DYNAMICAL MAPS AND NONNEGATIVE PHASE-SPACE DISTRIBUTION FUNCTIONS IN QUANTUM MECHANICS

R. JAGANNATHAN, R. SIMON, E.C.G. SUDARSHAN and R. VASUDEVAN
The Institute of Mathematical Sciences, Madras 600113, India

Received 4 December 1986; accepted for publication 11 December 1986

Using the formalism of dynamical maps it is shown that if a quantum measurement process is to be described in terms of a nonnegative phase-space distribution obtained by smoothing the Wigner distribution of the quantum state, then the smoothing kernel characterizing the measuring apparatus cannot be an arbitrary Wigner distribution.

The Wigner distribution method provides a phase-space description of quantum mechanics which resembles in several aspects the classical description [1]. It consists of associating with a quantum state described by a density matrix $\hat{\rho}$ a c-number distribution function $W(q, p)$ over the classical phase space through a well-defined procedure. Then if hermitian operators are associated with classical dynamical variables in accordance with the Weyl correspondence [2], the expectation value of a hermitian operator $\hat{O}$ in a state $\hat{\rho}$ is given *exactly* by the phase-space average of the corresponding classical dynamical variable $O(q,p)$ with $W(q,p)$ as the weight function:

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \int\int \mathrm{d}q\,\mathrm{d}p\; W(q,p)O(q,p)\;. \qquad (1)$$

The expression on the right-hand side of (1) is formally identical to the corresponding classical expression. However, as is well known, $W(q, p)$ is not a true probability distribution over the phase space, since it is not pointwise nonnegative in general. In order to make this expression for averages look truly classical-like, one goes from the Wigner distribution $W(q, p)$ to a corresponding "smoothed Wigner distribution" $W_s(q, p)$ which is nonnegative over the entire phase space, or equivalently alters the correspondence rule suitably [3–12]. The obvious way to obtain such a nonnegative distribution, and the one invariably used, is to convolve the Wigner distribution of the state with another Wigner distribution $W_A(q, p)$; the convolution of *any two* Wigner distributions results in a phase-space function which is pointwise nonnegative. The convolution process itself is then understood to be the result of the coarse graining which is invariably affected by the measuring apparatus, or present in any semiclassical description of quantum processes.

The purpose of this letter is to show that this smoothing procedure can be profitably interpreted as a dynamical map [13–17]. This interpretation leads in a natural way to the following important question: can an arbitrary Wigner distribution be used as the smoothing kernel $W_A(q,p)$? We show that the answer to this question is in the negative.

We consider a system with one degree of freedom; generalization to many degrees of freedom is straightforward. Further, we choose units such that $\hbar = 1$. The Wigner distribution is related to the density matrix through the following invertible transformation:

$$W(q,p) = \frac{1}{2\pi}\int \mathrm{d}\sigma\, \exp(\mathrm{i}\sigma p) \times \langle q - \tfrac{1}{2}\sigma|\hat{\rho}|q + \tfrac{1}{2}\sigma\rangle\;. \qquad (2)$$

Since $\hat{\rho}$ is hermitian, has unit trace, and is nonnegative, one deduces from (2)

$$W(q,p)^* = W(q,p)\;,$$

$$\mathrm{Tr}(\hat{\rho}) = \int\int \mathrm{d}q\,\mathrm{d}p\; W(q,p) = 1\;,$$

$$\mathrm{Tr}(\hat{\rho}\hat{\rho}') = \int\int \mathrm{d}q\,\mathrm{d}p\; W(q,p)\,W'(q,p) \geqslant 0\;, \qquad (3)$$

where $\hat{\rho}'$ is an arbitrary density matrix and $W'(q, p)$ is its Wigner distribution.

