Zero-energy modes, charge conjugation, and fermion number

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States with a half-integer fermion number occur when a fermionic field coupled to a soliton possesses a zero mode. This paper spells out the circumstances under which one can retain an integer fermion number as also a charge-conjugation-invariant ground state. It is necessary to make the representation reducible but it is kept irreducible by introducing an additional operator.

The quantum theory of the free Fermi-Dirac field was developed by Jordan and Wigner[1] in 1928 and may be outlined as follows: Expand the field in terms of (idealized) stationary states as the (generalized) sum

$$\psi(x,t)=\sum_{n=0}^{\infty}[a_n u_n(x)e^{-iE_n t}+b_n^\dagger v_n(x)e^{iE_n t}] \ ,$$

$$E_n>0 \ .$$

Then the anticommutation relations

$$\{\psi(x,t),\psi^\dagger(y,t)\}=\delta(x-y) \ ,$$

$$\{\psi(x,t),\psi(y,t)\}=\{\psi^\dagger(x,t),\psi^\dagger(y,t)\}=0 \ ,$$

take the simple form

$$\{a_n,a_{n'}^\dagger\}=\{b_n,b_{n'}^\dagger\}=\delta_{n,n'} \ ,$$

$$\{a_n,a_{n'}\}=\{b_n,b_{n'}\}=\{a_n,b_{n'}\}=\{a_n,b_{n'}^\dagger\}=0 \ .$$

The fermion-number operator N may be defined by the commutation with ψ and $\psi^\dagger$, and its fiducial value for any state $|\ \rangle$ as

$$[N,\psi]=-\psi, \quad [N,\psi^\dagger]=\psi^\dagger \ ,$$

$$N|\ \rangle=\nu|\ \rangle, \quad |\nu|<\infty \ .$$

It then follows that the spectrum of N is

$$N=\nu,\nu\pm1,\nu\pm2,\ldots,$$

by virtue of the commutation relations.

For the standard representation, $|\ \rangle$ is the vacuum state $|0\rangle$, and $\nu=0$. We have

$$N|0\rangle=0, \quad N=0,\ \pm1,\ \pm2,\ldots,$$

and an operator realization is

$$N_1=\tfrac{1}{2}\int(\psi^\dagger\psi-\psi\psi^\dagger)\,dx \ .$$

The anticommutation relations admit of the automorphism of charge conjugation:

$$\psi(x,t)\to C\psi^*(x,t) \ ,$$

$$\psi^\dagger(x,t)\to\psi^T(x,t)C^\dagger \ ,$$

$$a\rightleftarrows b, \quad a^\dagger\rightleftarrows b^\dagger \ .$$

Under this automorphism we want

$$N\to -N \ .$$

The standard representation above admits of this automorphism as an inner automorphism.[2]

It was discovered that nonstandard (amyriotic) representations of the Fermi field exist[3] in all of which the operator N_1 is unbounded. Thus N_1 is useless, and we may define N_2 as that which "differs from N_1 by an infinite constant." By choosing N_2 to be zero for a chosen state of the system the spectrum of N_2 coincides with that of N_1 in the standard state. It is to be noted that none of these irreducible nonstandard representations admits the charge-conjugation automorphism. If we want to have a nonstandard (amyriotic) representation which includes the charge conjugation as an inner automorphism, we need a reducible representation.

In addition to such nonstandard representations a new phenomenon comes about when the field is not free but in suitable external fields. For example, when a Dirac field is in a strong Coulomb potential which has binding energies large enough for the negative and positive energies to cross, the energy levels are no longer exclusively real, and the standard representations are not valid.

