CHAPTER 14

# Three Perspectives on Light

E.C.G. SUDARSHAN

*Institute of Mathematical Sciences*
*Madras 600 113, India*

*Progress in Quantum Field Theory*
*Edited by*
*H. Ezawa and S. Kamefuchi*

# Contents

# 1. Classical optics

## 1.1. Pencils of light

The starting point of classical optics is the pencil of light. A pencil of light is a collection of rays of light all emanating from the vertex of the pencil. By removing the vertex a pencil of light becomes a parallel beam of light while by making the vertex wander over a small area or volume we get a diffuse pencil. The rays of light are the primary undefined element; and in classical optics it is the basic building block, yet it being understood that single rays are unrealizable idealizations. Along with a ray we define its specific intensity which is a measure of the quantity of light traversing space per unit solid angle around the light ray per unit area normal to the ray. Since light comes in different colours the specific intensity depends on the colour:

$$\mathrm{d}I = I(\omega, n, \boldsymbol{r})\, \mathrm{d}\omega\, \mathrm{d}\Omega\, \mathrm{d}A.$$

The behaviour of a pencil of rays in classical optics is visualized easily in terms of the behaviour of the rays of the pencil. In empty space (or any uniform medium) light rays proceed in straight lines leading to the rectilinear propagation of beams of light. For a generic pencil the rays of light, proceeding along straight lines diverge and hence the specific intensity should decrease inversely as the area grows: this leads to the familiar inverse square law of photometry. For a diffuse pencil the law of photometry is more complicated. The general theory is the theory of radiative transfer, which usually includes scattering and absorption in addition to free propagation.

The first modifications to free propagation of a pencil of light are the reflection and refraction at plane surfaces. The laws are simply stated in terms of light rays. For reflection, the incident ray, the reflected ray and the normal to the surface of reflection at the point of incidence are in the same plane; and the angles of reflection and incidence are equal and opposite. For refraction, the incident ray, the refracted ray and the normal to the surface at the point of incidence are in the same plane; and the sine of the angle of incidence divided by that of the angle of refraction is the relative index of refraction. In both cases this leads to a propagation modification which leads to (virtual) images of the vertex, with respect to which the inverse square law will hold, as far as the reflected or refracted intensity is concerned.

The reflection and refraction at curved surfaces is obtained from the above by considering infinitesimal areas of the surfaces and replacing them by their tangent planes. An important class of curved reflecting surfaces are spherical mirrors, concave or convex. In this case we can show that any paraxial pencil with vertex $u$ goes into a paraxial pencil with vertex $v$ (provided the solid angle subtended is small!) with the mirror formula

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f} \tag{1.1}$$

with $f=\pm\frac{1}{2}R$ for concave and convex mirrors, respectively. The corresponding thing in refraction involves the thin lens which is a region of refracting medium bounded by two spherical surfaces of large radii $R_1$, $R_2$. If the refractive index is $\mu$ then the incident paraxial pencil vertex $u$ and the emergent paraxial pencil vertex $v$ are related by

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right). \tag{1.2}$$

It is an interesting question to ask to what extent these formulae apply in wave optics or whether they do not apply at all.

The question of transmission of a pencil through a medium whose refractive index varies arbitrarily as a function of position is the natural generalization. The geometrical optics of such a situation can be worked out as a problem in Hamiltonian mechanics.

When a paraxial pencil encounters a thin lens it is converted into a paraxial pencil with a different vertex. Hence the action of a set of thin lenses separated by finite distances of free propagation is to transform a paraxial pencil into another pencil. These transformations form a group which is $SL(2, \mathrm{R})$ for an axially symmetric system of lenses and the group $Sp(4, \mathrm{R})$ for the general case. A systematic analysis shows that the general $SL(2, \mathrm{R})$, $Sp(4, \mathrm{R})$ optical systems can be synthesized with a prescribed finite number of lenses separated by finite regions of free propagation. A similar analysis can be developed for small curvature mirrors placed paraxially.

### *1.2. Scaling laws*

In contrast to these phenomena of geometrical optics, phenomena of a different kind also occur. These are best understood in terms of wave optics. The simplest of these involve diffraction: the departure from rectilinear propagation and the redistribution of light energy so as to form diffraction fringes. A narrow aperture produces large diffraction fringes. The diffraction exhibited at large distances scale: if $\Delta$ is a typical linear dimension (fringe width, separation between successive rings, etc.) and $D$ the distance of the pattern from the diffracting edges, then

$$\frac{\Delta}{D}=\text{independent of } D=S, \tag{1.3}$$

the independent constant $S$ being linear in the wavelength. The diffraction pattern scales, as if the pattern is caused by a pencil of light rays!

The other typically wave phenomenon is that of interference. If light passes through two adjacent slits (coherently) instead of uniform illumination on a far screen, we have a pattern of striations called interference fringes. These are equidistant obeying the scaling law

$$\frac{\Delta}{D} = S = \frac{\lambda}{d}, \tag{1.4}$$

where $d$ is the separation between the slits and $\lambda$ is the wavelength of light. Again the wave interference pattern scales as if it is due to rays of light propagating in straight lines. A more sophisticated interference pattern is got by using a diffraction grating: again we find the typical scaling law.

Interference involving superposition of amplitudes and hence the possibility of mutual cancellation does not require the beams to be separated by small spatial distances. In the Michelson interferometer the beams are separated substantially and interference still takes place.

We have talked about the treatment of reflection and refraction in geometrical optics. In that case the fraction of reflected and refracted intensities at a dielectric interface could not be calculated but should be given: but in the wave theory we can calculate the reflected and refracted angles and the reflected and refracted intensities.

Another case of practical importance where interference effects play an essential role is to identify the standing wave modes of a nearly perfect laser. When the laser emits light we expect to see only those modes which do not cancel out due to interference.

Finally the wave aspects of light are exhibited in the phenomena of polarization, birefringence and optical activity. All these find a natural explanation in terms of transverse waves and the transformations imposed by a suitable homogeneous medium on them. The propagation is still rectilinear but the polarization aspects are essential.

