An exactly solvable quantum field theory in three dimensions

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We construct a quantum field theoretic model in three space dimensions and show that its spectrum can be exactly calculated. We also show that all the eigenvectors of the Hamiltonian can be obtained by a recursive procedure.

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1. INTRODUCTION

Nonperturbative phenomena in quantum field theory are obviously of great importance. One such phenomenon, the formation of bound states, is an example of an unsolved problem almost as old as quantum field theory itself. The study of solvable quantum field theoretic models is of considerable interest since it may shed light on mechanisms responsible for some nonperturbative features of more realistic theories.

There exists considerable literature[1] on solvable quantum field theories in one space and one time dimension. To the best of our knowledge, there are no solvable models in three space and one time dimension. In this paper, we present a nonrelativistic (in the sense that negative frequencies are absent) field theory in three space dimensions for which the spectrum of the Hamiltonian can be exactly calculated. Moreover, we demonstrate that all the eigenstates of the Hamiltonian can be obtained by a recursive procedure. Thus, the S-matrix element for any process is, in principle, exactly calculable.

This paper is organized as follows. In the next section, we write down the Hamiltonian and the equations of motion for our model. It is noted that the vector space of states is a countable union of noncombining subspaces, each labeled by the values of two conserved quantum numbers, $\mathscr{N}_1$ and $\mathscr{N}_2$ (see Eqs. 2.3). In Sec. 3, it is shown that the spectrum of the Hamiltonian in any subspace is simply related to that in the subspaces with the same value of $\mathscr{N}_1$ but with $\mathscr{N}_2$ differing by one unit. In Sec. 4, it is shown that the spectrum of the Hamiltonian and the corresponding eigenstates in the family of subspaces with $\mathscr{N}_2 = 0$ can be readily obtained by explicitly solving the equations of motion. The complete spectrum can then be obtained from this by using the recursive procedure described in Sec. 3. In Sec. 5, we show that, for fixed $\mathscr{N}_1$, the eigenstates in the subspace labeled by $\mathscr{N}_2$ can be obtained from the corresponding states in the subspace labeled by $\mathscr{N}_2 - 1$. In doing so, we find that certain "eigenvalues" of the Hamiltonian (as obtained in Secs. 3 and 4) are spurious in that the corresponding eigenvector is null. We end with some concluding remarks in Sec. 6.

2. THE FERMIONIC LEE MODEL[2]

The model we consider consists of two fermion[3] fields N and θ interacting with a boson field V via a Yukawa-type interaction. It is assumed that the fields N and V are infinitely massive so that their energy is independent of the momentum. For simplicity, we assume the θ particle is massless. The Hamiltonian for the system is given by

$$H = m_0 V^\dagger V + \int d^3l\, l a^\dagger(l) a(l) + \int d^3l f(l) [V^\dagger N a(k) + a^\dagger(l) N^\dagger V]. \tag{2.1}$$

The quantization rules are,

$$\begin{aligned} &\{N,N^\dagger\} = [V,V^\dagger] = 1,\\ &\{a(k),a^\dagger(l)\} = \delta(\vec{k}-\vec{l}),\\ &\{N,N\} = [V,V] = \{a(k),a(l)\} = 0,\\ &\{N,a(k)\} = \{N,a^\dagger(k)\} = [N,V] = [N,V^\dagger]\\ &\qquad = [a(k),V] = [a(k),V^\dagger] = 0, \end{aligned} \tag{2.2}$$

together with their Hermitian conjugates.

The operators,

$$\mathscr{N}_1 = \mathscr{N}_V + \mathscr{N}_N, \tag{2.3a}$$

and

$$\mathscr{N}_2 = \mathscr{N}_\theta - \mathscr{N}_N, \tag{2.3b}$$

with

$$\mathscr{N}_V = V^\dagger V, \tag{2.4a}$$

$$\mathscr{N}_N = N^\dagger N, \tag{2.4b}$$

and

$$\mathscr{N}_\theta = d^3k\, a^\dagger(k) a(k), \tag{2.4c}$$

commute with the Hamiltonian. The vector space of states is, therefore, a countable union of disjoint subspaces labeled by the eigenvalues of $\mathscr{N}_1$ and $\mathscr{N}_2$.