More recently, Jackiw and Rebbi[4] have shown that in the approximation in which a Dirac field is in a soliton background, zero-energy modes are present, so that

$$\psi(x,t)=\sum_n[a_n u_n(x)e^{-iE_n t}+b_n v_n(x)e^{+iE_n t}]+c\,\omega(x) \ .$$

With $\omega(x)$ being a zero-energy solution, the irreducible representation admitting of charge conjugation has states with a half-integer fermion number. This arising of a half-integer fermion number in a theory in which all fields have integer fermion numbers is, as they rightly point out, truly remarkable. They also show that this puzzling result obtains whenever a Dirac equation possesses a nondegenerate fermion-number self-conjugate zero-energy bound state.

In this paper we wish to study the question as to how in a theory where fermion numbers are finite we have such a nonstandard representation, and under what circumstances could we obtain the quantization of a Fermi field with zero-energy modes in which a vacuum with a zero fermion number obtains. For simplicity of presentation, we discuss the one-dimensional example studied by Jackiw and Rebbi[4] and then generalize.

In a theory of a scalar field ϕ and a spinor field ψ with a

Lagrangian density of the form

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}(\lambda^2/g^2)(1-g^2\phi^2)^2+i\bar{\psi}\gamma^\mu\partial_\mu\psi-G g\bar{\psi}\phi\psi\ ,$$

a classical solution when ψ is absent is

$$\phi_c(x)=(1/g)\tanh\lambda x\ .$$

If we consider the Dirac equation in this classical background field,

$$i\gamma^\mu\partial_\mu\psi-Gg\phi_c\psi=0\ ,$$

it has one normalizable zero-energy solution $\omega(x)$. Given this we could consider a mode expansion

$$\psi(x,t)=\sum_n[a_n u_n(x)e^{-iE_n t}+b_n v_n(x)e^{-iE_n t}]+c\omega(x)\ ,$$

as mentioned above. The standard charge-conjugation-odd fermion charge density is

$$\rho_1(x)=\tfrac{1}{2}[\psi^\dagger(x)\psi(x)-\psi(x)\psi^\dagger(x)]\ ,$$

which gives the fermion number

$$\begin{aligned}N_1&=\int\rho_1(x)\,dx\\&=\tfrac{1}{2}\sum_n(a_n^\dagger a_n-a_n a_n^\dagger)-\tfrac{1}{2}\sum_n(b_n^\dagger b_n-b_n b_n^\dagger)+\tfrac{1}{2}(c^\dagger c-cc^\dagger)\\&=\sum_n(a_n^\dagger a_n-b_n^\dagger b_n)+c^\dagger c-\tfrac{1}{2}\ .\end{aligned}$$

It follows that with this choice the zero-energy states of the fermion in the solitonic field have a fermion number $+\frac{1}{2}$ or $-\frac{1}{2}$, accordingly as $c^\dagger$ or c annihilates the state. This is the discovery of Jackiw and Rebbi; they further point out that this situation is quite general and is applicable to more general models and to three spatial dimensions: As long as there is a zero-energy mode half-integral fermion numbers obtain. This model is charge-conjugation invariant.

The fermion numbers of all states including the nonzero-energy modes are half-integral. For example, if there is a second fermion added to the $N_1=\frac{1}{2}$ state, its fermion number would be $\frac{3}{2}$, while if we took the $N_1=-\frac{1}{2}$ state and added a fermion of nonzero energy to it, we would get $N_1=\frac{1}{2}$ for that new state. Thus one unique zero-energy mode makes all states have a half-integer fermion number and become doubly degenerate.

In this irreducible representation the no-fermion old soliton state is not included: If it were included, we would encounter a contradiction. If N_3 is a number operator which has the property of being odd under charge conjugation and if the soliton background state $|0\rangle$ is unique, then it must follow that

$$N_3|0\rangle=0\ .$$

Being a number operator, N_3 must satisfy

$$[N_3,\psi]=-\psi,\quad [N_3,\psi^\dagger]=\psi^\dagger\ ,$$

and hence in any state derived from $|0\rangle$ by any number of actions of ψ and $\psi^\dagger$ this number operator has integral eigenvalues. It follows that in any representation of the Fermi field for which the primary soliton background state is cyclic the fermion number must be an integer.