## 2. *Towards quantum theory in optics*

When we come to the photoelectric process and the Compton effect in X-ray scattering we see the corpuscular aspect of light: the individual photons seem involved in both processes. In the case of the Compton effect the kinematics of a collision between a photon and an electron fully describe the process though the calculation of the cross section for scattering and thus the angular distribution depends on a relativistic wave optical calculation. The case of the photoelectric effect is simpler since only the infinitesimal time counting probability need be obtained from wave optics: the rest involves usual probability calculus. While in photocounting we need the stochastic average of higher order products of the

amplitudes, the other phenomena so far discussed including interference and diffraction are bilinear in the amplitudes. This has a profound effect on the degree to which classical wave theory gives the complete description with no modification coming from quantum effects for these phenomena.

The two classes of phenomena appear different and need different descriptions. The possibility of reuniting geometrical and wave optics began with the work of Walther, was revived by Wolf and has been completed by our group. The basic idea is to define a pencil of rays to associate it with every statistical wave field and see the extent to which the light rays behave as in geometrical optics. In the next section we reformulate statistical wave optics.

### 2.1. *Partial coherence and two-point functions*

In this section we shall systematically ignore polarization effects and deal with "scalar" wave optics. In this case the primary field quantity is a complex fluctuating field $\phi(\boldsymbol{r}, t)$ with an unspecified distribution, but with a zero mean. We define the two-point correlation function

$$\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2; t_1, t_2) = \langle \phi^*(\boldsymbol{r}_1, t_1)\, \phi(\boldsymbol{r}_2, t_2) \rangle. \tag{2.1}$$

It is conventional to take $\phi$ to be an "analytic signal" consisting only of positive frequency terms and consequently $\phi^*$ of only negative frequency terms. [Polarization can be included if we choose so by taking two-component fields $\phi_\lambda$ for the two (circular) polarizations.] For a stationary ensemble the two-point function $\tilde{\Gamma}$ depends only on the difference of the two times $t_1 - t_2$: it is clear that in this case we need to use generalized harmonic analysis on the field quantity. We shall assume this restriction throughout this paper:

$$\left\langle \phi^*\left(\boldsymbol{r}_1, t + \frac{\tau}{2}\right) \phi\left(\boldsymbol{r}_2, t - \frac{\tau}{2}\right) \right\rangle = \tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2; \tau). \tag{2.2}$$

$\tilde{\Gamma}$ may be thought of as a matrix $\tilde{\Gamma}(\tau)$ in the two indices $\boldsymbol{r}_1$, $\boldsymbol{r}_2$. The two-point function in the frequency domain is the Fourier transform

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \omega) = \frac{1}{2\pi} \int \mathrm{e}^{-\mathrm{i}\omega\tau}\, \tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2; \tau)\, \mathrm{d}\tau. \tag{2.3}$$

$$\tilde{\Gamma}(-\tau) = \tilde{\Gamma}^\dagger(\tau), \quad \Gamma(\omega) = \Gamma^\dagger(\omega). \tag{2.4}$$

When we have free propagation in space the two-point function clearly satisfies two second order wave equations:

$$\left(\nabla_1^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \left(\nabla_2^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2; t). \tag{2.5}$$

Changing coordinates

$$\boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \quad \boldsymbol{r}' = \boldsymbol{r}_1 - \boldsymbol{r}_2,$$

we deduce

$$\left(\tfrac{1}{4}\nabla^2 + \nabla'^2 + \frac{\omega^2}{c^2}\right)\Gamma\left(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}',\ \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}';\ \omega\right) = 0, \tag{2.6}$$

$$\nabla \cdot \nabla'\, \Gamma\left(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}',\ \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}';\ \omega\right) = 0. \tag{2.7}$$

Since the field $\phi(\boldsymbol{r}, t)$ is an analytic signal we get a stronger first order equation:

$$\sqrt{-\nabla_1^2}\ \tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \tau) = \mathrm{i}\frac{\partial}{\partial \tau}\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \tau), \tag{2.8}$$

which shows that given $\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \tau)$ at any time $\tau$, it is defined for all subsequent times. Since the spatial distribution is connected to the wave number and thus the colour, we find the relation between colour and frequency. The operator $\sqrt{-\nabla^2}$ is to be understood to be the integral operator

$$\left(\sqrt{-\nabla^2} f\right)(\boldsymbol{r}) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}')\, \mathrm{d}^3 r' \,|\, k \,|\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k \tag{2.9}$$

and hence samples the local colour.

If the two-point function in the frequency domain $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ factorizes we talk of two-point coherence:

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \omega) = \varphi^*(\boldsymbol{r}_1, \omega)\ \varphi(\boldsymbol{r}_2, \omega). \tag{2.10}$$

In such a case we could deal with $\varphi(\boldsymbol{r}, \omega)$ as a classical wave field subject to an appropriate propagation law. More generally we have an eigenmode description:

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \omega) = \sum_\alpha \varphi_\alpha^*(\boldsymbol{r}_1, \omega)\ \varphi_\alpha(\boldsymbol{r}_2, \omega) = \sum_\alpha \Gamma_\alpha \tag{2.11}$$

and $\Gamma$ could then be considered as the incoherent mixture of the pure two-point functions $\Gamma_\alpha(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \omega)$.

The two-point function $\Gamma$ shares some of the properties of specific intensity in that all ensemble averages are taken; yet contains relative phase information between separated space points in the field of illumination. This relative phase information is important as far as propagation is concerned. So the ensemble averaged propagating quantity $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \omega)$ is the appropriate quantity for statistical wave optics.

Moreover the total intensity is equal to the two-point function specialized to $\boldsymbol{r}_1 = \boldsymbol{r}_2$ apart from some kinematic factor $\rho(\omega)$ of the frequency depending upon the

normalization of $\phi$:

$$\Gamma(\boldsymbol{r}, \boldsymbol{r}; \omega) = I(\boldsymbol{r}; \omega)\, \rho(\omega). \tag{2.12}$$

In the usual case of optical propagation it is usual to fix the frequency $\omega$ and consider spatial propagation, say in the $z$ direction. Since by the requirement of stationarity in time of the ensemble, the different frequency components simply add in the two-point function we may consider the idealized monochromatic case when we really mean the quasimonochromatic case without any essential loss of generality. In this case the equation of "motion" is elliptic:

$$\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \omega^2\right)\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \omega) = 0. \tag{2.13}$$