We denote the eigenstates of the free Hamiltonian [the bilinear operator part of (2.1)] by $|\ \rangle$ whereas the eigenstates of the complete Hamiltonian are denoted by $|\ \rangle\rangle$. The phases of the free Hamiltonian eigenstates are defined by[4]

$$|V^a N\theta(k_1)\cdots\theta(k_b)\rangle \equiv (V^\dagger)^a a^\dagger(k_1)\cdots a^\dagger(k_b) N^\dagger|0\rangle, \tag{2.5a}$$

and

$$|V^a\theta(k_1)\cdots\theta(k_b)\rangle \equiv (V^\dagger)^a a^\dagger(k_1)\cdots a^\dagger(k_b)|0\rangle. \tag{2.5b}$$

For simplicity of notation, we shall label the exact eigenstates by their eigenvalues λ, suppressing all other labels that specify the state.

In the $\mathscr{N}_1 = a$, $\mathscr{N}_2 = b$ subspace (or, for short, the $V^a - \theta^b$ sector) there are just two[5] coupled amplitudes,

$$\psi_\lambda(k_1,\ldots,k_b) \equiv \langle\langle\lambda\,|V^a\theta(k_1),\ldots,\theta(k_b)\rangle \tag{2.6a}$$

and

$$\phi_\lambda(k_1,\ldots,k_{b+1}) \equiv \langle\langle\lambda\,|V^{a-1}N\theta(k_1),\ldots,\theta(k_{b+1})\rangle. \tag{2.6b}$$

The equations of motion satisfied by these can be readily obtained from the Hamiltonian. We find,

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$$(\lambda - am_0 - k_1 \cdots - k_b)\psi_\lambda(k_1,k_2,\ldots,k_b)$$
$$= (-1)^b a \int d^3 l f(l) \phi_\lambda(l,k_1,\ldots,k_b), \tag{2.7a}$$

and

$$[\lambda - (a-1)m_0 - k_1,\ldots,k_{b+1}]\phi_\lambda(k_1,\ldots,k_{b+1})$$
$$= (-1)^{b+1}[-f(k_1)\psi_\lambda(k_2,\ldots,k_{b+1})$$
$$+ f(k_2)\psi_\lambda(k_1,k_3,\ldots,k_{b+1})$$
$$+ \cdots (-)^i f(k_i)\psi_\lambda(k_1,k_2,\ldots,k_{i-1},k_{i+1},\ldots,k_{b+1}) + \cdots]. \tag{2.7b}$$

It is easily seen that for $a = b = 1$ these reduce to the equations of motion in the V-θ sector of the fermionic Lee model as given in Ref. 3.

Our purpose is to find those values of λ for which Eqs. (2.7) yield nontrivial solutions. To this end, we construct[6] the states $V^\dagger N|\lambda\rangle\rangle$ and $a^\dagger(k)|\lambda\rangle\rangle$ which reside in the subspace characterized by $(\mathcal{N}_1, \mathcal{N}_2 + 1)$ if the state $|\lambda\rangle\rangle$ was in the subspace labeled by $(\mathcal{N}_1, \mathcal{N}_2)$. By considering the action of the Hamiltonian on these new states, we are able to relate the spectral values in the $(\mathcal{N}_1, \mathcal{N}_2)$ subspace with those in the $(\mathcal{N}_1, \mathcal{N}_2 + 1)$ subspace. This forms the subject of the next section.

3. THE RELATIONSHIP BETWEEN THE SPECTRA IN THE $(\mathcal{N}_1, \mathcal{N}_2)$ AND $(\mathcal{N}_1, \mathcal{N}_2 + 1)$ SUBSPACES

As discussed at the conclusion of the previous section, we proceed by considering an eigenstate $|\lambda\rangle\rangle$ of the Hamiltonian in the particular subspace with $(\mathcal{N}_1, \mathcal{N}_2) = (a,b)$. It is clear that the states $a^\dagger(p)|\lambda\rangle\rangle$ and $V^\dagger N|\lambda\rangle\rangle$ belong to the subspace with $(\mathcal{N}_1, \mathcal{N}_2) = (a, b+1)$. The action of the Hamiltonian on these states can be readily calculated using Eqs. (2.1) and (2.2) to be

$$(H - \lambda - p)a^\dagger(p)|\lambda\rangle\rangle = f(p)V^\dagger N|\lambda\rangle\rangle \tag{3.1a}$$

and[7]

$$(H - \lambda - m_0)V^\dagger N|\lambda\rangle\rangle = \int d^3k f(k) a^\dagger(k)\mathcal{N}_1|\lambda\rangle\rangle$$
$$= a\int d^3k f(k) a^\dagger(k)|\lambda\rangle\rangle. \tag{3.2b}$$