It is convenient to combine the real q, p into a complex quantity $z = (q + ip)/\sqrt{2}$ and write $2\pi W(q, p)$ as $\omega(z)$.
Thus $\int\int dq\, dp/2\pi$ will be denoted $\int d^2z/\pi$ as usual [18]. Further, for convenience of formal manipulations we will rewrite (2) in terms of the basic hermitian operators $\hat{\Omega}(z)$ of the Weyl correspondence [19]:

$$\hat{\Omega}(z) = \int \frac{d^2\xi}{\pi} \exp(z\xi^* - z^*\xi)\hat{D}(\xi) ,$$

$$\xi = (\tau + i\sigma)/\sqrt{2};$$

$$\hat{D}(\xi) = \exp(\xi\hat{a}^\dagger - \xi^*\hat{a}), \quad \hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}. \tag{4}$$

Among the many interesting properties of $\hat{\Omega}(z)$ and $\hat{D}(z)$ we note only the following:

$$\mathrm{Tr}[\hat{\Omega}(z)\hat{\Omega}(\xi)] = \pi\delta(z - \xi) ,$$

$$\hat{D}(\xi)^\dagger\hat{\Omega}(z)\hat{D}(\xi) = \hat{\Omega}(z - \xi) . \tag{5}$$

In terms of $\hat{\Omega}(z)$ we have

$$\omega(z) = 2\pi W(q, p) = \mathrm{Tr}[\hat{\rho}\hat{\Omega}(z)] ,$$

$$\hat{\rho} = \int \frac{d^2z}{\pi} \omega(z)\hat{\Omega}(z) . \tag{6}$$

The fact that the Wigner distribution cannot be pointwise nonnegative for every state can be seen most easily by taking in (3) a pair of orthogonal states. In fact the only pure states for which the Wigner distribution does not assume a negative value anywhere in the phase space turn out to be those with gaussian wavefunctions [20,21].

As already noted, the smoothed Wigner distribution is defined as the convolution of the Wigner distribution under consideration using as kernel a Wigner distribution $\omega_A(z)$ characteristics of the measuring apparatus:

$$\omega_s(z) = \int \frac{d^2\xi}{\pi} \omega_A(z - \xi)\omega(\xi)$$

$$\equiv \int \frac{d^2\xi}{\pi} \omega(z - \xi)\omega_A(\xi) . \tag{7}$$

For *any two* Wigner distributions $\omega(z)$ and $\omega_A(z)$ the smoothed distribution $\omega_s(z)$ is a pointwise nonnegative function. This becomes transparent if we rewrite (7) in the form

$$\omega_s(z) = \mathrm{Tr}[\hat{\rho}_A\hat{D}(z)\hat{\mathscr{P}}\hat{\rho}\hat{\mathscr{P}}\hat{D}(z)^\dagger]$$

$$= \mathrm{Tr}[\hat{\rho}\hat{D}(z)\hat{\mathscr{P}}\hat{\rho}_A\hat{\mathscr{P}}\hat{D}(z)^\dagger] , \tag{8}$$

where $\hat{\mathscr{P}}$ is the parity operator. That (7) and (8) are equivalent can be checked by substituting for $\hat{\rho}_A$ and $\hat{\rho}$ from (6) and making use of (5). For instance, we have

$$\mathrm{Tr}[\hat{\rho}_A\hat{D}(z)\hat{\mathscr{P}}\hat{\rho}\hat{\mathscr{P}}\hat{D}(z)^\dagger]$$

$$= \int\int \frac{d^2\xi d^2}{\pi^2} \omega_A(\xi') \mathrm{Tr}[\hat{\Omega}(\xi')\hat{\Omega}(z - \xi)]$$

$$= \int \frac{d^2\xi}{\pi} \omega_A(z - \xi)\omega(\xi) . \tag{9}$$

Since convolution with $\omega_A(z)$ produces a linear map $\omega(z) \to \omega_s(z)$ and in view of the linearity of the defining relationship between $\hat{\rho}$ and $\omega(z)$ it follows that it induces a linear map on the density matrix itself. It is natural to ask if this map is a dynamical map.

Dynamical maps are the most general linear transformations [‡1] on the vector space of hermitian operators on state vectors which preserve the hermiticity, unit trace, and nonnegativity properties of the density matrices and hence any theory of measurement or evolution (including the irreversible ones) should necessarily conform to the framework of dynamical maps.

