# Unified geometrical approach to relativistic particle dynamics

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Models for systems of relativistic particle dynamics are reviewed in terms of a geometrical setting for constraint dynamics. They are derived from the same grand abstract space by means of a common reduction procedure and are put in correspondence with invariant subgroups of the Poincaré group. A new model corresponding to the identity subgroup is also discussed.

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## I. INTRODUCTION: ON THE DESCRIPTION OF BECOMING

Dynamics is the expression of flow by stringing together sequences of configurations together each labelled by a time evolution parameter according to an explicit rule. The collections of configurations so strung together in a well-ordered sequence constitute trajectories of the system, and each trajectory has certain configurational functionals characterizing them. These would be the constants of motion. In this account the configurations are the conventional coordinate space together with the velocity fibers: whatever constitutes the initial specification to make use of Newton's formulation of the equations of motion.

When such ideas are to be implemented for a relativistic system, we do encounter some new problems. Traditionally, we consider clock time as the time evolution parameter, and a configuration is defined by considering simulataneous specification of coordinates and velocities. In relativistic theory this poses a problem since distant simultaniety is not relativistically invariant. If we insist, nevertheless, on using clock time and a canonical formalism, the no-interaction theorem tells us that the only relativistically invariant descriptions could be for noninteracting systems only. We must therefore be prepared to consider other alternatives.

A satisfactory alternative is to consider a time evolution parameter defined dynamically rather than kinematically. Dynamical evolution is with respect to a temporal parameter that has different significance in different states of motion. The dynamical evolution is self-referring and "the time" is independent of the external reference frames.

It turns out that the temporal parameter so defined, being Lorentz-invariant, must have a generator of dynamical evolution which is also Lorentz-invariant, and is different from any of the ten generators of the Poincaré group. In this 11 parameter generator formalism it has been found possible to construct interacting relativistic systems with invariant world lines.

The natural mechanism for bringing about such a description is to make use of the Dirac constraint formalism starting with a system with excess degrees of freedom and systematically reducing them by imposing constraints. Among those constraints we include one which explicitly depends on a parameter $\tau$, which then gets identified with being the evolution parameter. We have thus the curious situation in which motion is generated by constraints.

In the recent literature there have been a number of such models constructed; they are of three kinds depending upon how the initial configuration and phase spaces are chosen. Each such group made use of a primary set of dynamical variables and a set of constraints. In the first kind of models each individual particle is described by four pairs of canonical variables. A system of $2N$ constraints are then imposed to produce $3N$ pairs of canonical variables and an evolution parameter to describe $N$ particles in motion. In the second kind of model a pair of 4-vectors represent spacetime specification of a uniformly moving "center" of the system and the total 4-momentum of the system, respectively. The constraints then relate these quantities to the particle configurations. In the third kind of model the new collective variables introduced are a Lorentz matrix and its canonical conjugate carrying the burden of the inertial frame. Constraints can then be used to obtain interacting relativistic particles describing world lines.

Each of these kinds of models has its own number of starting variables and judiciously chosen constraints. It would be desirable to have a systematic method of dealing with all three models and to see if there are other possibilities of a similar kind.

The present paper is devoted to this task. We start with grand abstract configuration space $\widetilde{\Sigma}$ consisting of the semidirect product of the Lorentz group with the product of $N$ 4-vectors. This configuration space thus has $4N + 10$ dimensions. The phase space has twice this dimension. We then take an invariant subgroup $G$ of the Poincaré group $P$ and take the equivalence classes.

$$\Sigma = \widetilde{\Sigma}/G$$

as the configuration space of a model. It turns out that by

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choosing $G$ to be $P$ itself, the Lorentz subgroup $\alpha$, and the translation subgroup $T^4$, respectively, we get the three kinds of models mentioned above. By choosing the identity subgroup of $P$ we are able to generate another kind of model.

Much of our previous work as well as that of other authors are stated in traditional language of canonical mechanics. For making the ideas accessible to a wider group of people to whom modern differential geometry is a standard tool as well as to expose the essential geometric aspects of the developments, we have carried out our formulation in the language of differential geometry.

The plan of the paper is as follows: Sec. II recapitulates the essential background to establish notation and provide the setting. The world line condition is formulated in its general form in Sec. III. The grand configuration space is introduced in Sec. IV along with the equivalence classes which realize the four kinds of formalisms. In Sec. V we construct the phase spaces and the choice of constraints to build up a suitable family of sections of the fiber bundle for each of the models. Some remarks in Sec. VI conclude the paper.

## II. A GEOMETRICAL SETTING FOR CONSTRAINT DYNAMICS

In dealing with constraint dynamics, the situation we are presented with is the following.

On a given $2n$-dimensional manifold[1] $\Gamma = T^*\Sigma$ a set of real functions $K_1,...,K_k$ is given. By choosing a value for each one of them a hypersurface $M$ in $\Gamma$ is determined. We consider the smooth map

$$\kappa\colon\ \Gamma \to \mathbb{R}^k,$$

$$\gamma \to (K_1(\gamma),...,K_k(\gamma)),$$

and by fixing a value, say $0 \in \mathbb{R}^k$, we get

$$M = \kappa^{-1}(0) = \{\gamma \in \Gamma\colon K_1(\gamma) = \cdots = K_k(\gamma) = 0\}.$$

We assume $M$ to be a submanifold of $\Gamma$, of codimension $k$. If $0 \in \mathbb{R}^k$ is a regular value for $\kappa$, then $M$ is a submanifold.

By means of the symplectic structure $\omega$ on $\Gamma$ we can define Poisson brackets and associate vector fields with functions. The vector field $X_f$ associated with the function $f$ is defined by the relation

$$L_{X_f} g = \{f, g\}$$

for any function $g$. An equivalent definition is given by

$$i_{X_f}\omega = df$$

if $\omega$ is the symplectic form of $\Gamma$.

A set of vector fields $X_1,...,X_r$ spans a tangent subspace for each point of $\Gamma$ on considering span $\{X_1(\gamma),...,X_r(\gamma)\}$. Such spaces will constitute the tangent space of a submanifold if and only if the relations.

$$[X_i, X_j] = c_{ij}^m X_m \tag{2.1}$$

are satisfied, with the $c_{ij}^m$ being functions on $\Gamma$. This is the Frobenius theorem.

A vector field $X$ can be evaluated at points of $M$. If it turns out that $X(m)$ is tangent to $M$ for any $m \in M$, we will say that $X$ is tangent to $M$.

With the above set of functions we will associate the vector fields $X_{K_j}$ and inquire about the relation (2.1). It is simple to prove that they satisfy the condition of the Frobenius theorem if and only if the following relations hold:

$$d\{K_i, K_j\} = c_{ij}^m\, dK_m.$$

The $c_{ij}^m$ will then be functions of the $K_i$. We say in this case that the $K_i$ form a function group. Such a situation leads to a foliation on $\Gamma$ and the relevant analysis has been carried out in Ref. 2, to which we will refer extensively in what follows.

Here we do not require the $K_i$ to form a function group; nevertheless, we shall show how, starting with the vector fields $X_{K_i}$ restricted to $M$, we can generate a set of vector fields tangent to $M$ and satisfying the condition for the Frobenius theorem.