If $|\mu\rangle\rangle$ is an eigenstate of the Hamiltonian in the $(a, b+1)$ sector, we easily obtain the equations of motion for the amplitudes

$$\sqrt{a}\langle\langle\mu|a^\dagger(p)|\lambda\rangle\rangle \equiv \eta_{\lambda\mu}(p) \tag{3.3a}$$

and

$$\langle\langle\mu|V^\dagger N|\lambda\rangle\rangle \equiv \sigma_{\lambda\mu}, \tag{3.3b}$$

to be

$$(\mu - \lambda - p)\eta_{\lambda\mu}(p) = \sqrt{a} f(p)\sigma_{\lambda\mu} \tag{3.4a}$$

and

$$(\mu - \lambda - m_0)\sigma_{\lambda\mu} = \sqrt{a}\int d^3k f(k)\eta_{\lambda\mu}(k). \tag{3.4b}$$

These are formally identical to the equations of motion in the lowest noninteracting sector of the Lee model,[2] with a coupling constant enhanced by a factor $\sqrt{a}$. It easily follows from Eqs. (3.4) that

$$\alpha_a(\mu - \lambda)\sigma_{\lambda\mu} = \sqrt{a}\int d^3k f(k)\delta(\mu - \lambda - k)$$
$$\equiv \sqrt{a}\tilde{f}(\mu - \lambda), \tag{3.5}$$

where

$$\alpha_a(z) \equiv z - m_0 - a\int d^3k \frac{f^2(k)}{z - k}. \tag{3.6}$$

It follows immediately that for μ to be in the spectrum of the Hamiltonian, σ_H, we must have either

(i) $\mu = \lambda + k$, $k > 0$, corresponding to the "scattering" solution[8] or

(ii) $\alpha_a(\mu - \lambda) = 0$, corresponding to the "discrete" solution.[8]

Before proceeding to analyze these further, we list some properties of the function α_a, regarded as a function of the complex variable z.

(i) $\alpha_a(z)$ is analytic in the z plane except for a cut along the positive real axis.

(ii) $\alpha_a(z)$ has no complex zeros.

(iii) For real, negative values of x, $\alpha'_a(x) > 0$ and $\alpha_a(-\infty) < 0$.

(iv) Again, for $x < 0$, $\alpha_a(x) - \alpha_1(x) \geqslant 0$.

In particular, if $\alpha_1(0^-) > 0$, $\alpha_a(0^-) > 0$ $\forall a$. This ensures a zero of $\alpha_a(x)$ if $\alpha_1(x)$ has a zero. The monotonicity of α_a ensures that the zero, if it exists, is unique. In this paper, we will assume that in the lowest nontrivial sector there is a discrete point in the spectrum at M_1, i.e., $\alpha_1(M_1) = 0$, $M_1 < 0$. It then immediately follows that for all a there exists M_a such that $\alpha_a(M_a) = 0$, with $M_a > M_b$ iff $a < b$.

We have thus shown that if $\lambda \in \sigma_H$, μ is not a solution to Eqs. (3.4) unless

$$\mu = \lambda + M_a, \tag{3.7a}$$

or

$$\mu = \lambda + k, \quad k > 0. \tag{3.7b}$$

In other words, $\mu \notin \sigma_H$ unless it is of the form (3.7). We will see in Sec. 5 that not all values of μ of the type (3.7) are in the spectrum. Before proceeding to do so, we first find the spectrum in the $(\mathcal{N}_1, \mathcal{N}_2) = (a,0)$ sector of the model. The remainder of the spectrum can be obtained using Eq. (3.7). We proceed to do so in the next section.

