Gaussian–Maxwell beams

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Gaussian-beam-type solutions to the Maxwell equations are constructed by using results from relativistic front
analysis, and the propagation characteristics of these beams are analyzed. The rays of geometrical optics are shown
to be the trajectories of energy flow, as given by the Poynting vector. The longitudinal components of the field
vectors in the direction of the beam axis, though small, are shown to be essential for a consistent description.

1. INTRODUCTION

In the context of wave optics, scalar Gaussian beams are
exact solutions to the paraxial (parabolic) wave equation,1
and they have received substantial interest because they
form the fundamental modes of laser cavities with spherical
mirrors. But one knows that light beams consist of electro-

magnetic-field excitations and hence are governed by Max-
well’s equations. In partial recognition of this fact, it is
sometimes suggested that the scalar function representing
the Gaussian beam should be interpreted as the amplitude
of the electric-field vector assumed to be polarized in the
same direction everywhere.2 In other words, one imagines
a plane-polarized Gaussian beam with the electric field (and
also the magnetic vector) transverse to the beam axis at all
points. It is easy to see that such an interpretation is not
consistent with Maxwell’s equations. The divergence-free
nature of the electric field requires that there be no spatial
variation of this field vector in the direction of its polariz-
ation, but in a Gaussian beam the field amplitude does vary in
all directions. Thus it seems desirable to augment the scalar
Gaussian beam so that it forms an actual (paraxial) solution
to the Maxwell equations.

We presented3,4 recently in this journal a procedure for
describing, in a way consistent with Maxwell’s equations, the
action of an optical system on the electric- and the magnetic-
field vectors, given its action on the field amplitude of con-
ventional scalar optics. This procedure unfolds in a natural
way through a relativistic front5–7 analysis of the Maxwell
equations. In the present paper, we use this procedure to
derive Gaussian-beam-type solutions to Maxwell’s equa-
tions. To distinguish these beams from the scalar ones, the
former will be called Gaussian–Maxwell beams.

We begin in Section 2 by describing the rule for going from
scalar to vector optics, derived in Ref. 3. The field in the
latter case is to be written as a six-element column vector so
that the optical system becomes a 6 × 6 matrix operator.
This is followed in Section 3 by a brief summary of those

properties of scalar Gaussian beams that are needed later in
this paper. Gaussian–Maxwell beams are constructed and
their basic properties analyzed in Section 4. The field vec-
tor components parallel to the beam axis, though small, are
shown to be essential for a consistent description. Section 5,
our final section, contains some concluding discussions.

2. TRANSITION FROM SCALAR TO VECTOR
OPTICS

In paraxial scalar optics it is customary to choose the posi-
tive z axis as the direction about which the field propa-
gates. The field of a fixed frequency ω is then specified through its
(complex) amplitude distribution U(x⊥) over a transverse
plane (z = constant), where x⊥ = (x, y) is a two-dimensional
variable and a phase factor exp(−iωt) is suppressed. The
effect of an optical system is to map the field distribution in
one transverse plane (the input plane) to another plane (the
output plane) and is represented by an integral operator
Ω(x⊥, P⊥):

\[ U_{\text{out}}(x⊥) = \Omega(x⊥, P⊥)U_{\text{in}}(x⊥), \]

\[ P_a = -i \frac{\partial}{\partial x_a}, \quad a = 1, 2. \quad (2.1) \]

When Ω has no dependence on P⊥, we have the various
amplitude transmittances, the thin lens and the Gaussian
aperture being special cases. For a thin lens of focal length f
we have, for example,

\[ \Omega_b(x⊥) = \exp\left(\frac{-ik}{2f} x_a x_a\right). \quad (2.2) \]

where k = ω/c and summation over a is implied. On the
other hand, for free propagation over a distance d, Ω is a
function only of P⊥, and we have8

\[ \Omega_c(P⊥) = \exp\left(\frac{id}{2k} P_a P_a\right). \quad (2.3) \]
When Eq. (2.2) is used in Eqs. (2.1) and written out in integral form, one obtains the Fresnel propagation formula. When amplitude transmittances are stored in sequence separated by free-propagation sections, the resulting system has an operator with nontrivial dependence on $x_\perp$ and $P_\perp$. For first-order systems, which incidentally can always be synthesized by using thin lenses and free-propagation sections, this operator reduces to the generalized Huygens integral.