Any dynamical map can be brought to the canonical form [13,14]

$$\hat{\rho} \to \hat{\rho}' = \sum_\alpha \eta(\alpha)\zeta(\alpha)\hat{\rho}\zeta(\alpha)^\dagger , \tag{10}$$

where $\eta(\alpha)$ are a set of scalars and $\zeta(\alpha)$ a set of operators on the Hilbert space $\mathscr{H}$ of state vectors. The requirement that for every density matrix $\hat{\rho}$, $\hat{\rho}'$ given by (10) be a density matrix is equivalent to the following conditions on the set $\{\eta(\alpha), \zeta(\alpha)\}$ characterizing the map:

[‡1] If the density matrices are $n \times n$ matrices, say, then the dynamical maps are $n^2 \times n^2$ matrices.

$$\eta(\alpha)^* = \eta(\alpha) ,$$

$$\sum_\alpha \eta(\alpha) \hat{\zeta}(\alpha)^\dagger \hat{\zeta}(\alpha) = \hat{I} ,$$

$$\sum_\alpha \eta(\alpha) | \langle \psi | \hat{\zeta}(\alpha) | \phi \rangle^2 \geqslant 0 ,$$

$$\forall | \psi \rangle, | \phi \rangle \in \mathscr{H} . \quad (11)$$

The real scalars $\eta(\alpha)$ need not be all positive. As an important example of dynamical maps for which the $\eta(\alpha)$ are not all positive, we cite the map $\hat{\rho} \to \hat{\rho}^T$, where $\hat{\rho}^T$ is the matrix transpose of ρ [15]. Those maps for which all the $\eta(\alpha)$ are positive are said to be "completely positive". Given a linear map on the density matrices, it is the last condition in (11) which is sometimes difficult to test. But if the map is completely positive then the last condition is automatically met. As a last remark on (11), we note that the more familiar hamiltonian evolution trivially fits into this framework with only one $\eta(\alpha)$ nonzero (in fact equal to unity) and the corresponding $\hat{\zeta}(\alpha)$ unitary.

To interpret the smoothing procedure within the framework of dynamical maps, first rewrite (7) in terms of density matrices using (5) and (6):

$$\hat{\rho}_s = \int \frac{d^2 z}{\pi} \omega_s(z) \hat{\Omega}(z)$$

$$= \int \frac{d^2 \xi}{\pi} \omega_A(\xi) \hat{D}(\xi) \hat{\rho} \hat{D}(\xi)^\dagger . \quad (12)$$

Now (12) has the same form as (10), and we readily identify

$$\eta(\alpha) \to \omega_A(\xi), \quad \hat{\zeta}(\alpha) \to \hat{D}(\xi) ,$$

$$\sum_\alpha \to \int \frac{d^2 \xi}{\pi} . \quad (13)$$

Clearly, the first two conditions in (11) are satisfied for any Wigner distribution $\omega_A(\xi)$. The third condition now reads

$$\int \frac{d^2 \xi}{\pi} \omega_A(\xi) | \langle \psi | \hat{D}(\xi) | \phi \rangle |^2 \geqslant 0 ,$$

$$\forall | \psi \rangle, | \phi \rangle \in \mathscr{H} . \quad (14)$$

If the Wigner distribution $\omega_A(z)$ is pointwise nonnegative, then (14) is trivially satisfied and the smoothing process is a dynamical map, in fact a completely positive one. This is the case when $\omega_A(z)$ is gaussian. However, (14) is not satisfied by every Wigner distribution $\omega_A(z)$ implying that the smoothing process with an arbitrary Wigner distribution as kernel is not a dynamical map in general. For instance if $\hat{\rho}_A = |1\rangle\langle 1|$ the first excited state of an oscillator of unit mass and unit angular frequency, the corresponding Wigner distribution $\omega_A(z)$, is

$$\omega_A(z) = 2(4|z^2| - 1) \exp(-2|z|^2) . \quad (15)$$

Now choosing $|\psi\rangle = |\phi\rangle = |1\rangle$, the factor $\langle \psi | \hat{D}(z) | \phi \rangle$ in (14) can be evaluated to be