4. THE $(\mathcal{N}_1, \mathcal{N}_2) = (a,0)$ SUBSPACE

In the V^a sector of the model, the equations of motion (2.7) reduce to

$$(\lambda - am_0)\psi_\lambda = a\int d^3l f(l)\phi_\lambda(l), \tag{4.1a}$$

and

$$[\lambda - (a-1)m_0 - k]\phi_\lambda(k) = f(k)\psi_\lambda. \tag{4.1b}$$

Once again, carrying out the same manipulations that led to Eq. (3.5), we find that the spectrum consists of one discrete state with $\lambda = (a-1)m_0 + M_a$ and a continuum of scattering states with $\lambda = (a-1)m_0 + p$, $p \geqslant 0$. The corresponding (unnormalized) state vectors can easily be calculated. We find,

$$|B^{(a)}\rangle\rangle = |V^a\rangle + a \int \frac{d^3k f(k)}{M_a - k} |V^{a-1}N\theta(k)\rangle, \quad (4.2a)$$

for the discrete state and

$$|\lambda\rangle\rangle = \frac{\tilde{f}(\xi)}{\alpha_a^+(\xi)} |V^a\rangle + \int d^3k \left[\delta(\xi - k) + \frac{af(k)\tilde{f}(\xi)}{\xi - k + i\epsilon}\right] |V^{a-1}N\theta(k)\rangle, \quad (4.2b)$$

with $\xi = \lambda - (a-1)m_0$ for the continuum states. Here $\alpha_a^{\pm}(\xi) + \alpha_a(\xi \pm i\epsilon)$. The spectrum in this subspace, therefore, consists of

(i) a discrete point at $\lambda = (a-1)m_0 + M_a$

and

(ii) a continuum starting at $(a-1)m_0$ and extending to infinity.

It is interesting to notice that the discrete point in the spectrum occurs at $\lambda = (a-1)m_0 + M_a$. $(a-1)$ bare V particles act as mere spectators whereas only one of them is bound by the interaction, with an effective strength increased by a factor $\sqrt{a}$. This can also be seen from the wavefunction, since Eq. (4.2a) can be rewritten as[9]

$$|B^{(a)}\rangle\rangle = |V^{a-1}\rangle \otimes \left[\sqrt{a}\, |V\rangle + a \int d^3k \frac{f(k)}{M_a - k} |N\theta(k)\rangle\right]. \quad (4.3)$$

The presence of the other V particles merely enhances the effective coupling constant by the abovementioned factor.

This phenomenon of "limited interaction" can be naively understood if we recognize that the bare V particles can interact only in the presence of N particles. For more than one V particle to directly interact, more than one N particle would have to be present, which is not possible because of the fermionic nature of N coupled to the no-recoil structure of the Hamiltonian.

The spectrum in the subspace $(\mathcal{N}_1, \mathcal{N}_2) = (a,1)$ can be obtained from that in the $(a,0)$ subspace using Eqs. (3.7). In fact by repeating the procedure b times, we can obtain the spectrum in any arbitrary subspace. It is obvious that this procedure would lead to, among other things, eigenvalues of the form cM_a or "cM_a + continuum" with $c \neq 1$ in contradiction[10] with the considerations of the previous paragraph. The elimination of these spurious eigenvalues forms the subject of the next section.

5. THE ELIMINATION OF SPURIOUS SOLUTIONS

As we have pointed out in the last section, and also in our earlier papers,[3,6] it is the solution to the equations of motion for the Källén–Pauli amplitudes that yield nontrivial eigenvectors of the Hamiltonian. It is quite possible[3,6] to have nontrivial solutions to equations of motion for any auxiliary set of amplitudes which lead to trivial solutions for the Källén–Pauli system. It is, therefore, essential to check whether the solutions to Eqs. (3.4) really correspond to genuine eigenstates of the system. We proceed as follows.

For any eigenstate $|\lambda\rangle\rangle$ in the $\mathcal{N}_1 = a$ subspace, we define the state $|\mu\rangle\rangle$ by

$$|\mu\rangle\rangle = \int d^3p\, \eta_{\lambda\mu}(p) a^\dagger(p) |\lambda\rangle\rangle + \frac{\sigma_{\lambda\mu}}{\sqrt{a}} V^\dagger N |\lambda\rangle\rangle. \quad (5.1)$$

It is easily verified that

$$H|\mu\rangle\rangle = \mu|\mu\rangle\rangle, \quad (5.2)$$

when Eqs. (3.4) are satisfied. $|\mu\rangle\rangle$ is thus a new eigenstate. There are two types of eigenstates $|\mu\rangle\rangle$ that can be obtained from $|\lambda\rangle\rangle$ [See Eqs. (3.7)]. These are

(i) the "scattering" type with $\mu = \lambda + k$

and

(ii) the "discrete" type with $\mu = \lambda + M_a$.

