of the survival amplitude in this time interval.

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