Given the illumination on any $z =$ constant surface we must know if the light is going in the positive or negative direction. If we identify the wave as forward propagating (positive direction) we still have to deal with the instability of the solution of an elliptic differential equation with respect to initial condition. If we choose the initial condition to be specified on a $z =$ constant plane for a forward propagating wave there are some finer details in the $z =$ constant plane for which the $z$-component of the wave number is imaginary,

$$k_z^2 = \omega^2 - k_x^2 - k_y^2. \tag{2.14}$$

So as soon as the $x$, $y$ details become finer than $|k| = \omega/c$ then $k_z$ becomes imaginary. The factor $e^{ik_z z}$ hence exponentially dies or grows. The latter choice is unsatisfactory and discarded. The $ik < 0$ terms die down rapidly and are called "evanescent waves" and may be neglected as far as propagation is concerned. Consequently for optical propagation the transverse wave numbers (detailing the spatial variation on the $z =$ constant plane) cover a compact circle

$$0 \le k_x^2 + k_y^2 < \frac{\omega^2}{c^2}. \tag{2.15}$$

Since the region of illumination is usually compact it follows that optical transmission has a finite channel capacity directly proportional to the area and the square of the frequency.

## 3. *Pencils of rays in wave optics*

### *3.1. The Wolf function*

We have not yet established a connection with the specific intensity $I(\boldsymbol{r}, \hat{\boldsymbol{n}}, \omega)$ but only to the total intensity

$$I(\boldsymbol{r}, \omega) = \int \mathrm{d}\Omega_{\hat{n}}\, I(\boldsymbol{r}, \hat{\boldsymbol{n}}, \omega). \tag{3.1}$$

To establish this relation and the connection with pencils of rays in geometrical optics we introduce the Wolf function

$$W(\boldsymbol{r}, \boldsymbol{p}, \omega) = \frac{1}{(2\pi)^3} \int e^{-i\boldsymbol{p}\cdot\boldsymbol{r}'} \Gamma(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}'; \omega)\, d^3r'. \tag{3.2}$$

By the hermiticity property of $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \omega)$ it follows that the Wolf function $W(\boldsymbol{r}, \boldsymbol{p}, \omega)$ is real. Its partial integrals are positive:

$$\int W(\boldsymbol{r}, \boldsymbol{p}; \omega)\, d^3p = I(\boldsymbol{r}, \omega)\, \rho(\omega) > 0, \tag{3.3}$$

$$\int W(\boldsymbol{r}, \boldsymbol{p}, \omega)\, d^3r = J(\boldsymbol{p}, \omega)\, \rho(\omega) > 0. \tag{3.4}$$

But $W(\boldsymbol{r}, \boldsymbol{p}; \omega)$ itself need not be definite. The quantity $J(\boldsymbol{p}, \omega)$ is the integrated intensity of light streaming in the direction $\hat{\boldsymbol{p}}$. Now, apart from the kinematic factor $\rho(\omega)$ the quantities $I(\boldsymbol{r}, \omega)$ and $J(\boldsymbol{p}, \omega)$ are the integrals of the specific intensity $I(\boldsymbol{r}, \hat{\boldsymbol{n}}, \omega)$ over $\hat{\boldsymbol{n}}$ and $\boldsymbol{r}$, respectively; it is natural to make the identification of $W(\boldsymbol{r}, \boldsymbol{p}; \omega)$ with the ray pencil density in wave optics:

$$I(\boldsymbol{r}, \hat{\boldsymbol{n}}) = W(\boldsymbol{r}, \boldsymbol{p}). \tag{3.5}$$

Two points are to be noted with regard to this identification: First, by virtue of the equation of motion of $\Gamma$,

$$(p^2 - \omega^2/c^2 - \tfrac{1}{4}\nabla^2)\, W(\boldsymbol{r}, \boldsymbol{p}) = 0. \tag{3.6}$$

So the momentum $p$ is not quite the same as $\omega/c$ but reduced from it by a quantity which depends on how rapidly the illumination varies; in a field of constant illumination this effect disappears. In most practical cases these are essentially edge effects. The other point is more startling: $W(\boldsymbol{r}, \boldsymbol{p})$ is not pointwise positive. It can become negative in small regions of the $\boldsymbol{r}$, $\boldsymbol{p}$ domain. Wave optics deals not only with pradipa (shining) rays but also with tamasic (dark) rays. The regions over which such nonpositivity is tolerated is the basic phase cell: if $P(\boldsymbol{r}, \boldsymbol{p})$ is any Wigner–Moyal distribution function

$$P(\boldsymbol{r}, \boldsymbol{p}) = \int \psi^*(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}')\, \psi(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}')\, e^{i\boldsymbol{p}\cdot\boldsymbol{r}'}\, d^3r', \tag{3.7}$$

then it can be easily demonstrated that

$$\int P(\boldsymbol{r}, \boldsymbol{p})\, W(\boldsymbol{r}, \boldsymbol{p})\, d^3r\, d^3p > 0. \tag{3.8}$$

Since Wigner–Moyal functions themselves are negative over regions of a phase cell $\Delta r\,\Delta p \sim 1$ it follows that the same is true for the Wolf function. Note that unlike the Wigner–Moyal function, the Wolf function is not normalized but its normalization is the total integrated intensity [apart from the factor $\rho(\omega)$].

In many cases of interest in optics we need to consider the quasimonochromatic case and propagation from one plane with $z=$ constant to another plane with $z=$ constant. In this case the quantity of interest is $\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ restricted to the two transverse directions:

$$\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2, M), \quad M=\frac{\omega}{c}+k_z \sim \text{constant}. \tag{3.9}$$

We shall outline below why we choose "henochromaticity": $\omega/c+k_z \sim$ constant to monochromaticity since we are essentially interested in travelling with the wavefront and see the modifications that it undergoes on advancing along the $z$-axis. The corresponding Wolf function $W(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp)$ can be identified with the diffuse pencil of henochromatic rays which pass in the neighbourhood of the point $\boldsymbol{r}_\perp$ in the direction $\hat{\boldsymbol{p}}_\perp$ per unit solid angle and unit area. The conceptual synthesis of geometric optics and wave optics is now complete.