The corresponding solutions are

$$\sigma_{\lambda\mu} = \sqrt{Z_a},$$

$$\eta_{\lambda\mu}(p) = \frac{\sqrt{a} f(p) \sqrt{Z_a}}{M_a - p}, \quad (5.3)$$

with

$$Z_a = \frac{1}{\alpha_a'(M_a)},$$

for the discrete type, and

$$\sigma_{\lambda\mu} = \frac{\sqrt{a} f(k)}{\alpha^+(k)},$$

$$\eta_{\lambda\mu}(p) = \delta(k-p) + \frac{\sqrt{a} f(p) \sqrt{Z_a}\, f(k)}{\alpha^+(k)[k - p + i\epsilon]}, \quad (5.4)$$

where

$$\mu = \lambda + k,$$

for the continuum type.

Following the same procedure as was used to obtain $|\mu\rangle\rangle$ from $|\lambda\rangle\rangle$, we now proceed to obtain the state $|\nu\rangle\rangle$ from $|\mu\rangle\rangle$. There are then three possibilities for ν, viz. $\nu = \lambda + 2M_a$, $\nu = \lambda + M_a + l$ and $\nu = \lambda + k + l$. We have calculated the state vector $|\nu\rangle\rangle$ in terms of the vector $|\lambda\rangle\rangle$ for the three cases. We find that for $\nu = \lambda + 2M_a$ the state vector vanishes on account of our choice statistics for the particles. This is completely in keeping with our earlier observation that "$2M_a$" could only arise due to a simultaneous interaction of two V^s which we had rule out earlier. For the state with $\nu = \lambda + k + M_a$, we find

$$
\begin{aligned}
|\nu\rangle\rangle = {} & \sqrt{a}\sqrt{Z_a}\int d^3q\,\frac{f(q)}{M_a-q}\,a^\dagger(q)a^\dagger(k)|\lambda\rangle\rangle \\
& + a^{3/2}\,\frac{f(k)}{\alpha^+(k)}\sqrt{Z_a}\int d^3p\,d^3q\,\frac{f(p)f(q)}{(M_a-q)(k-p+i\epsilon)}\,a^\dagger(q)a^\dagger(p)|\lambda\rangle\rangle \\
& + \sqrt{a}\sqrt{Z_a}\,\frac{f(k)}{\alpha^+(k)}\int d^3q\left[\frac{f(q)}{M_a-q}-\frac{f(q)}{k-q+i\epsilon}\right]a^\dagger(q)V^\dagger N|\lambda\rangle\rangle \\
& + \frac{\sqrt{Z_a}}{\sqrt{a}}\,V^\dagger N a^\dagger(k)|\lambda\rangle\rangle.
\end{aligned}
\tag{5.5}
$$

The same state is obtained whether we choose $\mu=\lambda+M_a$, $\nu=\mu+k$ or $\mu=\lambda+k$, $\nu=\mu+M_a$. Finally, for the state with $\nu=\lambda+k+l$, we have

$$
\begin{aligned}
|\nu\rangle\rangle = {} & a^\dagger(l)a^\dagger(k)|\lambda\rangle\rangle + \left[\frac{af(l)}{\alpha^+(l)}\int d^3q\,\frac{f(q)}{l-q+i\epsilon}\,a^\dagger(q)a^\dagger(k)|\lambda\rangle\rangle-(k\leftrightarrow l)\right] \\
& + \frac{a^2f(k)f(l)}{\alpha^+(k)\alpha^+(l)}\int d^3p\,d^3q\,\frac{f(p)f(q)}{(l-q+i\epsilon)(k-p+i\epsilon)}\,a^\dagger(q)a^\dagger(p)|\lambda\rangle\rangle \\
& + V^\dagger\left[\frac{f(k)a^\dagger(l)}{\alpha^+(k)}-(k\leftrightarrow l)\right]N|\lambda\rangle\rangle \\
& + \frac{af(k)f(l)}{\alpha^+(k)\alpha^+(l)}\int d^3q\left[\frac{f(q)}{l-q+i\epsilon}-(k\leftrightarrow l)\right]a^\dagger(q)V^\dagger N|\lambda\rangle\rangle.
\end{aligned}
\tag{5.6}
$$

It is seen that $|\nu\rangle\rangle$ is antisymmetric under the change $(k\leftrightarrow l)$ as it should be.