If

$$i\colon M \to \Gamma$$

is the identification map, we can consider the 2-form $i^*\omega$ on $M$, which is the pullback of $\omega$ by $i$. In general, $i^*\omega$ is degenerate. If its rank is constant the vector fields on $M$ annihilated by it constitute an involutive distribution $\mathscr{D}$, i.e., they obey the Frobenius theorem. We will prove that they are combinations (with coefficients functions on $M$) of the $X_{K_i}$ evaluated on $M$. (Notice that in general the $X_{K_i}$ are not tangent to $M$.) They will be denoted by $Y$, and the hypothesis is that they satisfy

$$i_Y(i^*\omega) = 0.$$

This implies that

$$(i_Y\omega)_{|M} = 0$$

and therefore one can write

$$i_Y\omega = c_i\, dK_i \quad \text{(summed on } i\text{)}$$

or

$$Y = c_i\, X_{K_i}$$

with the $c_i$ being functions on $M$. (Here there is an abuse of notation, as $Y$ is actually a vector field on $M$, but we do consider it as a vector field on $\Gamma$.)

Such an expression for $Y$ implies

$$c_i\{K_i, K_j\} = 0 \quad \text{on } M \text{ for any } j = 1,...,k.$$

When a relation involving Poisson brackets is true only when evaluated on $M$, it is customary to replace the equality sign $=$ with the sign $\approx$ and it is said to be true in a weak sense. Thus our relations can be written as

$$c_i\{K_i, K_j\} \approx 0 \quad \text{for any } j = 1,...,k. \tag{2.2}$$

It is useful to define the antisymmetric matrix $A$:

$$A_{ij} = \{K_i, K_j\} \tag{2.3}$$

related to $i^*\omega$ by

$$\text{rank}\, A(m) = \text{rank}(i^*\omega)(m), \quad m \in M.$$

The set of $(c_i)$ can now be considered as nullvectors of $A_{|M}$ and the number of independent nonvanishing vector fields satisfying (2.2) turns out to be

$$d = \text{codim}\, M - \text{rank}\, A_{|M}.$$

If rank $A_{|M}$ is to be a constant on $M$, the vector fields $Y$ define an involutive distribution $\mathscr{D}$ on $M$ with the above dimension. This allows us to foliate $M$ and to consider

$$\mathscr{N} = M/\mathscr{D}.$$

In physics it is customary to assume $\mathcal{N}$ to be a manifold having the property that

$$\pi: M \to \mathcal{N}$$

is a submersion. It can be proved that $\mathcal{N}$ inherits a symplectic structure $\rho$, which allows us to call it the "reduced phase space" or "the frozen phase space."[3]

But so far no dynamics has been defined at all. This is done by introducing a one-parameter family of sections

$$\mathcal{N} \times \mathbb{R} \xrightarrow{\sigma} M.$$

From a global point of view this assumes that a section for $M \xrightarrow{\pi} \mathcal{N}$ does exist. (If the vector fields $Y$ integrate to a Lie group $\mathcal{G}$, such that the leaves of the submersion $\pi: M \to \mathcal{N}$ are diffeomorphic to $\mathcal{G}$, the existence of such a section requires the $\mathcal{G}$-bundle to be trivial.) It is on $\sigma(\mathcal{N} \times \mathbb{R}) \subset M$ that dynamics will be defined, not on $M$ itself. The leaves of $\pi$ are $d$-dimensional, and it turns out that $k + d$ is an even number. Therefore,

$$\dim \mathcal{N} = 2n - (k + d)$$

is even, and

$$\dim[\sigma(\mathcal{N} \times \mathbb{R}) \subset M] = 2n - (k + d) + 1, \quad d > 0.$$

Of course, if $d = 0$, then $\mathcal{N} = M$, $\dim \sigma(\mathcal{N} \times \mathbb{R}) = 2n - k$, and our procedure generates a dynamics (the trivial one), i.e., a one-parameter group of transformations on $M$, which is independent of $K_i$. But in general this is not the case and the set of $K_i$ has a further role. All possible dynamics that can be defined in such a fashion, corresponding to different choices of $\sigma$, have the property that the manifolds of states of motion are all diffeomorphic among themselves.

If $Y_1, Y_2, \ldots, Y_d$ are a basis of vector fields which span $\ker i^*$ each dynamical vector field $\Delta$ can be expressed as

$$\Delta = \alpha^i Y_i$$

with $\alpha^i$ functions on $M$. All this is restricted to the submanifold $\sigma(\mathcal{N} \times \mathbb{R}) \subset M$. This vector field $\Delta$ is tangent to the submanifold.

But another way to build up dynamics and the appropriate submanifold is commonly used in dealing with constraint dynamics. Besides the $K_i$ functions, another set of $d$ real functions $X_1, \ldots, X_d$ is chosen to constitute the smooth map

$$\mathbb{X}: \Gamma \times \mathbb{R} \to \mathbb{R}^d,$$
$$(\gamma, \tau) \to X^\tau(\gamma).$$

The requirement on the $X$ is that they are functionally independent and together with the $K_i$ define for each value of the parameter $\tau$ a $[2n - (k + d)]$-dimensional surface in $\Gamma$ on which $\omega$ turns out to be nondegenerate. To put it differently, the equations

$$c_m \{ \xi_m, \xi_n \} \approx 0$$
$$(m, n = 1, \ldots, k + d) \quad \text{(summed on } m)$$

(where $\xi_m$ stands for $K_1, \ldots, K_k; X_1, \ldots, X_d$) do not have nontrivial solutions. Then for each $\tau \in \mathbb{R}$ the surface generated by

$$\mathbb{K} \times \mathbb{X}^\tau: \Gamma \to \mathbb{R}^{d+k}$$

by taking the inverse image of $0 \in \mathbb{R}^{d+k}$ is of dimension $2n - (k + d)$. In this way one recovers what was earlier called $\sigma(\mathcal{N} \times \mathbb{R})$, as will be seen in the next section.

From the previous discussion it is clear that different $X_i$ define different dynamical systems even if all of them have diffeomorphic spaces of trajectories. Their carrier spaces may be different.

In many physical situations, the starting space $\Gamma$ carries a symplectic action $\overline{\mathscr{R}}$ of some Lie group $G$, i.e., $G$ acts on $\Gamma$ via canonical transformations. We ask ourselves what happens to such an action with respect to the constraint surface $M$. It is obvious that only that part of $G$ which maps $M$ onto itself is relevant as far as dynamics is concerned. If all the infinitesimal generators $X^G$ for $\overline{\mathscr{R}}$ happen to satisfy the relations

$$(i_{X^G}\, dK_i)_{|M} = 0 \quad (i = 1, \ldots, k)$$

then the action carries over to the manifold $M$. Furthermore, as the action of $G$ on $M$ preserves $i^*\omega$, it happens that $\mathcal{N}$ also will carry a $G$-action, $\overline{\mathscr{R}}$, which is symplectic with respect to the symplectic structure $\rho$. This statement follows from the fact that the vector fields $\widetilde{Y}$ defined by

$$i_{\widetilde{Y}}\omega = d(c_i K_i)$$

when restricted to $M$ coincide with

$$Y = c_i X_{K^i}.$$

Since $(\overline{\mathscr{R}})^*\omega = \omega$ and $M$ is invariant under $\overline{\mathscr{R}}$, we have also

$$(\overline{\mathscr{R}})^*\mathscr{D} = \mathscr{D}.$$

In fact $(\overline{\mathscr{R}})^*(i_X\omega) = i_{\overline{\mathscr{R}}_* }\omega$, if $X$ is a vector field on $\Gamma$.[4]