$$\langle \psi | \hat{D}(z) | \phi \rangle = \langle 1 | \hat{D}(z) | 1 \rangle$$

$$= (1 - |z|^2) \exp(-|z|^2/2) . \quad (16)$$

Finally, using (15) and (16) in (14) we deduce

$$\int \frac{d^2 \xi}{\pi} \omega_A(\xi) | \langle \psi | \hat{D}(\xi) | \phi \rangle |^2$$

$$= \int \frac{d^2 \xi}{\pi} 2(4|\xi|^6 - 9|\xi|^4 + 6|\xi|^2 - 1)$$

$$\times \exp(-3|\xi|^2) = -2/27 , \quad (17)$$

showing that in this case (14) is violated. Thus, the smoothing process with the kernel

$$\omega_A(z) = 2(4|z|^2 - 1) \exp(-2|z|^2)$$

is not a dynamical map.

If the smoothing kernel is such that the smoothing process is not a dynamical map, then there always exist Wigner distributions whose smoothed distribution (though pointwise nonnegative) will not be Wigner distributions. This means one can find in such cases *nonnegative* operators $\hat{O}$ whose averages as computed using the nonnegative smoothed distribtution will be *negative*:

$$\hat{O} \geqslant 0, \quad \int \frac{d^2 z}{\pi} \omega_s(z) \hat{O}(z) < 0 . \quad (18)$$

Since this is an undesirable situation, one should demand of the smoothing kernel the property that its convolution with any Wigner distribution is again a

Wigner distribution so that the smoothing process itself will be a dynamical map. While the often used gaussian kernel indeed effects a dynamical map, the set of all Wigner distributions which qualify to be used as smoothing kernel remains to be classified. We shall return to an analysis of this and related problems in a subsequent publication.

We wish to thank Dr. S.K. Basu for acquainting us with the work of Kuryshkin and himself and stimulating our interest in this problem.

References

[1] E.P. Wigner, Phys. Rev. 40 (1932) 749.
[2] J.E. Moyal, Proc. Cambridge Philos. Soc. Japan 45 (1949) 99.
[3] K. Husimi, Proc. Phys. Math. Soc. Japan 22 (1940) 264.
[4] C.L. Mehta and E.C.G. Sudarshan, Phys. Rev. B 138 (1965) 274.
[5] Y. Kano, J. Math. Phys. 6 (1965) 1913.
[6] J. McKenna and H.L. Frisch, Phys. Rev. 145 (1966) 93.
[7] V.V. Kuryshkin, Int. J. Theor. Phys. 7 (1973) 451.
[8] M.D. Srinivas and E. Wolf, Phys. Rev. D 11 (1975) 1477.
[9] N.D. Cartwright, Physica A 83 (1976) 210.
[10] R.F. O'Connell and E.P. Wigner, Phys. Lett. A 83 (1981) 145.
[11] K. Wodkiewicz, Phys. Rev. Lett. 52 (1984) 1064.
[12] S.K. Basu, Phys. Lett. A 114 (1986) 303.
[13] E.C.G. Sudarshan, P.M. Mathews and J.Rau, Phys. Rev. 121 (1961) 930.
[14] T.F. Jordan and E.C.G. Sudarshan, J. Math. Phys. 2(1961) 772.
[15] E.C.G. Sudarshan, in: From SU(3) to gravity, Festschrift in honour of Yuval Ne'eman, eds. E. Gotsman and G. Tauber (Cambridge Univ. Press, Cambridge, 1985).
[16] E.B. Davies, Quantum theory of open systems (Academic Press, New York, 1976).
[17] R. Simon, Opt. Commun. 42 (1982) 293.
[18] J.R. Klauder and E.C.G. Sudarshan, Fundamentals of quantum optics (Benjamin, New York, 1968).
[19] N. Mukunda, Pramana 11 (1978) 1.
[20] R.L. Hudson, Rep. Math. Phys. 6 (1974) 249.
[21] V.I. Tatarskii, Sov. Phys. Usp. 26 (1983) 311.