### 3.2. *Wave fronts and paraxial rays*

We briefly outline the "front form" for paraxial propagation. Given the free scalar wave equation

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\boldsymbol{r}, t)=0, \tag{3.10}$$

which is invariant under the full Poincaré group. If we choose the front-coordinate $\sigma=ct-z$ and its companion $\tau=(2c)^{-1}(ct+z)$ this wave equation can be rewritten

$$\frac{\partial^2}{\partial\sigma\,\partial\tau}\phi=\tfrac{1}{2}\nabla_\perp^2\phi. \tag{3.11}$$

The set of transformations that leave a front invariant is the extended Galilei group in $(2+1)$ dimensions; the 2 dimension being $\boldsymbol{r}$ and the 1 dimension being the propagation variable $\tau$. The "mass" of the Galilei group is $\mathrm{i}\,\partial/\partial\sigma$ shifting the front; while the "Hamiltonian" is given by

$$H=\frac{1}{2M}\left(-\nabla_\perp^2\right). \tag{3.12}$$

The wave equation becomes

$$\left(\mathrm{i}\frac{\partial}{\partial\tau}+\frac{1}{2M}\nabla_\perp^2\right)\phi=0. \tag{3.13}$$

It is clear that

$$M = \frac{1}{2}\left(\frac{\omega}{c} + k_z\right) + \frac{1}{2}\left(\frac{\omega}{c} - \mathrm{i}\frac{\partial}{\partial z}\right) = \mathrm{i}\frac{\partial}{\partial \sigma} \tag{3.14}$$

is the quantity to be kept constant. When this condition is satisfied we talk of henochromaticity. (For transverse electromagnetic field the helicity introduces some minor technical modifications, but the use of henochromaticity and the derivation of Galilei group is unchanged.)

When these same considerations are applied to the two-point function we get the equations of motion:

$$\left(\mathrm{i}\frac{\partial}{\partial \tau_1} + \frac{1}{2M}\nabla_{1\perp}^2\right)\tilde{\Gamma}(\boldsymbol{r}_{1\perp}, \boldsymbol{r}_{2\perp}, \tau_1 - \tau_2) = 0, \tag{3.15}$$

$$\left(\mathrm{i}\frac{\partial}{\partial \tau_2} + \frac{1}{2M}\nabla_{2\perp}^2\right)\tilde{\Gamma}(\boldsymbol{r}_{1\perp}, \boldsymbol{r}_{2\perp}, \tau_1 - \tau_2) = 0. \tag{3.16}$$

Note that stationarity is with respect to the propagation variable $\tau$. As before it follows that

$$W(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp, M) = \frac{1}{(2\pi)^2}\int \Gamma\left(\boldsymbol{r}_\perp + \tfrac{1}{2}\boldsymbol{r}'_\perp, \boldsymbol{r}_\perp - \tfrac{1}{2}\boldsymbol{r}'_\perp\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}'}\, \mathrm{d}^2 r'_\perp \tag{3.17}$$

satisfies:

$$\left[\tfrac{1}{2}(p^2 - \nabla^2) - M^2\right] W(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp, M) = 0. \tag{3.18}$$

Hence $p_\perp^2$ is essentially constant except where the Wolf function varies very rapidly. Also

$$\boldsymbol{p}_\perp \cdot \nabla_\perp\, W(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp, M) = 0, \tag{3.19}$$

showing that $\boldsymbol{p}_\perp$ is required to be tangent to the contours of $W$. (This is particularly important in total internal reflection and other such situations.)

### *3.3. Transformations of pencils of rays*

The use of the Wolf functions [and the geometrical pencil] in simple optical problems is based on the transformation law

$$W'(\boldsymbol{r}', \boldsymbol{p}') = W(\boldsymbol{r}, \boldsymbol{p}), \tag{3.20}$$

where $W'$ is the transformed Wolf function and $\boldsymbol{r}'$, $\boldsymbol{p}'$ the transform of the ray $\boldsymbol{r}$, $\boldsymbol{p}$. (We suppress the subscripts for $\boldsymbol{r}_\perp$, $\boldsymbol{p}_\perp$ henceforth.) At reasonable distance from a reflecting or refracting surface this result is verified by elementary means: the essential requirement that the incident and emergent beams should not overlap.

With a slightly lengthier calculation it could also be shown that the result holds for refraction through a prism.

The encounter of a Wolf pencil with a thin lens is a more interesting case. The phase advance of a wave due to a thin lens (thickness of the lens is $\delta_0$, refractive index $\mu$ and focal length $f$) is given by

$$\theta(\boldsymbol{r}_\perp) = \mu\delta_0 - r_\perp^2/2f, \tag{3.21}$$

provided $\delta_0 \ll f$ to satisfy the requirement of the lens being "thin". Remembering that $\boldsymbol{p}$ is conjugate to the phase difference between two fronts it follows that this phase advance implies a change of direction of the rays but no displacement:

$$\boldsymbol{r}' = \boldsymbol{r}, \quad \boldsymbol{p}' \neq \boldsymbol{p}. \tag{3.22}$$

The change in $\boldsymbol{p}$ can be determined by noting that the change in phase between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is

$$\theta(\boldsymbol{r}_2) - \theta(\boldsymbol{r}_1) = \frac{2\pi}{2f}(r_1^2 - r_2^2) = \frac{1}{2f}(\boldsymbol{r}_1 + \boldsymbol{r}_2)\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

Hence $\boldsymbol{p}$ undergoes an abrupt change

$$\boldsymbol{p}' - \boldsymbol{p} = -\frac{M}{f}\boldsymbol{r}. \tag{3.23}$$

On the other hand free propagation of the ray through distance $D$ leads to

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{D}{M}\boldsymbol{p}, \tag{3.24}$$

$$\boldsymbol{p}' = \boldsymbol{p}. \tag{3.25}$$

These can be put in matrix form

$$Q = \begin{pmatrix} x \\ p \end{pmatrix} \to Q' = \begin{pmatrix} x' \\ p' \end{pmatrix} \tag{3.26}$$

with

$$L(f) = \begin{pmatrix} 1 & 0 \\ -M/f & 1 \end{pmatrix}, \quad U(D) = \begin{pmatrix} 1 & D/M \\ 0 & 1 \end{pmatrix}.$$

These results relate very naturally to the important observation by Bacry and Cadhilac that the phase changes on free propagation, $\exp[(\mathrm{i}\,p^2/2M)D]$, and on encountering a lens, $\exp[\mathrm{i}x^2/2f)M]$, are elements of the group $SL(2, \mathrm{R})$ of which the generators are

$$\tfrac{1}{2}\boldsymbol{r}^2, \quad \tfrac{1}{2}\boldsymbol{p}^2, \quad \tfrac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\boldsymbol{p}). \tag{3.27}$$

Since these are the generators the collection of any number of lenses and free propagations can be computed in any representation of the $SL(2, \mathrm{R})$ group, in particular the fundamental representation. The carrier space of the Bacry–Cadhilac two-dimensional representations is the set of light rays. Light rays thus furnish a realization of the group properties of an axially symmetric optical system.