As we have already said, a dynamics is specified only after we have a section

$$\sigma: \mathcal{N} \times \mathbb{R} \to M$$

and it will be a dynamics on $\sigma(\mathcal{N} \times \mathbb{R})$. The submanifold $\sigma(\mathcal{N} \times \{0\}) \subset \sigma(\mathcal{N} \times \mathbb{R})$ can be thought of as the set of all possible Cauchy data for our dynamics. Furthermore, the projected action of $G$ on $\mathcal{N}$ gives an action of $G$ on $\sigma(\mathcal{N} \times \{0\})$ by setting

$$\mathscr{R}^*(g)\sigma(n, 0) = \sigma(\overline{\mathscr{R}}(g)n, 0), \quad n \in \mathcal{N}, \; g \in G.$$

This can be extended to $\sigma(\mathcal{N} \times \mathbb{R})$ by the relation

$$\mathscr{R}^*(g)\sigma(n, \tau) = \sigma(\overline{\mathscr{R}}(g)n, \tau).$$

It is obvious that $\mathscr{R}^*$ is equivariant with respect to the projection $\pi: M \to \mathcal{N}$ restricted to $\sigma(\mathcal{N} \times \mathbb{R}) \mapsto \mathcal{N}$. It is also clear that it depends on the section $\sigma: \mathcal{N} \times \mathbb{R} \to M$. Moreover, it is canonical with respect to the Poisson brackets on $\sigma(\mathcal{N} \times \mathbb{R})$ defined by the symplectic form $\pi_\tau^* \rho$ the pullback of the symplectic form $\rho$ on $\mathcal{N}$ by the map $\pi_\tau: \sigma(\mathcal{N} \cdot \{\tau\}) \mapsto \mathcal{N}$. This coincides with the usual action generated by Dirac brackets defined on all $\Gamma$ and restricted to $\sigma(\mathcal{N} \times \{\tau\})$.

But, to connect all this with the evolution of physical objects, it will be necessary to properly define the physical variables, namely positions and momenta in spacetime. In the following sections, maps $\phi_a$ and $\psi_a$ will be introduced, respectively, for the position and momentum 4-vectors of the $a$th particle. As the group $G$ involved will be the Poincaré group, it will have the usual action on them. We will denote it by $\mathscr{R}_{\rm reg}$.

We remark that as both dynamics and states of motion are given by the choice of a section $\sigma$, it is the above action $\mathscr{R}^*$ of the Poincaré group that is the physically relevant one.

In the following sections we are going to apply the above procedure to some specific models.

In some of the models the starting functions $K$ satisfy the relations

$$\{K_i, K_j\} = c_{ij}^m K_m \quad (i,j,m = 1,...,k),$$

i.e.,

$$\{K_i, K_j\} \approx 0.$$

They are then said to form a first class set of constraints. The additional functions $\chi$, meeting the previously stated requirements, are said to form, together with the $K$, a second class set of constraints. We have

$$\text{rank } A_{|M} = 0, \quad d = k,$$

and the determinant of the matrix

$$B_{m,n} = \{\xi_m, \xi_n\}_{|M}$$

reduces to $(\det|\{K_i, \chi_j\}|)^2$. The Poisson brackets are evaluated on $(\mathbb{K}\times\mathbb{X})^{-1}(0)$.

In other models the structure of the matrix $B$ allows us to carry out the reduction procedure through intermediate steps. For them $A_{|M}$ is singular and has nonzero rank $r$. A nonsingular submatrix $A'$, of even rank $r$, is then formed by a subset of the $K$, which are a second class system of constraints to begin with, so that Dirac brackets can be computed relative to them only. To have the final set of second class constraints, one adds to the remaining $K$ an equal number of $\chi$ satisfying the requirement

$$\det B \neq 0.$$

## III. WORLD LINE CONDITION

With the space $\mathscr{N}$ we can associate dynamics according to Sec. II. There we have seen that this dynamics is defined on $\sigma(\mathscr{N}\times\mathbb{R})\subset M$, not on $M$ itself. As already stated, in each model a map $\phi_a : \Gamma \to$spacetime will be introduced to denote the position 4-vector of particle $a$. By restricting $\phi_a$ to $\sigma(\mathscr{N}\times\mathbb{R})$, with each trajectory we associate a world line on spacetime. The physical interpretation of such world lines requires that this association has a definite Poincaré-covariant property. It is this requirement that is usually called the world line condition (WLC). The formal statement of this condition is as follows.

The association

$$n\in\mathscr{N}\mapsto\sigma(n,\mathbb{R})$$

defines a line in $\sigma(\mathscr{N}\times\mathbb{R})$ for each $n$. On such a set of lines we had defined a Poincaré group action $\mathscr{R}^*$ by setting

$$\mathscr{R}^*(g)\circ\sigma(n,\mathbb{R}) = \sigma(\widetilde{\mathscr{R}}(g)n,\mathbb{R}), \quad g\in G.$$

We can now state the WLC

$$\phi_a \circ \mathscr{R}^*(g)\circ\sigma(n,\mathbb{R}) = \mathscr{R}_{\text{reg}}(g)\circ\phi_a\circ\sigma(n,\mathbb{R}),$$

where $\mathscr{R}_{\text{reg}}$ is the usual action on the four-dimensional vector space of spacetime positions.

For computations it is convenient to express the WLC in a more explicit way in terms of parametrized lines. Recall the one parameter family of section $\sigma^\tau$, introduced in Sec II. By varying $\tau$, a line on $M$ is described for each $n$. Such a line is in turn projected for each $a$ onto $\mathbb{R}_a^4$ by $\phi_a$, thus yielding the world line of particle $a$:

$$c_a^n: \mathbb{R}\to\mathbb{R}_a^4,$$
$$c_a^n(\tau) = \phi_a\circ\sigma^\tau(n).$$

The WLC becomes in this context the requirement that the actions $\mathscr{R}_{\text{reg}}$ defined on each $\mathbb{R}^4$ and $\widetilde{\mathscr{R}}$ on $\mathscr{N}$ are physically consistent, in the sense that if $n' = \widetilde{\mathscr{R}}(g)n$, then there is a $\tau'$ such that

$$c_a^{n'}(\tau') = \mathscr{R}_{\text{reg}}(g)c_a^n(\tau). \tag{3.1}$$

Here $\tau'$ can depend on $\tau$, $g$, and $a$. This obviously poses conditions on $\sigma^\tau$.

To satisfy the WLC, we construct a section of $\pi: M\to\mathscr{N}$ in terms of the real functions $\chi$ of the previous section, and choose the $\chi$ suitably. We consider the subsets $(\mathbb{X}^\tau)^{-1}(0)\equiv N^\tau\subset\Gamma$. A first requirement is that

$$N^\tau\cap M\neq 0.$$

A second is that $N^\tau_{|M}$ be transversal with respect to the fibers of $\pi: M\to\mathscr{N}$. This condition is satisfied if no vector field exists in $\mathscr{D}$ with a flow tangent to $N^\tau_{|M}$.

While the first demand is met in all cases by requiring that the components of $\mathbb{X}^\tau$ constitute additional constraints not identically vanishing on $M$, the second one needs some elaboration.

