CONFIGURATION SPACE TOPOLOGY AND QUANTUM INTERNAL SYMMETRIES

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The inequivalent quantizations of a physical system with a general configuration space Q are studied, with particular emphasis on the case when $\pi_1(Q)$ is nonabelian. The necessary generalizations of the state space and the set of observables are given. Possible realizations of this scenario in quantum mechanics and quantum field theory are outlined.

When quantizing a physical system with configuration space [#1] Q, the standard procedure is to construct the fixed time quantum mechanical state vectors Ψ as ordinary functions from Q into the complex numbers $\mathbb{C}$. One can, however, generalize this notion of a state vector in two steps. First, we may consider certain "multiple-valued" functions from Q into $\mathbb{C}$. More specifically, when the argument of $\Psi(q)$ is taken around a loop in the space Q, Ψ can, in general, change by a phase. (Note that $|\Psi|^2$ must be single-valued). Secondly, we may further consider multipe-valued functions from Q into $\mathbb{C}^N$, $N\geqslant 1$. That is, we can construct the states Ψ as N-component vectors with components Ψ_n, $1\leqslant n\leqslant N$, and when $q\in Q$ is taken around a loop l in Q we get

$$\Psi_n(q)\xrightarrow{l}\sum_{m=1}^{N} U_{nm}(l)\Psi_m(q), \tag{1}$$

where $U_{nm}(l)$ are the components of a $N\times N$ unitary matrix $U(l)$. One can show [1] that the mapping from the (based) loop space of Q into the group $\mathscr{U}(N)$ defined by the matrices $U(l)$ above, must furnish an N-dimensional unitary representation of the fundamental group $\pi_1(Q)$ (i.e. U must be constant on any homotopy class of loops and also respect the standard multiplication between these classes) [#2]. In the quantum theory of such N-component objects, there is a superselection rule between any two state vectors which "transform under loops" according to distinct representations of $\pi_1(Q)$. Therefore, the full Hilbert space $\mathscr{H}$ of state vectors breaks up into a direct sum of subspaces $\{\mathscr{H}_\rho\}$ where each $\mathscr{H}_\rho$ only contains states which realize the fixed representation of $\pi_1(Q)$ labelled by ρ. If the representation ρ is reducible, then $\mathscr{H}_\rho$ breaks further into a direct sum of subspaces $\{\mathscr{H}_{\rho_i}\}$, where the ρ_i label the irreducible components of ρ. So we can achieve a decomposition of $\mathscr{H}$ into superselection sectors, each labelled by an irreducible unitary representation (IUR) of $\pi_1(Q)$. Let us denote the set of *all* finite-dimensional IUR's of $\pi_1(Q)$ by $\mathscr{R}$. The quantum theories defined by each of the Hilbert spaces $\mathscr{H}_\alpha$, $\alpha\in\mathscr{R}$, represent distinct quantizations of the original system. So we see that there is, in general, an "ambiguity" in quantizing a classical system with configuration space Q; the different choices of quantum representation being in

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[#1] We assume that Q is a path-connected manifold. If the true configuration space of the system is not path-connected, then it is understood that Q represents a fixed path-component. For simplicity, we also assume that the system is not interacting with any external field. The situation where such a field exists can be handled in a similar fashion.

[#2] More precisely, we should say that the mapping U from loops based at q into $\mathscr{U}(N)$ furnishes a unitary representation of $\pi_1(Q, q)$. However, the representation of $\pi_1(Q, q)$ realized by Ψ at $q\in Q$ determines (up to unitary equivalence) the representation of $\pi_1(Q, q')$ realized at any other $q'\in Q$. Throughout, when we use the work "representation" we will always mean "unitary equivalence class of representations".

one-to-one correspondence with the elements of $\mathscr{R}$ [#3].

If one is only interested in constructing theories with one-component state vectors, the so-called scalar quantum theories, then only one-dimensional representations of $\pi_1(Q)$ need be considered. Such representations are simply labelled by the group $\mathrm{Hom}(\pi_1(Q), \mathscr{U}(1))$ of all homomorphisms of $\pi_1(Q)$ into $\mathscr{U}(1)$. It was shown in ref. [2] that $\mathrm{Hom}(\pi_1(Q), \mathscr{U}(1))$ is trivial (i.e., contains only one element, namely the trivial homomorphism) if and only if $\pi_1(Q)$ is a perfect group [#4], or equivalently the first (integral) homology group $H_1(Q)=\{e\}$. So there is a unique scalar quantization of a system with configuration space Q if and only if $H_1(Q)$ is trivial. Many other interesting studies of scalar quantizations, both in general and for specific systems, have been carried out [1,4–13]. If $\pi_1(Q)$ is abelian, then all of its irreducible representations are one-dimensional and only scalar quantizations exist. However, in general this is not the case.