When the lenses, and more generally the optical system is not axially symmetric the group is $Sp(4, \mathrm{R})$ generated by the 10 elements

$$r_a r_b, \quad p_a p_b, \quad (r_a p_b + p_b r_a), \quad 1 \leq a, b \leq 2. \tag{3.28}$$

If the lenses are slightly off-axis these groups get extended by a semidirect product with two- and four-dimensional translations.

### *3.4. Group-theoretic synthesis of paraxial systems*

One advantage of the recognition of the group property is synthesis: any $SL(2, \mathrm{R})$ matrix can be realized by a set of lenses; similarly any $Sp(4, \mathrm{R})$ matrix can be realized by at most a set of lenses. These straightforward conclusions solve a long-standing problem in optics.

It is clear from the analysis in this section that if the lens system has phase function $\theta(\boldsymbol{r})$ which has higher than quadratic terms, the ray transformations no longer obtain. There is still a linear transformation of the Wolf function but it will have a more complex integral transform character. Incidentally the nonquadratic terms for large angles for a thin lens may be modified by redesigning the lens to be strictly quadratic.

A simple picture of the rays in wave optics may be obtained by recognizing that the rays are locally normal to the surface over which the two-point function is in phase. So if the "coherence patch" for any point is large the rays in the vicinity would be parallel, if it is too small the rays would have rapidly varying directions; and conversely. If we have an illumination patch of small dimensions, which propagates illumination over large distances, it is easy to see that the rays of light would tend to become parallel as the distance increases. But this should, in turn, imply that the coherence patch would grow by propagation. Now, this remarkable result was noticed by van Cittert and by Zernike and deduced from the rigorous diffraction theory by Wolf. Our considerations above show that it is the wave optical equivalent of the rectilinear propagation of light.

## *4. Non-classical pencils*

### *4.1. Interference and diffraction*

So far we considered in the context of the conceptual synthesis of geometrical and wave optics those situations in which both of them gave identical descriptions. There are however cases in which they differ and these are cases where the wave effects explicitly appear. These involve the diffraction and interference effects, the

near zone behaviour of light rays and the behaviour of the Wolf function when lens aberrations begin to play a significant role. In each of these cases geometric optics fails.

The simplest case of interference is the two-slit pattern. It is known from elementary optics that this leads to a far-field pattern of equidistant parallel fringes with a separation

$$\Delta = \frac{D\lambda}{d}. \tag{4.1}$$

The two-point function for the coherent sum of the amplitudes is

$$\phi(\boldsymbol{r}) = \phi_1(\boldsymbol{r}) + \phi_2(\boldsymbol{r}) \sim \xi\{e^{i\delta_1(\boldsymbol{r})} + e^{i\alpha} e^{i\delta_2(\boldsymbol{r})}\}, \tag{4.2}$$

where $\xi$ is the contribution of either slit in propagated amplitude, $\delta_1(\boldsymbol{r})$, $\delta_2(\boldsymbol{r})$ are the phases due to propagation,

$$\delta_{1,2} = 2\pi\sqrt{(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{d})^2 + D^2}\,/\lambda = \frac{2\pi}{\lambda}\left(D \pm \frac{1}{2}\frac{\boldsymbol{r}\cdot\boldsymbol{d}}{D} + \frac{1}{2}\frac{\boldsymbol{r}^2}{D}\right), \tag{4.3}$$

and $\alpha$ is a (constant) phase difference between the illuminations of the slit. Then

$$\delta_1(\boldsymbol{r}_1) - \delta_2(\boldsymbol{r}_2) = \frac{2\pi}{\lambda D}\left\{\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot\boldsymbol{d} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right\}$$

Hence

$$\begin{aligned}
\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \xi^*\xi\Bigg[&\exp\left\{\frac{i2\pi}{\lambda D}\left(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)\right)\right\} \\
&+ \exp\left\{\frac{i2\pi}{\lambda D}\left(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right)\right\} \\
&+ \exp\left\{\frac{i2\pi}{\lambda D}\left(\alpha + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot\boldsymbol{d} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)\right)\right\} \\
&+ \exp\left\{-\frac{i2\pi}{\lambda D}\left(\alpha + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot\boldsymbol{d} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right)\right\}\Bigg] \\
= \xi^*\xi\Bigg[&\exp\left\{\frac{2\pi i}{\Delta}\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2d}(\boldsymbol{r}_2 - \boldsymbol{r}_1)\right\} + \exp\left\{\frac{2\pi i}{\Delta}\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2d}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right\} \\
&+ \exp\left\{\frac{2\pi i}{\Delta}\left(\frac{\alpha}{d} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{d}\right)\right\} \\
&+ \exp\left\{-\frac{2\pi i}{\Delta}\left(\frac{\alpha}{d} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} + \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{d}\right)\right\}\Bigg].
\end{aligned} \tag{4.4}$$

Consequently

$$W(\boldsymbol{r}, \boldsymbol{p}) = \xi^*\xi\left\{\delta\left(\boldsymbol{p} - \frac{\boldsymbol{r} + \frac{1}{2}d}{\Delta d}\right) + \delta\left(\boldsymbol{p} + \frac{\boldsymbol{r} - \frac{1}{2}d}{\Delta d}\right) + 2\cos\left(\frac{\boldsymbol{r}}{\Delta} - \alpha\right)\delta\left(\boldsymbol{p} - \frac{\boldsymbol{r}}{\Delta d}\right)\right\}. \tag{4.5}$$

These correspond to *three* pencils of light, one from each of the slits and one appearing to proceed from the centre of the slits; recall that this is a far-field approximation when we can deal with scaling. The first two have only pradipa rays but the third one contains both pradipa and tamasic rays; however at no time are the intensities nonpositive. It would be worthwhile to demonstrate that in the wave theory the tamasic rays correspond to a *backflow of light energy*.