Referring to Sec. II, a vector field lying in $\mathscr{D}$ was seen to be $X_{\psi_{|M}}$, with $\psi$ being such that

$$\psi = c_i K_i \tag{3.2}$$

and

$$\{\psi, K_j\}_{|M} = 0, \quad \forall j = 1,...,k. \tag{3.3}$$

Hence

$$L_{X_\psi} K_j = c_i\{K_i, K_j\} = 0. \tag{3.4}$$

We proceed to determine the functions $c_i$. Equation (3.4) can be written as

$$(A\mathbf{c})_{|M} = 0, \tag{3.5}$$

where $\mathbf{c} = (c_1,...,c_k)$ and $A$ is the matrix (2.3). We recall that in all the models

$$\text{rank } A = r < k.$$

This allows us to choose $r$ components of $\mathbb{K}$ in terms of which the submatrix $A'$ of nonzero determinant can be built. They will be denoted $K'_l$ $(l = 1,...,r)$ and the remaining ones $K''_h$ $(h = 1,...,d)$ so that

$$\psi = c'_l K'_l + c''_h K''_h.$$

There are $\infty^d$ solutions of (3.5): the $c''$ can be arbitrarily chosen and the $c'$ are then computed as the unique solution of a linear inhomogeneous system of dimension $r$. A set of independent solutions is obtained by starting with each $K''_h$ in turn. We denote it by, $\psi_h$:

$$\psi_h = K''_h - (A')^{-1}_{ll'}\{K'_l, K''_h\}K'_{l'}.$$

The $\psi_h$ constitute a basis for first class constraints.

Returning now to the transversality condition, this can be formulated as the requirement that the equations

$$(b_h\{\psi_h, \chi_{h'}\}) = 0$$

with $b_h$ real functions on $\Gamma$, have only the trivial solution $b_h = 0$. This is possible iff

$$\det|\{\psi_h,\chi_{h'}\}_{|M}|\neq 0. \qquad (3.6)$$

We note at this point that

$$\{\psi_h,\chi_{h'}\}_{|M} = \{K''_h,\chi_{h'}\}^*_{|M}, \qquad (3.7)$$

the bracket on the right-hand side being the Dirac bracket, relative to the $K'$ only.

When

$$\text{rank}\, A_{|M} = 0,$$

there are no second class constraints (i.e., no $K'$), and Eq. (3.6) reduces to

$$\det|\{K_j,\chi_{j'}\}|\neq 0, \quad j,j' = 1,\ldots,k. \qquad (3.8)$$

In all the schemes considered $\chi_i^\tau$ are chosen so that all but one, say $\chi_d^\tau$, are $\tau$-independent and constitute a Poincaré-invariant set. The $\chi_i$ $(i = 1,\ldots,d-1)$ define a line on each fiber and $\overline{\mathscr{R}}_{|M}$ simply permutes these lines among themselves. Thus the WLC is satisfied because in this action on lines $\overline{\mathscr{R}}_{|M}$ and $\mathscr{R}^*$ agree.

Further imposing $\chi_d^\tau = 0$ then puts a parameter $\tau$ on each line which is not necessarily preserved under the $\overline{\mathscr{R}}_{|M}$ action. However, this leads us to define a value for $\tau'$ in terms of $\tau$, $g$ and other variables such that the WLC in the form (3.1) is satisfied.

## IV. THE CHOICE OF THE VARIABLES

In this section we will discuss the variables used in each model to describe systems of $N$ interacting particles.

The physical positions and momenta, in spacetime, will be denoted by 4-vectors $q_a^\mu$ and $p_a^\mu$ for the $a$th particle $(a = 1,\ldots,N)$. They transform under the action $\mathscr{R}_{\text{reg}}$ of $\mathscr{P}$ defined by

$$\mathscr{R}_{\text{reg}} = (L,b)q_a, = Lq_a + b$$

and

$$\mathscr{R}_{\text{reg}}(L,B)p_a = Lp_a,$$

where $L$ is a $4\times 4$ Lorentz matrix and $b$ a translation 4-vector. Let $\mathscr{L}$ denote the Lorentz group $\{L\}$ and $T^4$ the translation group $\{b\}$.

We start with an abstract space $\widetilde{\Sigma}$, on which proper actions of $\mathscr{P}$ will be defined. We will then show how the various models equipped with such $q_a$ and $p_a$ emerge.

Let us define

$$\widetilde{\Sigma} = \mathscr{P}\times\Sigma_0.$$

$\mathscr{P}$ is the Poincaré group and $\Sigma_0 = \otimes_{a=1,\ldots,N}\mathbb{R}_a^4$. Elements of $\widetilde{\Sigma}$ will be denoted $[(\Lambda,a),(x)]$, in which $(\Lambda,a)\in\mathscr{P}$ and $(x)$ stands for $x_1,\ldots,x_N$, $x_a$ being a vector in $\mathbb{R}_a^4$. The following action of $\mathscr{P}$ is defined:

$$\mathscr{R}^{(1)}(L,b)[(\Lambda,a),(x)] = [(\Lambda,a)(L,b)^{-1},(L,b)(x)],$$

where on the right-hand side the right action on $\mathscr{P}$ is given by group multiplication and the left, on $(x)$, is the $\mathscr{R}_{\text{reg}}$ on each $\mathbb{R}^4$, i.e.,

$$(L,b)x_a = \mathscr{R}_{\text{reg}}(L,b)x_a = Lx_a + b.$$

Endowed with such an action, $\widetilde{\Sigma}$ has the structure of a fiber bundle associated with the trivial principal bundle $\mathscr{P}$. It is therefore possible to consider equivalence classes with respect to $\mathscr{R}^{(1)}$ and obtain distinct spaces

$$\Sigma = \widetilde{\Sigma}/\mathscr{R}^{(1)}(g) \qquad (4.1)$$

corresponding to distinct subgroups $g$ of $\mathscr{P}$,

$$\Gamma = \mathscr{T}^*\Sigma,$$

such that the basic (abstract) variables are taken and the analysis of the previous section starts.

Another action of $\mathscr{P}$ on $\widetilde{\Sigma}$ commuting with $\mathscr{R}^{(1)}$ can be defined to make $\widetilde{\Sigma}$ a trivial principal $\mathscr{P}$-bundle. This is

$$\mathscr{R}^{(2)}(L,b)[(\Lambda,a),(x)] = [(L\Lambda,La+b),(x)].$$

Going to the quotient as in (4.1), it gives rise to an action $\mathscr{R}$ on $\Sigma$, which in turn can be lifted to $\Gamma$. The symplectic manifold $\Gamma$ therefore carries a symplectic action $\overline{\mathscr{R}}$ of $\mathscr{P}$.[5]

Maps will be seen to exist from $\Gamma$ to spacetime for the physical positions, i.e.,

$$q_a^\mu = \phi_a^\mu(\gamma), \quad \gamma\in\Gamma,$$

with the property that

$$\phi_a\circ\overline{\mathscr{R}}(L,b) = \mathscr{R}_{\text{reg}}(L,b)\circ\phi_a$$

and, analogously, for the momenta, i.e.,

$$p_a^\mu = \psi_a^\mu(\gamma), \quad \gamma\in\Gamma,$$

$$\psi_a\circ\overline{\mathscr{R}}(L,b) = \mathscr{R}_{\text{reg}}(L,b)\circ\psi_a.$$

The above physical maps need not be defined on the whole of $\Gamma$ but rather on the part $\sigma(\mathscr{N}\times\mathbb{R})$, where dynamics operates, i.e., where all the constraints are satisfied. Furthermore, it is there that the generalized mass shell relations

$$p_a^2 - m_a^2 - v_a = 0 \qquad (4.2)$$

will hold.

In what follows we will consider four models. Each of them corresponds to an invariant subgroup of $\mathscr{P}$ with respect to which the quotient (4.1) is taken. Four such subgroups are considered, namely $\mathscr{P}$ itself, the Lorentz group $\mathscr{L}$, the translations $T^4$, and the identity.