To make the transition to the vector case governed by Maxwell's equations, the field in a transverse plane is written as a six-component column $F(x_\perp)$ consisting of the components of the $E$ and $B$ vectors in place of the scalar function $U(x_\perp)$:

$$F(x_\perp) = \begin{pmatrix} E_x(x_\perp) \\ E_y(x_\perp) \\ E_z(x_\perp) \\ B_x(x_\perp) \\ B_y(x_\perp) \\ B_z(x_\perp) \end{pmatrix}. \quad (2.4)$$

Then for an optical system whose action on the scalar field is described by $\Omega(x_\perp, P_\perp)$, the corresponding $6 \times 6$ matrix operator to act on the vector field $F(x_\perp)$ is obtained by replacing $x_\perp$ in $\Omega(x_\perp, P_\perp)$ by $x_\perp + (1/k)G_\perp$:

$$\Omega(x_\perp, P_\perp) \rightarrow \Omega(x_\perp + \frac{1}{k}G_\perp, P_\perp); \quad G_\perp = \frac{1}{2} \begin{pmatrix} -S_2 & S_1 \\ -S_1 & -S_2 \end{pmatrix}, \quad G_\perp = \frac{1}{2} \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \quad (2.5)$$

The important point of course is that $\Omega(x_\perp + (1/k)G_\perp, P_\perp)$ acting on a paraxial $F(x_\perp)$ forming a solution to the Maxwell equations produces another $F(x_\perp)$ that is guaranteed to be a solution to these equations. As noted in Section 1, this rule emerges from a front-form analysis of the Maxwell equations. We have already applied this rule to the action of a thin lens on a paraxial Maxwell field and shown that it gives useful results. Further, we have shown that it leads to an unambiguous generalization of the scalar Fourier optics to the Maxwell-field case. More recently, we have used this rule to describe the action of polarization-sensitive linear optical systems on arbitrary paraxial Maxwell fields. In Section 4 of this paper, we adopt this rule to derive Gaussian-beam solutions to the Maxwell equations. But, in Section 3, we collect some results regarding the propagation characteristics of scalar Gaussian beams needed for this purpose.

### 3. SCALAR GAUSSIAN BEAMS

With the beam propagating along the $z$ axis, we can assume without loss of generality that the waist of the scalar Gaussian beam is in the transverse plane $z = 0$. The field in the waist plane is

$$U(x_\perp; z = 0) = \frac{(2/\pi)^{1/2}}{\sigma_0} \exp(-x_\perp^2/\sigma_0^2). \quad (3.1)$$

The field in a later plane $z = \text{constant}$ can be computed by using the Fresnel propagation operator [Eq. (2.2)]. One obtains

$$U(x_\perp; z) = \exp(-izP_\perp P_\perp/2k)U(x_\perp; 0) \times \exp[ikx_\perp^2/2R(z)],$$

$$R(z) = 1 + \left(\frac{k\sigma_0^2}{2z}\right)^2. \quad (3.2)$$

In Eqs. (3.2) we have suppressed an unimportant phase factor independent of the transverse coordinates $x_\perp$. Clearly, $R(z)$ is a measure of the beam width in the general transverse plane and is called the spot size. $R(z)$ is the radius of curvature of the phase front.

In terms of a complex $z$-dependent parameter $q$ called the complex radius of curvature and defined as

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{2i}{k\sigma(z)^2}, \quad (3.3)$$

we can write

$$U(x_\perp; z) = \frac{(2/\pi)^{1/2}}{\sigma(z)} \exp[ikx_\perp^2/2q]. \quad (3.4)$$

We will find this form particularly convenient in our analysis in Section 4.

### 4. GAUSSIAN–MAXWELL BEAMS

To derive Gaussian solutions to the Maxwell equations we first reinterpret the scalar field [Eq. (3.1)] as follows. We take a plane wave of unit amplitude propagating exactly along the $z$ axis. Clearly, for such a wave $U(x_\perp; z = 0) = 1$. At $z = 0$ we pass it through a Gaussian filter with amplitude transmittance

$$\Omega(x_\perp; z) = \frac{(2/\pi)^{1/2}}{\sigma_0} \exp(-x_\perp^2/\sigma_0^2). \quad (4.1)$$

Then what comes out of the filter will be the field [Eq. (3.1)].

With this reinterpretation, the generalization to the vector case is straightforward. Without loss of generality let us take the incident plane wave polarized in the $x$ direction, so that the column $F(x_\perp)$ equals $(1 \ 0 \ 0 \ 0 \ 0 \ 0)^T$. Using our rule [Eqs. (2.5)], the amplitude transmittance becomes a $6 \times 6$ matrix. Thus, we have for the field in the plane $z = 0$:

$$F(x_\perp; 0) = \frac{(2/\pi)^{1/2}}{\sigma_0} \exp\left[-\frac{(x_\perp + \frac{1}{k}G_\perp)^2}{\sigma_0^2}\right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.2)$$
On a later transverse plane, the field reads as

\[ F(x_{\perp}; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp \left( i k x_{\perp}^2 / 2q \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ (4.3)} \]

The matrix exponential in Eq. (4.3) can be easily evaluated in closed form by virtue of the properties of the matrices \( G \).