Most of the calculations so far given we have studied the rays of light in the far zone. Since our introduction of light rays and the Wolf function is exact it should be possible to deal with the near zones also. In the near zones we would have rapid variation of the amplitudes and hence the two-point correlation functions would vary rapidly. By virtue of the above equation the rays would lie along surfaces of constant Wolf function. This implies, in turn, that light rays will not travel along straight lines but would bend appreciably to accommodate the effective rapid variation imposed on the Wolf function. For the two-slit pattern we saw that the far-field illumination appeared to come from three pencils: the third pencil beginning in the middle of the two slits must have rays that were contributed by the two slits which bend in a suitable fashion. A still simpler case is given by the case of pure reflection. Take a wide beam with asymptotically parallel rays; asymptotically the emergent beam also has parallel rays obeying the simple laws of reflection. But what about the rays in the region near the reflecting surface? In this region both the beams contribute and we do have a bending of the rays of light. Bending of light rays arise wherever the intensity varies significantly.

### *4.2. Aberration*

Strictly speaking if the paraxial system contains terms beyond the quadratic in the phase change there would be "aberrations" in geometric optics. Even for paraxial beams point objects would not have point images. These aberrations have been classified in geometrical optics. But in the corresponding wave optics formulation there is no strict correspondence: While a pencil will still go into a pencil, it is no larger by the transformations of individual rays. Instead we have a new pencil which is an integral transform of the original pencil. The strict correspondence between geometrical optics and wave optics is no longer valid.

Nevertheless the usefulness of the geometric approximation is violated only if the cubic and higher order terms in the phase shift vary by appreciable amounts over the scale of the coherence patch. Such domains would simulate vertices of supplementary pencils which provide corrections to geometrical optics. This was seen above in the context of wave like (interference and diffraction) effects. The larger

the coherence patch, the greater the importance of aberrations and wave effects and greater the departure from classical geometrical optics behaviour.

## 5. *Intensity fluctuations*

### 5.1. *Gaussian beams*

So far the discussion has been about the kinematics of wave optics. The intensity of illumination is a normalizing constant in these discussions. There are however, reasons to expect this to be an idealization, if the light derives from many independent sources whose individual contributions have random relative phases. The net amplitude would then be a member of an esemble of complex amplitudes whose real and imaginary parts are independently distributed with zero-mean Gaussians of equal dispersion:

$$p(A)\,\mathrm{d}^2A = \frac{1}{\pi\mathscr{I}}\exp\left\{-\frac{A^*A}{\mathscr{I}}\right\}\mathrm{d}^2A. \tag{5.1}$$

One could then ask for direct experimental verification of such a Gaussian distribution for natural light; and ask for departure from such distributions for coherent sources, in particular laser sources.

We recognize that for Gaussian light the distribution of the intensity,

$$I = A^*A,$$

is given by the Rayleigh distribution

$$p(I)\,\mathrm{d}I = \frac{1}{\mathscr{I}}\mathrm{e}^{-I/\mathscr{I}}\,\mathrm{d}I. \tag{5.2}$$

For such a distribution the mean intensity $\langle I\rangle$ and the mean square intensity $\langle I^2\rangle$ are related by the simple expression:

$$\langle I^2\rangle = 2\mathscr{I}^2 = 2\langle I\rangle^2,$$

so that the intensity fluction is

$$\langle I^2\rangle - \langle I\rangle^2 = (\Delta I)^2 = \langle I\rangle^2. \tag{5.3}$$

If there were $N$ uncorrelated intensities $I_1, I_2, \ldots$ then the total intensity

$$I = I_1 + I_2 + \ldots + I_N$$

has a fluctuation

$$(\Delta I)^2 = (\Delta I_1)^2 + \ldots + (\Delta I_N)^2 = N(\Delta I_1)^2 = \frac{1}{N}\langle I\rangle^2. \tag{5.4}$$

So if we knew that the light beam was Gaussian but contained a number of uncorrelated "modes" the measurement of the fluctuation would enable us to determine the effective number of contributing modes.

For statistical optical fields we still have the field of illumination defined by the two-point correlation function (assuming stationarity):

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; t) = \langle \phi^*(\boldsymbol{r}_1, t+\tau)\ \phi(\boldsymbol{r}_2, \tau)\rangle. \tag{5.5}$$

Since $\Gamma$ is a Hermitian symmetric form under rather general conditions we can have an eigen-mode decomposition

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \sum_\alpha \eta_\alpha\ \varphi_\alpha^*(\boldsymbol{r}_1)\ \varphi_\alpha(\boldsymbol{r}_2)\ \mathrm{e}^{-\mathrm{i}\omega_\alpha t}. \tag{5.6}$$

With the quantities $\varphi_\alpha$ being normalized and $\eta_\alpha$ being suitable positive numbers. By expanding

$$\phi(\boldsymbol{r}, t) = \sum_\alpha c_\alpha\ \varphi_\alpha(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\omega_\alpha t} + \sum_\beta b_\beta\ \chi_\beta(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\omega_\beta t} \tag{5.7}$$

in terms of a complete set of orthonormal modes $\{\varphi_\alpha, \chi_\beta\}$, the two-point function implies

$$\langle c_\alpha^* c_\beta\rangle = \eta_\alpha \delta_{\alpha\beta}, \quad \langle b_\alpha^* b_\beta\rangle = 0. \tag{5.8}$$

It then follows that the modes $\chi_\beta$ are not excited at all and may be omitted from further discussion. The statistical wave field could thus be specified by the correlated distribution of two random complex variables $c_\alpha$. A general Gaussian field is then a multivariate Gaussian in the complex variables $c_\alpha$. The correlated distribution of any finite number of distinct amplitudes can be obtained by integrating out the irrelevant amplitudes and thus obtaining the marginal distributions. In particular if the mean intensities at the points 1 and 2 have the same mean, then the intensity correlation can be related to the two-point amplitude correlation by

$$\langle I_1(t_1)\ I_2(t_2)\rangle = \langle I_1\rangle^2 + |\Gamma_{12}(t_1 - t_2)|^2. \tag{5.9}$$

Hence the two-point correlation function (or, rather its absolute magnitude) can be determined by intensity interferometry. We note that the existence of a positive intensity correlation does not necessarily depend on any quantum property but applies equally well to classical statistical wave fields. But surely there are quantum effects, but we should look elsewhere for them. But before deriving this let us remind ourselves about the vector nature of the electromagnetic field.