### A. The model I[6–9]

The equivalence classes are taken with respect to $\mathscr{P}$, i.e.,

$$\Sigma = \widetilde{\Sigma}/\mathscr{R}^{(1)}(\mathscr{P})$$

and each of them can be represented by a set of $N$ 4-vectors $(z)$, so that

$$\Sigma\simeq(\mathbb{R}^4)^{\otimes N}.$$

In fact, the class to which $[(\Lambda,a),(x)]$ belongs contains also $[(L,0),(\Lambda,a)(x)]$ and if

$$(z) = (\Lambda,a)(x)$$

this can be denoted $\{(z)\}$.

The other variables in $\Gamma = T^*\Sigma$ are $(\eta)$, the canonical conjugates to $(z)$. So a point in $\Gamma$ is represented by $\{(z);(\eta)\}$. The action $\overline{\mathscr{R}}$ can be seen to be

$$\overline{\mathscr{R}}(L,b)\{(z);(\eta)\} = \{(Lz+b);(L\eta)\}.$$

This allows us to identify these variables with the physical spacetime positions and momenta. The relations (4.2) will enter in the definition of $M$.

### B. The model II[10,11]

Here the subgroup to be taken in (4.1) is the Lorentz group $\mathscr{L}$ and

$$\Sigma = \widetilde{\Sigma}/\mathscr{R}^{(1)}(\mathscr{L}).$$

Since

$$\mathscr{R}^{(1)}(L,0)[(\Lambda,a),(x)] = [(\Lambda L^{-1},a),(Lx)],$$

one sees that $[(\Lambda,a),(x)]$ is equivalent to $[(1,a),(\Lambda x)]$. Thus the elements of $\Sigma$ can be denoted $\{Q,(z)\}$, the $Q$ and $z_a$ $(a = 1,...,N)$ being 4-vectors, $z = \Lambda x$, so that

$$\Sigma \simeq (\mathbb{R}^4)^{\otimes (N+1)}.$$

The additional variables for $\Gamma = T^*\Sigma$ will be $R$ and $(\eta)$, the canonical conjugates to $Q$ and $(z)$. A point of $\Gamma$ may be written $\{Q,(z);R,(\eta)\}$. The action of $\overline{\mathscr{R}}(L,b)$ on it gives $\{LQ+b,(Lz);LR,(L\eta)\}$. The physical variables

$$q_a = Q + z_a, \quad p_a = R + \eta_a$$

transform with $\mathscr{R}_{\rm reg}$ but are not canonically conjugate. The relations (4.2) are satisfied once all the constraints on $\Gamma$ have been imposed, i.e., when the sections $\sigma$ have also been introduced.

### C. The model III[12]

The equivalence classes are taken with respect to the translation group $T^4$, i.e.,

$$\Sigma = \widetilde{\Sigma}/\mathscr{R}^{(1)}(T^4).$$

Since

$$\mathscr{R}^{(1)}(1,b)[(\Lambda,a),(x)] = [(\Lambda,a - \Lambda b),(x+b)],$$

we have

$$[(\Lambda,a),(x)] = [(\Lambda,0),(x + \Lambda^{-1}a)].$$

This allows us to denote a point of $\Sigma$ by $\{\Lambda,(z)\}$ where

$$z_a = x_a + \Lambda^{-1}a.$$

This gives

$$\Sigma \simeq \mathscr{L} \times (\mathbb{R}^4)^{\otimes N}.$$

The variables for $\Gamma = T^*\Sigma$ include those for $\Sigma$ and the "momentum" variables $S_{\mu\nu} = -S_{\nu\mu}$ and $(\eta)$, which are conjugate to $\Lambda^{\mu}_{\ \nu}$ and $(z)$, respectively. The nonvanishing Poisson brackets are

$$\{z_{a\mu},\eta_{b\nu}\} = \delta_{ab}\delta_{\mu\nu},$$
$$\{\Lambda^{\mu}_{\ \nu},S_{\alpha\beta}\} = g_{\nu\beta}\Lambda^{\mu}_{\ \alpha} - g_{\nu\alpha}\Lambda^{\mu}_{\ \beta},$$
$$\{S_{\mu\nu},S_{\alpha\beta}\} = g_{\mu\alpha}S_{\nu\beta} - g_{\nu\alpha}S_{\mu\beta} + g_{\mu\beta}S_{\alpha\nu} - g_{\nu\beta}S_{\alpha\mu}.$$

As far as $\overline{\mathscr{R}}$ is concerned, we see that

$$\mathscr{R}^{(2)}(L,b)[(\Lambda,0),(x+\Lambda^{-1}a)]$$
$$= [(L\Lambda,b),(x+\Lambda^{-1}a)]$$
$$\simeq [(L\Lambda,0),(x+\Lambda^{-1}a + (L\Lambda)^{-1}b)]$$

so that

$$\overline{\mathscr{R}}(L,b)\{\Lambda,(z);S,(\eta)\} = \{L\Lambda,(z+(L\Lambda)^{-1}b);LSL^{-1},(\eta)\}.$$

The position variables in spacetime are defined as

$$q_a = \Lambda z_a,$$

and these transform by means of the action on $\Gamma$ as under $\mathscr{R}_{\rm reg}$.

The physical energy–momenta are

$$p_a^{\mu} = \Lambda^{\mu}_{\ j}\eta_a^j + \Lambda^{\mu}_{\ 0}[m_a^2 + V_a(z) + \boldsymbol{\eta}_a\cdot\boldsymbol{\eta}_a]^{1/2}.$$

This allows us to satisfy the relations (4.2). Such $p_a$ transform properly as

$$p_a \rightarrow Lp_a$$

since the $v_a(z)$ will be chosen to be functions of the differences $z_b - z_c$.

### D. The model IV

The equivalence classes are taken with respect to the identity subgroup so that

$$\Sigma = \widetilde{\Sigma} = \Sigma^0 \times \mathscr{P}.$$

The variables of $\Sigma$ are then $\Lambda$, $Q$, and $(z)$, where $\Lambda\in\mathscr{L}$ and $Q$ and $(z)$ are vectors in $\mathbb{R}^4$. The variables of $T^*\Sigma$ are those of $\Sigma$ and the "momentum" variables $S_{\mu\nu} = -S_{\nu\mu}$, $R$, $(\eta)$. Here $R_\mu$ is conjugate to $Q_\mu$ and $\eta_{a\mu}$ is conjugate to $z_{a\mu}$ in the usual sense while $S_{\mu\nu}$ is the four-dimensional "angular momentum" conjugate to $\Lambda^{\mu}_{\ \nu}$. The Poisson brackets are the same as for model III with the addition of

$$\{Q_\mu,R_\nu\} = \delta_{\nu\mu}.$$

The physical position and momentum variables are given by

$$q_a = \Lambda z_a + Q, \quad p_a = \Lambda\eta.$$

The action of the physical (geometrical) Poincaré group is given by $\mathscr{R}_{\rm reg}$. Under this action $q_a$ and $p_a$ transform as they should:

$$\mathscr{R}_{\rm reg}(L,b)q_a = Lq_a + b,$$
$$\mathscr{R}_{\rm reg}(L,b)p_a = Lp_a.$$

Note that $z_a$ and $\eta_a$ are invariant under $\mathscr{R}_{\rm reg}$. The mass shell relations (4.2) will hold as a consequence of the definition of $M$.

## V. REDUCED PHASE SPACES AND SECTIONS

To see how the four models fit within the geometrical setup of Sec. II, we will construct the reduced phase space $\mathscr{N}$ for each of the four models following the procedure outlined before. The additional step will be to consider the choice of the constraints $\mathbb{X}$ to build up a family of sections of the bundle $\pi: M \rightarrow \mathscr{N}$.