We have

\[ G_x G_\perp = G_y G_\perp, \]

\[ G_y G_\perp = 0, \]

\[ G_x G_\perp = 0. \]

Hence Eq. (4.3) reduces to

\[ F(x_{\perp}; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp \left( i k x_{\perp}^2 / 2q \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ (4.5)} \]

Substituting for \( G_\perp \) from Eqs. (2.5) we have to the lowest nontrivial order

\[ F(x_{\perp}; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp \left( i k x_{\perp}^2 / 2q \right) \begin{bmatrix} 0 & 0 & -x & 0 & 0 & y & 0 \\ 0 & 0 & -y & 0 & -x & 0 & 0 \\ x & y & 0 & -y & 0 & x & 0 \\ 0 & 0 & -y & 0 & -x & 0 & 0 \\ 0 & 0 & x & 0 & 0 & -y & 0 \\ y & -x & 0 & x & 0 & y & 0 \end{bmatrix} \text{ (4.6)} \]

In terms of components this reads as

\[ E_1(x_{\perp}; z) = B_2(x_{\perp}; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp(i k x_{\perp}^2 / 2q), \]

\[ E_2(x_{\perp}; z) = B_1(x_{\perp}; z) = 0, \]

\[ E_3(x_{\perp}; z) = -\frac{x}{q} E_1(x_{\perp}; z), \]

\[ B_3(x_{\perp}; z) = -\frac{y}{q} B_1(x_{\perp}; z). \]

These are the \( E \) and \( B \) fields of a Gaussian–Maxwell beam.

After reinserting the \( z, t \)-dependent phase factors, the reader can directly verify that these fields indeed form paraxial solutions to the Maxwell equations.

The transverse components \( E_1, B_2 \) propagate in the same way as the scalar beam [Eq. (3.4)], with the waist again at \( z = 0 \). But the longitudinal components behave differently.

Even at \( z = 0 \), \( E_2 \) and \( B_3 \) do not have Gaussian dependence on \( x_{\perp} \):

\[ E_2(x_{\perp}; 0) = -(2/\pi)^{1/2} \frac{2 i x}{\pm (k_0 z_{\perp}^2)} \exp(-x_{\perp}^2 / 2q_0^2), \]

\[ B_3(x_{\perp}; 0) = -(2/\pi)^{1/2} \frac{2 i y}{\pm (k_0 z_{\perp}^2)} \exp(-y_{\perp}^2 / 2q_0^2). \]

As they propagate, \( E_2 \) and \( B_3 \) stay proportional to the respective transverse components and to the transverse distances from the beam axis measured in the direction of polarization of those transverse components. Since \( q \) is, in general, complex, the longitudinal components have a part in phase with the transverse ones and a part in phase quadrature. The field is elliptically polarized, \( E \) in the \( xz \) plane and \( B \) in the \( yz \) plane.

Close to the waist plane, i.e., for \( z \ll k_0 z_{\perp}^2 / 2 \), we have

\[ \sigma(z) \approx \sigma_0, \quad R(z) = \frac{1}{z} (k_0 z_{\perp}^2 / 2), \]

and hence

\[ \frac{1}{q} \approx \frac{2i}{k_0 z_{\perp}^2}. \]

Thus in this region \( E_2 \) and \( B_3 \) are in phase quadrature with \( E_1 \) and \( B_y \), respectively. Therefore \( E \) and \( B \) are elliptically polarized with minor axis along the beam axis. Sufficiently far away from the waist plane, i.e., in the Fraunhofer region \( z \gg k_0 z_{\perp}^2 / 2 \), we have

\[ q \approx z, \]

so that \( E \) and \( B \) are linearly polarized and are transverse to the radial direction.