If $\boldsymbol{A}(\boldsymbol{r}, t)$ is the positive frequency part of the vector potential, the related two-point function is a tensor

$$\langle A_j^*(\boldsymbol{r}_1, t+\tau)\ A_k(\boldsymbol{r}_2, \tau)\rangle = \Gamma_{jk}(\boldsymbol{r}_1, \boldsymbol{r}_2, t). \tag{5.10}$$

For the free electromagnetic field the vector potential is transverse: consequently there need be no gauge ambiguity and the tensor two-point correlation satisfies

$$\frac{\partial}{\partial x_{1j}}\Gamma_{jk}=0, \quad \frac{\partial}{\partial x_{2k}}\Gamma_{jk}=0. \tag{5.11}$$

For higher order functions representing many-point correlation similar results hold. Instead of the vector potential sometimes people use the electric field: the correlation functions would then have similar properties but different physical dimensions.

Having seen how the vector electromagnetic field may be handled we may, for many practical reasons, deal with a "scalar electromagnetic field".

### 5.2. Quantum ensembles

We recognize that traditional optics dealt with only various aspects of two-point functions: either it involved photometry or study of interference or diffraction effects. Since the two-point function is always a Hermitian nonnegative kernel in both classical and in quantum theory it would suggest that there is essentially *no quantum property* that could be discerned *from traditional optics*. This is indeed so! We shall obtain it as a corollary in the sequel.

To fix our ideas we may deal with only a single mode of the wave field:

$$\boldsymbol{A}(\boldsymbol{r}, t) \sim a\,\boldsymbol{A}_1(\boldsymbol{r}, t), \tag{5.12}$$

where $\boldsymbol{A}_1(\boldsymbol{r}, t)$ is a definite normalized mode and we need the distribution of the dynamical variable $a$, which for a quantum theory is an annihilation operator, though it is a complex-valued random variable in a classical theory. For the quantum theory the operator $a$ and its adjoint $a^\dagger$ satisfy the communication relation

$$aa^\dagger - a^\dagger a = 1. \tag{5.13}$$

It follows that every representation of $a$, $a^\dagger$ is infinite dimensional. It can then be shown that there is a unique state $|0\rangle$ on which $a$ vanishes:

$$|0\rangle = 0. \tag{5.14}$$

The state $|0\rangle$ is normalized:

$$\langle 0|0\rangle = 1.$$

It is now possible to construct a continuously infinite number of normalized states $|z\rangle$ which are eigenstates of $a$:

$$a|z\rangle = z|z\rangle, \quad \langle z|z\rangle = 1. \tag{5.15}$$

They are called coherent states and given by

$$|z\rangle = \exp(za^\dagger - z^* a)\ |0\rangle. \tag{5.16}$$

These states have the property that the expectation values of any operator $(a^\dagger)^m a^n$ is simple:

$$\langle z|(a^\dagger)^m a^n|z\rangle = (z^*)^m z^n. \tag{5.17}$$

So as long as we define operators to be normal-ordered (all creation operators to the left of all the annihilation operators) the expectation value would be given as if both the creation and the annihilation operators are respectively specified to be $z^*$ and $z$. So if a statistical state $\rho$ is expressed as the weighted average

$$\rho = \int \Phi(z)\,|z\rangle\langle z|\,\mathrm{d}^2 z \tag{5.18}$$

of the projection to the coherent states, any normal ordered operator $\chi(a^\dagger, a)$ can be given the value

$$\begin{aligned}\langle \chi(a^\dagger, a)\rangle &= \mathrm{Tr}(\chi(a^\dagger, a)\rho)\\ &= \int \chi(z^*, z)\ \Phi(z)\ \mathrm{d}^2 z.\end{aligned} \tag{5.19}$$

Thus $\Phi(z)$ acts as the real valued probability distribution for the complex random variable $z$.

The remarkable fact is that this is always possible with a suitable enlargement of the weight $\Phi(z)$. Clearly $\Phi(z)$ cannot be the nonnegative measure it would be in classical theory; we expect some departure from classical theory. This diagonal representation theorem tells us that quantum optics is classical statistical optics with the probability measure replaced by a diagonal distribution function.

To compute $\Phi(z)$ we first determine its characteristic function

$$\begin{aligned}\chi(\xi) &= \int \exp(\xi z^* - \xi^* z)\ \Phi(z)\ \mathrm{d}^2 z\\ &= \int \langle z|\mathrm{e}^{\xi a^\dagger}\,\mathrm{e}^{-\xi a}|z\rangle\ \Phi(z)\ \mathrm{d}^2 z\\ &= \mathrm{Tr}\left(\mathrm{e}^{\xi a^\dagger}\,\mathrm{e}^{-\xi^* a}\rho\right).\end{aligned} \tag{5.20}$$

But

$$\chi(\xi) = \mathrm{Tr}\left(\mathrm{e}^{-\xi^* a}\, \mathrm{e}^{\xi a^\dagger}\, \mathrm{e}^{\xi^*\xi}\, \rho\right)$$

$$= \mathrm{e}^{\xi^*\xi}\, \mathrm{Tr}\left(\mathrm{e}^{-\xi^* a} \int |z\rangle\langle z|\, \mathrm{e}^{\xi a^\dagger}\, \rho\right) \frac{\mathrm{d}^2 z}{\pi}$$

$$= \mathrm{e}^{\xi^*\xi} \int \mathrm{e}^{(\xi z^* - \xi^* z)}\, k(z)\, \mathrm{d}^2 z$$

$$= \mathrm{e}^{\xi^*\xi}\, \Omega(\xi), \tag{5.21}$$

where $\Omega(\xi)$ is the characteristic function of

$$k(z) = \frac{1}{\pi} \langle z | \rho | z \rangle. \tag{5.22}$$

In deriving the above relation we have used the easily derived identity

$$\frac{1}{\pi} \int |z\rangle\, \langle z|\, \mathrm{d}^2 z = 1. \tag{5.23}$$

Since $\Omega(\xi)$ is easily calculated, so is $\chi(\xi)$ and by inverting it we could get the distribution function $\Phi(z)$.