The dimension of the $\mathscr{N}$'s turns out to be always $6N$; this is another reason to call them phase spaces. Another common feature is that the map $\mathbb{K}$ is taken to be invariant under the Poincaré group, which therefore renders $M$ invariant.

### A. The model I

The phase space $\Gamma$ is of dimension $8N$. The $\mathscr{P}$-invariant submanifold $M$ is constructed by introducing the set of $N$

real-valued functions on $\Gamma$,

$$\mathbb{K} = \{K_a\},$$

$$K_a = p_a^\mu p_{a\mu} - m_a^2 - v_a, \quad a = 1,\ldots,N,$$

having the following properties:

— the zero value is in the image of each of them;
— $(dK_1 \wedge \cdots \wedge dK_N)(m) \neq 0 \quad \forall\, m \in M \equiv \mathbb{K}^{-1}(0)$
(i.e., zero is a regular value for $\mathbb{K}$);
— each of them is $\mathscr{P}$-invariant.

$M \equiv \mathbb{K}^{-1}(0)$ is then a submanifold of $\Gamma$. Since $\dim \Gamma = 8N$, we have $\dim M = 7N$.

The $v_a$ satisfy the requirement[6–9]

$$\{K_a, K_b\} = 0, \quad a,b = 1,\ldots,N.$$

Therefore, the matrix $A$ vanishes; and

$$d = \dim \mathscr{D} = N.$$

The vector fields $X_a$ which generate $\mathscr{D}$ are then defined through the relations

$$i_{X_a}\omega = dK_a.$$

The dimension of each leaf is $N$; hence

$$\dim \mathscr{N} = \dim M / \mathscr{D} = 6N.$$

A point in each leaf, depending on a parameter $\tau$, is obtained by imposing the constraints

$$\chi_a = \left(\sum_{b=1}^{N} p_b\right)\cdot(q_{a+1} - q), \quad a = 1,\ldots,N-1,$$

$$\chi_N = \left(\sum_{b=1}^{N} p_b\right)\cdot q_1 - \tau.$$

As shown in the references quoted, they form, together with the $K_a$ a second class system of constraints; therefore,

$$\det|\{K_a, \chi_b\}|_M| \neq 0, \quad a,b = 1,\ldots,N.$$

Since $A_{|M} = 0$, our transversality condition (3.8) coincides with the above.

### B. The model II

The phase space $\Gamma$ has dimension $8N + 8$. The construction of the $\mathscr{P}$-invariant submanifold $M$ is made by introducing $2N + 5$ functions:

$$K_a^{(1)} = P\cdot z_a, \quad K_a^{(2)} = P\cdot \eta_a, \quad a = 1,\ldots,N,$$

$$K_i^{(3)} = \sum_{a=1}^{N} \eta_{ai}, \quad i = 1,2,3,$$

$$K^{(4)} = \sqrt{P^2} - \sum_{a=1}^{N} (m_a^2 - \eta_a^2 + v_a)^{1/2},$$

$$K^{(5)} = \sum_{a=1}^{N} \eta_{a0}.$$

$$\mathbb{K} \equiv (K_a^{(1)}, K_a^{(2)}, K_i^{(3)}, K^{(4)}, K^{(5)}),$$

$$M = K^{-1}(0).$$

The "potentials" $v_a$ are taken to be $\mathscr{P}$-invariant functions of $z_b - z_c$ and $\eta_b$. Only $2N + 4$ of them are functionally independent as, for instance, $K^{(5)}$ is a combination of the $K_i^{(3)}$ due to the $K_a^{(2)}$ vanishing; however,

$$(dK_1^{(1)} \wedge \cdots \wedge dK_N^{(1)} \wedge dK_1^{(2)} \wedge \cdots \wedge dK_N^{(2)} \wedge dK_1^{(3)} \wedge \cdots \wedge dK^{(4)} \wedge dK^{(5)})(m) \neq 0$$

for all $m \in M$. We have

$$\operatorname{codim} M = 2N + 4.$$

Again the zero value is regular and $M$ is $\mathscr{P}$-invariant since $\mathscr{R}$ either leaves the components of $\mathbb{K}$ invariant or permutes them among themselves.

The $(2N+5)$-dimensional antisymmetric matrix $A$, the elements of which are the Poisson brackets of components of $\mathbb{K}$, has the form

$$A = \begin{array}{c} \\ K_a^{(1)} \\ \\ K_a^{(2)} \\ \\ K_i^{(3)} \\ \\ K^{(4)} \\ K^{(5)} \end{array}
\left[\begin{array}{c|c|ccc|c|c}
K_a^{(1)} & K_a^{(2)} & & K_i^{(3)} & & K^{(4)} & K^{(5)} \\
\hline
0 & \begin{matrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{matrix} & \begin{matrix} P_1 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} P_2 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} P_3 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} x_1 \\ \vdots \\ x_N \end{matrix} & \begin{matrix} P_0 \\ \cdot \\ \cdot \end{matrix} \\
\hline
\begin{matrix} -c & & 0 \\ & \ddots & \\ 0 & & -c \end{matrix} & 0 & \begin{matrix} 0 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} 0 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} 0 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} x_{N+1} \\ \vdots \\ x_{2N} \end{matrix} & \begin{matrix} 0 \\ \cdot \\ \cdot \end{matrix} \\
\hline
\begin{matrix} -P_1 & \cdots\cdots \\ -P_2 & \cdots\cdots \\ -P_3 & \cdots\cdots \end{matrix} & \begin{matrix} 0 & \cdots\cdots \\ 0 & \cdots\cdots \\ 0 & \cdots\cdots \end{matrix} & \cdot\cdot & \cdot & \cdot & \begin{matrix} 0 \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \\
\hline
-x_1 \ \cdots \ -x_N & -x_{N+1} \ \cdots \ -x_{2N} & 0 & \cdots & \cdots & \cdot\cdot & \cdot \\
\hline
-P_0 \ \cdots\cdots & 0 \ \cdots\cdots & \cdots & \cdots & \cdots & \cdots & \cdot\cdot
\end{array}\right],$$

where

$$c = P^\mu P_\mu,$$
$$x_a = \{P\cdot z_a, K^{(4)}\},$$
$$x_{a+N} = \{P\cdot \eta_a, K^{(4)}\}.$$

To compute the latter matrix elements and then to evaluate them on $M$, use is made of the $\mathscr{P}$ invariance of the $v_a$. This means that both $x_a$ and $x_{a+N}$ are combinations of terms each of which has $P\cdot z_2$ or $P\cdot\eta_a$ as factors; then they vanish on $M$.

To compute the rank of $A$ on $M$, we note that its minor $A'$, formed by the first $2N$ rows and $2N$ columns, is nonsingular. We then act on the remaining rows and columns, adding to the elements of a line those of other parallel lines multiplied by suitable constants, to transform $A$ on $M$ into a new matrix $\overline{A}$ having the same rank:

$$\overline{A} = \left[\begin{array}{c|ccc} & P_1 & 0 & \cdots \\ A' & 0 & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ \hline -P_1 \quad 0 \quad \cdots\cdots & & & \\ 0 & & & \\ \cdot & & 0 & \\ \cdot & & & \\ \cdots\cdots\cdots & & & \end{array}\right]$$

the rank of which cannot be $2N+1$ it being antisymmetric. Since $\det A' \neq 0$, we conclude that

$$\operatorname{rank} A = 2N;$$
$$\dim \mathscr{D} = \operatorname{codim} M - \operatorname{rank} A = 4.$$

Since

$$\dim M = 8(N+1) - (2N+4) = 6N+4,$$

we have

$$\dim \mathscr{N} = \dim M/\mathscr{D} = 6N.$$

The set of constraints $K'$ leading to the nonsingular matrix $A'$ is made up of the $K_a^{(1)}$ and $K_a^{(2)}$ $(a = 1,\dots,N)$.