More insight is gained by computing the Poynting vector. One obtains

\[ S = \text{Re} \begin{bmatrix} E_1 & B_2 \end{bmatrix} = \frac{2}{\pi} \frac{1}{\sigma(z)^2} \begin{bmatrix} x_{\perp} \\ R(z) \end{bmatrix}. \]

First, we note that the parts of the longitudinal components that are in phase quadrature with the transverse ones do not contribute to \( S \). To see another interesting aspect let us define a two-parameter family of curves

\[ x_{\perp} = \left( x_{\perp} / R(z) \right)^{1/2}, \]

where the pair of parameters \((x_{\perp}, z)\) label a curve and \( z \) is a parameter along a curve. These curves are well known in the context of scalar Gaussian beams. They represent the geometrical-optics rays normal to the phase curvature and account for the manner in which the beam diverges. The tangent at any point \((x_{\perp}, z)\) to the curve passing through that point can be easily calculated by noting from Eqs. (4.10) and (3.2) that

\[ \frac{dx_{\perp}}{dz} = -\frac{x_{\perp}}{R(z)}. \]

Thus the tangent vector at \((x_{\perp}, z)\) is proportional to \([x_{\perp} / R(z), 1]\), which is exactly the direction of the Poynting vector \( S \) at that point. We find that the Poynting vector is
along the rays of geometrical optics so that these rays are indeed the trajectories of energy flow. It should be noted that the small longitudinal components are essential for the consistency of the theory because in their absence the energy flow would have been axial everywhere, contrary to the notion of a diverging beam.

To conclude this section we note that it is consistent with Maxwell's equations to have a Gaussian beam with \( E_2 = B_3 = 0 \). However, it is inconsistent with these equations to require a Gaussian beam for which both \( E_2 \) and \( E_3 \) vanish and only \( E_1 \neq 0 \).

5. CONCLUDING DISCUSSION

We have seen that a Gaussian beam polarized in the same direction everywhere is not compatible with Maxwell's equations. The Maxwell field that is closest to the usual scalar Gaussian beam is the Gaussian–Maxwell beam given in Eqs. (4.7). The divergence-free condition on the \( \mathbf{E} \) and \( \mathbf{B} \) vectors imply that spatial modulation necessarily affects polarization. In other words, the polarization of the field cannot be chosen independently of its spatial variation. Thus a Gaussian-varying transverse component necessarily results in the longitudinal component in Eqs. (4.7).

Once the choice of polarization of the transverse component is made, the Gaussian–Maxwell beam is completely specified by the complex parameter \( q \), as can be seen from Eqs. (4.7). Thus these beams form a three-parameter family labeled by the orientation of the transverse component of the electric vector (the transverse polarization) and the two real parameters \( s \) and \( R \) constituting \( q \) or, alternatively, by the transverse polarization, the spot size at the waist, and the location of the waist. Our analysis in Section 4 shows that the transverse polarization does not change under free propagation. It is easy to show that it is also unaffected by action of thin lenses, and, in view of the theorem established in Ref. 7 and already noted in Section 2, it follows that it is an invariant under action by any first-order system. Thus we conclude that the \( abcd \) law of Kogelnik governing the transformation of \( q \) completely describes the transformation properties of all the field components of Gaussian–Maxwell beams under the action of first-order systems.

Gaussian–Schell–model (GSM) beams have received much interest in recent times. It has been shown that GSM beams can be obtained as incoherent superposition of coherent Gaussian beams and their higher-order Hermite–Gaussian beams. In view of this result, extension of our analysis to the GSM case presents no difficulty.

The rule represented in Eqs. (2.5) is evidently central to the later results of the paper. It was established in Ref. 3 through an analysis of the form of the angular-spectrum representation of paraxial vector electromagnetic fields.

Since submission of the manuscript of this paper, the authors' attention has been drawn to possible applications of the analysis to several questions related to problems of wave propagation in inhomogeneous media that are of interest in ocean acoustics, seismology, and other fields. These include construction of the vector wave theories corresponding to the extended parabolic (wide-angle) scalar theories, development of paraxial wave theory for the vector equations of linear elasticity theory, and application to the Gaussian-beam approach for ray-optics computations. We plan to analyze these questions in a subsequent paper.

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REFERENCES AND NOTES

1. See, for example, H. Kogelnik and T. Li, "Laser beams and resonators," Proc. IEEE 54, 1312–1329 (1966), where further references can be found.


5. The front form of dynamics was first introduced in P. A. M. Dirac, "Forms of relativistic dynamics," Rev. Mod. Phys. 21, 392–399 (1949). It has been used for a systematic analysis of paraxial wave optics, both the scalar case and the vector case, in Refs. 8 and 7.


12. For an analysis of this rule, see H. Bercy, "Group theory and paraxial optics," invited paper presented at the XIII International Colloquium on Group Theoretical Methods in Physics, University of Maryland, May, 1984, where it has been called the MSS postulate. It was postulated in Ref. 7 and subsequently proved in Ref. 3.