We saw that in any irreducible charge-conjugation-invariant representation the fermion numbers were half-integral. As long as we want charge-conjugation invariance, we must realize the Fermi fields in a reducible manner.

This recognition is strengthened by the following observation: Since both N_3 and N_1 satisfy both the commutation relations and oddness under charge conjugation, the difference between them must be a neutral element also odd under charge conjugation. If

$$\nu=(N_3-N_1)\ ,$$

then

$$[\nu,\psi]=0,\quad [\nu,\psi^\dagger]=0\ ,$$

and since $\nu\to-\nu$ under charge conjugation, ν is not a numerical constant but must take a half-integral value when acting on the soliton background vacuum. On charge conjugation the vacuum goes into itself, and hence N_3 must remain zero. But N_1, which was $-\frac{1}{2}$, now goes to $+\frac{1}{2}$; consequently ν must now change from $+\frac{1}{2}$ to $-\frac{1}{2}$.

Since these are the only values that ν need take, we recognize that it is very similar to the charge-conjugation-invariant Fermi oscillator occupation number

$$\tfrac{1}{2}(c^\dagger c-cc^\dagger)\ .$$

Without any loss of generality, we can construct operators d and $d^\dagger$ which connect the state $\nu=-\frac{1}{2}$ with $\nu=+\frac{1}{2}$ and vice versa, and

$$\nu=\tfrac{1}{2}(dd^\dagger-d^\dagger d),\quad \{d,d^\dagger\}=1,\quad d^2=d^{\dagger 2}=0\ .$$

By means of a Klein construction[5] we can make

$$\{d,a\}=\{d,b\}=\{d,c\}=\{d,a^\dagger\}=\{d,b^\dagger\}=\{d,d^\dagger\}=0\ .$$

Thus the representation space of ψ and $\psi^\dagger$, which admits of charge conjugation, is a reducible representation; it may be realized as an irreducible representation of the extended system $\psi,\psi^\dagger$, $d,d^\dagger$, and the number operator N_3 as the number operator for this extended system:

$$N_3=\sum_n(a_n^\dagger a_n-b_n^\dagger b_n)+(c^\dagger c-d^\dagger d)\ .$$

It would be instructive to examine the full set of states of this extended system with only integral fermion numbers realizing both charge conjugation and a cyclic vacuum. Since the realization of ψ is not irreducible, a larger degeneracy is expected than in the Jackiw-Rebbi realization without a cyclic vacuum. The vacuum has zero energy and is unique; but there are three other zero-energy states, one with $N_3=1$, $\nu=\frac{1}{2}$, one with $N_3=-1$, $\nu=-\frac{1}{2}$, and one with $N_3=0$. (Note that the vacuum state and the new $N_3=0$, state are, respectively, even and odd under charge conjugation, somewhat in the spirit of K_2^0 and K_1^0 neutral kaon states.[6]) All finite-energy states, built on these four zero-energy states, are fourfold degenerate.

It is also necessary to point out that starting out with the soliton background state $|0\rangle$, we cannot build a charge-conjugation-invariant set of states with ψ and $\psi^\dagger$ acting any number of times on $|0\rangle$. This follows from the observation that $(c^\dagger c-cc^\dagger)$ being odd under charge conjugation must vanish on the vacuum state, but it cannot, since its eigenvalues are ± 1. Thus we must include the additional operators which make the realization of ψ and $\psi^\dagger$ reducible.

The spectrum of states is to be contrasted with the Jackiw-Rebbi theory[4] in which every (finite- or zero-) energy state is twofold degenerate.

The considerations apply to any theory in which a zero-energy fermion state obtains in a background classical state as Jackiw and Rebbi point out: Our vacuum-cyclic representations also obtain in each of these cases. The demonstration is elementary and therefore not reproduced here.

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