The four remaining functionally independent $K$ form the $K''$ set. To these are added four constraints $\chi$ to form a second class set. (The explicit form for $\chi$ is discussed in Ref. 11.) The transversality condition (3.7) involves the Dirac bracket $\{K'',\chi\}^*$ relative to the $K'$ constraints only. This is satisfied as a previous analysis of this model shows.[11]

### C. The model III

The phase space $\Gamma$ has dimension $8N+12$. Here $2N+5$ functions are introduced to construct the $\mathscr{P}$-invariant submanifold $M$ They are

$$K_\alpha^{(1)} = z_\alpha^0 - z_{\alpha+1}^0, \quad K_\alpha^{(2)} = \eta_\alpha^0 - \eta_{\alpha+1}^0, \quad \alpha = 1,\dots,N-1,$$
$$K_i^{(3)} = \sum_{a=1}^N \eta_a^i, \quad i = 1,\dots,3,$$
$$K_i^{(4)} = \tfrac{1}{2}\epsilon_{ikl}S^{kl} + \sum_{a=1}^N (\mathbf{z}_a \wedge \boldsymbol{\eta}_a)_i,$$
$$K^{(5)} = \eta_1^0 + v(\Delta z) + \sum_{a=1}^N (m_a^2 + \boldsymbol{\eta}^2 + v_a)^{1/2}.$$

They are either invariant or transformed into each other under $\mathscr{P}$.

Furthermore the wedge product of their differentials is a $(2N+5)$-form which does not vanish on $M \equiv \mathbf{K}^{-1}(0)$. This is therefore a $\mathscr{P}$-invariant submanifold of dimension $6N+7$.

The antisymmetric matrix $A$ of dimension $(2N+5)$ has the following form:

$$A = \begin{array}{c|c:c:c:c:c|} & K^{(1)} & K^{(2)} & K^{(3)} & K^{(4)} & K^{(5)} \\ K^{(1)} & 0 & A^{(1)} & 0 & 0 & \begin{smallmatrix}1\\0\\ \vdots\end{smallmatrix} \\ \hdashline K^{(2)} & -A^{(1)} & 0 & 0 & 0 & \vdots \\ \hdashline K^{(3)} & 0 & 0 & 0 & A^{(2)} & \vdots \\ \hdashline K^{(4)} & 0 & 0 & -A^{(2)} & A^{(3)} & \vdots \\ \hdashline K^{(5)} & -1\ 0\ \cdot & \cdots & \cdots & \cdots & \cdot\ \cdot \end{array}$$

Here

$$A^{(1)} = \begin{vmatrix} 2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & 2 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 \end{vmatrix},$$

$$A^{(2)} = \begin{vmatrix} 0 & K_3^{(3)} & -K_2^{(3)} \\ -K_3^{(3)} & 0 & K_1^{(3)} \\ K_2^{(3)} & -K_1^{(3)} & 0 \end{vmatrix},$$

and

$$A^{(3)} = \begin{vmatrix} 0 & K_3^{(4)} & -K_2^{(4)} \\ -K_3^{(4)} & 0 & K_1^{(4)} \\ K_2^{(4)} & -K_1^{(4)} & 0 \end{vmatrix}.$$

Evaluated on $M$,

$$\operatorname{rank} A \leqslant 2(N-1)+1.$$

A further reduction to

$$\operatorname{rank} A \leqslant 2(N-1)$$

is obtained since an antisymmetric matrix of odd order has vanishing determinant. Direct computation shows $\det A^{(1)} \neq 0$. Therefore,

$$\operatorname{rank} A = 2(N-1).$$

In this case $\mathscr{D}$ has dimension $(2N+5) - 2(N-1) = 7$ and again $\dim \mathscr{N} = 6N$.

The set of $K'$ is formed by the $2(N-1)$ $K_\alpha^{(1)}$ and $K_\alpha^{(2)}$. Imposing only these constraints means restricting the analysis to a subset $M'$ of $\Gamma$ having dimension $6N+14$. If $i_{M'}$ is the identification map

$$i_{M'}: M' \to \Gamma,$$

then the original symplectic form $\omega$[17],

$$\omega = \sum_{a=1}^{N} \sum_{\mu=0}^{3} dz_a^\mu \wedge d\eta_{a\mu} + \omega',$$

when pulled back to $M'$ gives

$$i_{M'}^* \omega = \sum_{a=1}^{N} \sum_{i=1}^{3} dz_a^i \wedge d\eta_a^i - N\, dz_{10} \wedge d\eta_{10} + \omega',$$

where $\omega'$ pertains to the variables $\Lambda_\mu^{\ \nu}$ and $S_{\mu\nu}$. This 2-form on $M'$ is seen to be nondegenerate as a consequence of the $K'$ being second class. Introducing new variables to replace $z_{10}$ and $\eta_{10}$,

$$Q = \sqrt{N} z_{10}, \quad R = \sqrt{N} \eta_{10},$$

we can write

$$i_{M'}^* \omega = \sum_{a=1}^{N} \sum_{i=1}^{3} dz_a^i \wedge d\eta_a^i - dQ \wedge dR + \omega',$$

This is actually the starting symplectic form for the model described in Ref. 12 since the relation between a symplectic form

$$\omega = \tfrac{1}{2}\, \omega_{\mu\nu}(\xi)\, d\xi^\mu \wedge d\xi^\nu$$

and its associated Poisson brackets

$$\{f, g\} = \omega^{\mu\nu}(\xi) \frac{\partial f}{\partial \xi^\mu} \frac{\partial g}{\partial \xi^\nu}$$

is given by

$$\omega^{\mu\nu} \omega_{\nu\lambda} = \delta_\lambda^\mu .$$

To form the section $\sigma(\mathscr{N} \times \mathbb{R})$ we need to make specific choice of $\chi$ as described in Ref. 12.

### D. The model IV

The dimension of $T^*\Sigma$ is $8N + 20$ so that a second class system of $2N + 20$ constraints is required to obtain $\dim \mathscr{N} = 6N$. We may choose them to be the following:

$$K_\mu^{(1)} = R_\mu - \sum_{a=1}^{N} p_{a\mu}, \quad \mu, \nu = 1, ..., 4,$$

$$K_{\mu\nu}^{(2)} = (Q \wedge R)_{\mu\nu} + S_{\mu\nu} - \sum_{a=1}^{N} (q_a \wedge p_a)_{\mu\nu},$$

$$K_a^{(3)} = \eta_a^2 - m_a^2 - v_a, \quad a = 1, ..., N,$$

$$\chi_\mu^{(1)} = \sum_{a=1}^{N} \epsilon_a z_{a\mu}, \quad \left( \sum_{a=1}^{N} \epsilon_a = 1,\ \epsilon_a > 0 \right),$$

$$\chi_\alpha^{(2)} = z_{1\alpha} - z_{2\alpha}, \quad \alpha \leqslant 2,$$

$$\chi_\alpha^{(3)} = z_{1\alpha} - z_{3\alpha}, \quad \alpha \leqslant 1,$$

$$\chi^{(4)} = z_{10} - z_{40},$$

$$\chi_\alpha^{(5)} = R \cdot (q_\alpha - q_N), \quad \alpha = 1, ..., N-1,$$

$$\chi^{(5)} = R \cdot q_N - \tau .$$

Here we choose $v_a$ in $K_a^{(3)}$ to be functions only of the internal variables $z_a$ and $\eta_a$. We choose them to be also invariant under the "Poincaré" group with generators $\Sigma\, \eta_a$, $\Sigma\, (z_a \wedge \eta_a)$ and adjust their functional dependence so that the $K_a^{(3)}$ form a first class set. (This is always possible.[9]) With such a choice $K^{(1)}, K^{(2)}$, and $K^{(3)}$ together form a first class set of $(N + 10)$ constraints.