It is quite clear that $\Phi(z)$ cannot be always a true probability function since we can have excited states in which only a finite number of photons exist. These states are not vacuum states, yet not expressible in terms of pointwise positive function $\Phi(z)$. The only positivity condition satisfied by $\Phi(z)$ is of the form

$$\int |f(z)|^2\, \Phi(z)\, \mathrm{d}^2 z > 0 \tag{5.24}$$

for all entire functions $f(z)$. By taking traces of the expression

$$\rho = \int \Phi(z)\, |z\rangle\, \langle z|\, \mathrm{d}^2 z, \tag{5.25}$$

we get

$$\int \Phi(z)\, \mathrm{d}^2 z = 1. \tag{5.26}$$

Any $\Phi(z)$ which satisfies these two relations give an allowed quantum ensemble.

In place of a single mode ensemble if we chose an arbitrary number of modes, the coherent states and distribution functions respectively generalize:

$$|z\rangle \to \quad |z_1, z_2, \ldots\rangle,$$

$$\Phi(z) \to \Phi(z_1, z_2, \ldots),$$

$$\int |f(z)|^2 \, \Phi(z) \, \mathrm{d}^2 z > 0$$

$$\to \int |f(z_1, z_2, \ldots)|^2 \, \Phi(z_1, z_2, \ldots) \, \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2 \cdots > 0. \tag{5.27}$$

We can now take expectation values of arbitrary normalized products of field operators $F(\phi^\dagger(.), \phi(.)) \to F(v^*(.), v(.))$:

$$\phi(\boldsymbol{r}, t) \to v(\boldsymbol{r}, t) = \sum z_\alpha \varphi_\alpha(\boldsymbol{r}, t), \quad \phi^+(\boldsymbol{r}, t) \to v^*(\boldsymbol{r}, t) = \sum z_\alpha^* \varphi_\alpha^*(\boldsymbol{r}, t). \tag{5.28}$$

The vector fields $\boldsymbol{A}(\boldsymbol{r}, t)$ would have vector mode functions $\varphi_\alpha$ and hence (complex!) vector representatives $\boldsymbol{v}$, $\boldsymbol{v}^*$.

We have thus achieved the synthesis of statistical classical wave optics and quantum optics. It is not necessary to do quantum optics ab initio for those problems in which statistical classical wave optics theory has been worked out. Apart from the replacement of the probability distribution over complex analytic signals by the diagonal distribution $\Phi(z)$ there are no quantum corrections! We see thus the two fundamental theorems of optics which serve the unification:

(I) *For traditional optics making use of only intensities and two-point functions, quantum optics and classical wave optics predictions are indistinguishable.*

(II) *Every quantum ensemble is characterized by a diagonal distribution function in terms of which the results of classical wave ensembles can be transcribed to give the exact predictions of quantum optics, normal ordering of operators being understood.*

In particular the equations of motion for the multiple correlation functions and the geometric results of propagation through passive systems are unaltered.

To find typically quantum behaviour we should therefore look for possible consequences of the failure of pointwise positivity.

### *5.3. Photocounts: sub-Poisson counting distributions*

Earlier we have remarked that for the Gaussian ensemble (which is pointwise positive!) the intensity correlation is positive:

$$\langle I_1 I_2 \rangle - \langle I_1 \rangle \langle I_2 \rangle = |\Gamma_{12}|^2. \tag{5.29}$$

If on the other hand we took a quantum ensemble of one-photon states only, the

normal ordered quantity $I_1 I_2 \sim a^\dagger a^\dagger a a$ will have vanishing expectation value while $I_1$ and $I_2 \sim a^\dagger a$ would have nonzero value. So we have in this case an anticorrelation! But for any classical distribution this is not possible. Thus while Bose particles have positive correlations, it is nevertheless possible for intensity correlations to be negative. Furthermore such negative correlations are typically quantum effects, somewhat reminiscent of negative ray densities being typical of wave optic ray pencils.

A more interesting case where quantum effects show up is photon counting. If the counter registering a count is proportional to the intensity $I(t)$, say $\lambda I(t)$, then the number of counts in the interval $0 < t < T$ would be a random variable $n$ with the distribution

$$p(n) = \frac{\mu^n}{n!} \mathrm{e}^{-\mu}, \tag{5.30}$$

with

$$\mu = \chi \int_0^T I(t)\, \mathrm{d}t.$$

In particular if the intensity is a constant:

$$p(I)\, \mathrm{d}I = \delta(I - I_0)\, \mathrm{d}I, \tag{5.31}$$

then the photocount distribution is the Poisson distribution:

$$\begin{aligned} \pi(n) &= \int p(n, I)\, p(I)\, \mathrm{d}I \\ &= \frac{(\bar{n})^n}{n!} \mathrm{e}^{-\bar{n}}, \end{aligned} \tag{5.32}$$

with a counting variance

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle = \bar{n}. \tag{5.33}$$

For a Gaussian amplitude example for which

$$p(I)\, \mathrm{d}I = \frac{1}{\mathscr{I}} \mathrm{e}^{-I/\mathscr{I}}\, \mathrm{d}I, \tag{5.34}$$

the photocount distribution becomes the Bose distribution:

$$\pi(n) = (1 - x) x^n = \frac{1}{\bar{n} + 1} \cdot \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n, \tag{5.35}$$

with a counting variance

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = 2\bar{n}. \tag{5.36}$$

This variance is double the Poisson variance. Any innovation on the Poisson distribution by making a random variable of the Poisson parameter would increase the variance from the Poisson fluctuation. The Gauss–Rayleigh case illustrated above is an example.

On the other hand a quantum ensemble can have a lower variance. For example if we have a state of a fixed number of photons, the variance is zero! More generally any sub-Poisson counting statistics is a definite indication of a quantum ensemble. A wide class of such sub-Poisson states is provided by the so-called "squeezed states".

## 6. *Concluding remarks*

The classical and quantum ensembles share the formal structure but positivity conditions become more subtle in the quantum case. But subject to this understanding the already developed classical statistical optics provides the proper framework for quantum optics also. This is reminiscent of the use of light rays for wave optics.

Thus we have shown how to unify geometrical ray optics, classical wave optics and quantum optics. In this expository presentation we could not cover many fascinating aspects of statistical optics like speckle and holography nor deal with aberrations or active systems. Even in the selected topics we have not been able to cover the quantum field theory of light rays, i.e. of correlation between light rays. Perhaps these are better reserved for the celebration of Hiroomi Umezawa's seventieth birthday.

Manuscript received October 7, 1985