We have thus seen that if $\lambda\in\sigma_H$, $\mu\in\sigma_H$ iff

$$\mu=\lambda+k,\quad k>0, \tag{5.7a}$$

or

$$\mu=\lambda+M_a, \tag{5.7b}$$

provided $\lambda\neq E+M_a$, $E\in\sigma_H$.

Since the spectrum in the $(\mathcal{N}_1,\mathcal{N}_2)=(a,0)$ sector is exactly known, and since any state in the $(\mathcal{N}_1,\mathcal{N}_2)=(a,b)$ sector can be reached by a sufficient number of applications of the operators $a^\dagger(p)$ and $V^\dagger(N)$. The spectrum in the $\mathcal{N}_1=a$ subspace thus consists of the points M_a, k, M_a+k, $M_a+k+l,\cdots$, where $k,l,\cdots,\geqslant 0$.

6. CONCLUDING REMARKS

In all renormalizable quantum field theories in three space dimensions, the scattering of fermions occurs via an exchange of a boson. In this paper, we have studied a considerably simplified version of one such theory and obtained the exact spectrum of the Hamiltonian. Furthermore, we have, in principle, obtained all the Hamiltonian eigenstates. The exact S matrix for this theory is, therefore, calculable. Although some features of our calculation were peculiar to the particular model (such as the "mass nonrenormalization" of a bare V particle in the presence of other V particles), all the results we obtained were in keeping with our intuitive expectations. Although all our considerations have been confined to a non-relativistic framework, it is hoped that some of the results obtained here may serve to elucidate some nonperturbative features of more realistic quantum field theories.

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[1]A. Zamolodchikov and A. Zamolodchikov, Nucl. Phys. **B 133**, 525 (1978) and references therein; R. Shankar and E. Witten, Nucl. Phys. B **141**, 349 (1978); for a review, see M. Karowski, Phys. Rep. **49**, 229 (1979). See also references therein.

[2]T. D. Lee, Phys. Rev. **95**, 1329 (1954); K. D. Friedrichs, Commun. Pure Appl. Math. **1**, 361 (1948); G. Källén and W. Pauli, K. Dan. Vidensk. Selsk. **30**, No. 7 (1955).

[3]X. Tata, E. C. G. Sudarshan, C. B. Chiu, and C. C. Chiang, Phys. Rev. D (in press).

[4]There can be at most one bare N particle since N is fermionic and does not carry momentum.

[5]This is so because the N content of the bare states is bounded by one. The corresponding situation for the Bosonic Lee model is much more complicated. For details of the solution in the two lowest sectors, see Ref. 6.

[6]M. Bolsterli, Phys. Rev. **166**, 1760 (1968); C. A. Nelson, J. Math. Phys. **13**, 1051 (1972); C. C. Chiang, C. B. Chiu, E. C. G. Sudarshan, and X. Tata, Phys. Rev. D (in press).

[7]For the case of the Lee model with all the fields quantized as bosons, the right-hand side of Eq. (3.2b) would read $\int d^2k f(k)a^\dagger(k)(N^\dagger N - V^\dagger V)|\lambda\rangle\rangle$. The reason why the spectrum of the fermionic Lee model is soluble is that an eigen-operator appears in place of $N^\dagger N - V^\dagger V$.

[8]The terms "scattering" solution and "discrete" solution are used in a loose sense. We mean the solutions corresponding to these eigenvalues would be the ones corresponding to the scattering and discrete solutions in the lowest nontrivial sector of the model.

[9]The factor $\sqrt{a}$ appears in front of $|V\rangle$ in the square bracket since $\langle V^a|V^a\rangle = a\|\,|V^{a-1}\rangle\otimes|V\rangle\|^2$.

[10]The fact that such spurious eigenvalues occur is not surprising in view of the considerations of Refs. 3 and 6. There, we had found solutions to equations of motion for the overcomplete auxiliary amplitudes for which there was no corresponding solution to the Källén–Pauli equations of motion. For the full dynamical content of the Lee model to be realized, the equations of motion for the overcomplete amplitudes had to be solved subject to the constraints discussed in the abovementioned papers. Equations (3.4) are akin to the overcomplete amplitude equations and hence it is not surprising to find solutions to these which do not correspond to any eigenstates of the system.