The remaining constraints $\chi$ turn this first class set into a second class set. Of these, $\chi^{(1)}$ to $\chi^{(4)}$ are generalizations of those in model III. The functions $\epsilon_a$ are functions only of the internal variables $(z)$ and $(\eta)$ and are thus invariant under the physical Poincaré group. In the free particle limit $v_a \to 0$, they become the "renormalized energies" so that the usual free particle trajectories are recovered as in Ref. 12. The conditions $\chi^{(2)}$ to $\chi^{(4)}$ are designed to fix a Lorentz frame, and thus they are conjugate to $K^{(2)}$. For $N \leqslant 3$ they are clearly inadequate: They must then be replaced by some other "frame fixing" condition. Conditions $\chi^{(5)}$ are the familiar constraints conjugate to $K^{(3)}$.

Since $K^{(1)}$ to $K^{(3)}$ form a first class set $\mathbb{K}$ and the $(N + 10) \times (N + 10)$ matrix of their Poisson brackets with the constraints $\mathbb{X}$ is by construction nondegenerate, it is clear that the $(2N + 20) \times (2N + 20)$ matrix of Poisson brackets is nondegenerate. That is, the constraints $\mathbb{K}$ and $\mathbb{X}$ form a second class set. To be precise, there are degeneracies in these matrices whenever $\chi^{(2)}$ to $\chi^{(4)}$ fail to fix a frame, for instance, when $z_1$, $z_2$, and $z_3$ are parallel. Such situations have to be handled as in Ref. 12.

Thus $M = \mathbb{K}^{-1}(0)$ has dimension $7N + 10$ and the distribution $\mathscr{D}$ has dimension $N + 10$ and is formed by the vector fields $X_K$. The transversality condition for the $\sigma$ defined in terms of $\chi$ reduces to (3.8) and is satisfied as $K$ and $\chi$ form a second class set.

We note the following. The constraints $K^{(1)}$ and $K^{(2)}$ ensure that in the reduced phase space the generators of the physical Poincaré group have the desirable expressions $\Sigma\, p_a$ and $\Sigma\, q_a \wedge p_a$. Also, by virtue of the constraints $\chi^{(1)}$, $Q$ becomes the weighted average $\Sigma\, \epsilon_a q_a$ as in other models.[10,12]

## VI. DYNAMICS AS A GATHERING OF MANY INTO A SYSTEM

In the present paper we have started with a grand configuration in which we have a private world to each particle with a 4-vector all to itself and a Lorentz matrix describing the inertial frame. At this stage we had no particles and no motion, no interaction, and no dynamics: We need to generate some *togetherness* and some *self-referral* mechanism to introduce evolution. *Interaction comes from togetherness.*

To form a *system*, this "preparticle" collection has to give up part of its free-wheeling style and subject themselves to some constraints. It is from such constraints that the dynamical system specification and even the notion of dynamical evolution and the evolution parameter emerge.

In this paper we show many alternate patterns to the same goal and how the intermediate stage formulations appear drastically different. We also see in the course of time that not all constraints are on the same footing. Some are gauge constraints which change only the language of description; but some are essential constraints. *Changing the latter means changing the physical system.*

It is fairly straightforward to make choice of the constraints so that the world line condition is satisfied thus fulfilling one of the elementary requirements on relativistic interacting systems. But it was essential to go beyond the ten-parameter descriptions to the generalized *11-parameter form of Dirac's relativistic dynamics*.

In all this discussion the question of separability for systems with more than two particles has not been answered. We have addressed ourselves to this question elsewhere.[13]

In conclusion, we wish to stress the unifying power of geometry allowing us to view different models for relativistic interacting particles from a common perspective. The emphasis on the role of geometry in description of nature goes back to Plato, and this point of view has been enriched over the centuries by many illustrious scholars.[14] We hope that our work is in keeping with this tradition.

[1]We refer to R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Addison-Wesley, Reading, Mass., 1978) for the theory and notation of the calculus on manifolds.
[2]G. Marmo, E. J. Saletan, and A. Simoni, Nuovo Cimento B **50**, 1 (1979).
[3]Peter G. Bergman and Arthur Komar, "The Hamiltonian in Relativistic Systems of Interacting Particles," Syracuse University, 1980; F. Rohrlich, Phys. Rev. D **25**, 2576 (1982).
[4]Ref. 1, p. 116.
[5]Ref. 1, p. 180.
[6]Ph. Droz-Vincent, Lett. Nuovo Cimento **1**, 839 (1969); **1**, 206 (1973); Phys. Scripta **2**, 129 (1970); Rep. Math. Phys. **8**, 79 (1975); Ann. Inst. H. Poincaré **27**, 407 (1977); Phys. Rev. D **19**, 702 (1979).
[7]I. T. Todorov, "Dynamics of Relativistic Point Particles as a Problem with Constraints," Commun. of the JINR, EZ-10125, Dubna, 1976.
[8]A. Komar, Phys. Rev. D **18**, 1881, 1887, 3017 (1978); **19**, 2908 (1979).
[9]E. C. G. Sudarshan, N. Mukunda, and J. N. Goldberg, Phys. Rev. D **23**, 2218 (1981).
[10]F. Rohrlich, Ann. Phys. **117**, 292 (1979); Physica A **96**, 290 (1979); M. J. King and F. Rohrlich, Ann. Phys. **130**, 350 (1980).
[11]N. Mukunda and E. C. G. Sudarshan, Phys. Rev. D **23**, 2210 (1981).
[12]A. P. Balachandran, G. Marmo, N. Mukunda, J. S. Nilsson, A. Simoni, E. C. G. Sudarshan, and F. Zaccaria, Nuovo Cimento A **67**, 121 (1982).
[13]A. P. Balachandran, D. Dominici, G. Marmo, N. Mukunda, J. S. Nilsson, J. Samuel, E. C. G. Sudarshan, and F. Zaccaria, Phys. Rev. D **26**, 3492 (1982).
[14]G. Galilei, Il Saggiatore, **VI**, 232 (1623); S. Lie, "Zur Theorie der Transformationsgruppen," Christ. Forh. Aar. 1888 Nr. 13, Ges. Abh. BNdv, XXIII (especially pp. 554–7); S. Lie and F. Engle, *Theorie der Transformationsgruppen*, (Teubner, Liepzig, 1888–1893), Vols. I–III (in particular Vol. II); E. Cartan, *Leçons sur les invariants integraux* (Hermann, Paris, 1922); C. Caratheodory, *Calculus of Variations and Partial Differential Equation of the First Order (Part I)* (Holden Day, San Francisco, 1965); A. Lichenerowicz, J. Differential Geom. **12**, 253 (1977); R. Herman, *Interdisciplinary Mathematics* (Math. Sci. Press, Brookline, Mass., 1975, 1977), Vol. 14, Chap. 14, and Vol. 15, p. 52.