An alternative, geometrical (and more precise) way of viewing the above results is to note that the generalized state vectors in eq. (1) can be considered as sections of a complex vector bundle over Q which possesses a natural $\mathscr{U}(N)$ connection [1]. The requirement that U furnish a representation of $\pi_1(Q)$ is equivalent to requiring that this connection be flat. The irreducible complex vector bundles over Q with flat $\mathscr{U}(N)$ connection, $N\geqslant 1$, are known to be in one-to-one correspondence with the elements of $\mathscr{R}$ [1,14] and the quantum theory defined by the bundle associated with any $\alpha\in\mathscr{R}$ is the same as that on the irreducible sector $\mathscr{H}_\alpha$ considered above [#5].

On each $\mathscr{H}_\alpha$, $\alpha\in\mathscr{R}$, we now construct a complete set of operators. Such a set should involve the following three things. First, the "position operator" $\hat{q}$. Second, the finite displacement operators $\hat{T}^\sigma_{qq'}$, which take $\Psi(q)$ to $\Psi(q')$ along the path σ from q to q'. And finally, the "internal symmetry" operators $\tau(\gamma)$, $\gamma\in\pi_1(Q)$, which are $N\times N$ matrices ($N=\dim\alpha$) forming an IUR of $\pi_1(Q)$ and satisfying (for all $1\leqslant n\leqslant N$)

$$\Psi_n(q\gamma)=\sum_{m=1}^{N}\tau_{nm}(\gamma)\Psi_m(q). \tag{2}$$

(Here $q\gamma$ means q taken around a loop in the homotopy class γ). The position operator $\hat{q}$ may be simply represented as multiplication by the coordinate q. We also have

$$\hat{T}^\sigma_{qq'}=\mathrm{P}\exp\left(\mathrm{i}\int_\sigma \mathrm{D}\right), \tag{3}$$

where P is the path ordering symbol and

$$\mathrm{D}=-\mathrm{i}\,\mathbb{1}\,\delta/\delta q-A(q). \tag{4}$$

Here $A(q)$ is a local (matrix) expression for the natural flat $\mathscr{U}(N)$ connection on the vector bundle associated with $\mathscr{H}_\alpha$. (Note that the operators $\hat{T}^\sigma_{qq'}$ depend only on the homotopy class of σ). It is also clear that since α is irreducible, the matrices $U(l)$ (eq. (1)) provide the internal symmetry operators τ. However, since for a loop l based at q we have $T^l_{qq}=U(l)$, we see that $\hat{q}$ and the $\hat{T}$'s alone form a complete set of operators. So one sees explicitly the internal symmetry as an expression of the topology of Q.

The above scenario is of more than formal interest. There are many physically interesting systems, both finite and infinite dimensional, which possess topologically nontrivial configuration spaces Q. We now present some examples of systems which have $\pi_1(Q)$ nonabelian and therefore may possibly realize the above quantizations with internal symmetry. As our first example we consider an asymmetric rigid rotator in three-space. This situation has possible applications to the excitations of strongly deformed nuclei [17]. The relevant configuration space is the orbit space S^3/Q_8 where S^3 is the three-sphere coordinatized by the three Euler angles and Q_8 is the group of quaternions of order 8 [3] which has a natural, free action on S^3. Standard techniques in algebraic topology [18] yield $\pi_1(S^3/Q_8)\cong Q_8$. S^3/Q_8 is one of the simplest manifolds with a nonabelian fundamental group. There are five IUR's of Q_8, four of which are one-dimensional ($\mathrm{Hom}(Q_8, \mathscr{U}(1))\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$). The

[#3] We are ignoring possible ambiguities in quantization which are of dynamical origin, e.g., operator ordering ambiguities. We are only interested in classifying kinematical quantization ambiguities.

[#4] A group G is called perfect if $[G, G]\cong G$, where $[G, G]$ is the commutator subgroup of G. See ref. [3].

[#5] The above classification arises as well in geometric quantization [13] where it can be seen from the nonuniqueness of the symplectic potential giving rise to a fixed symplectic two-form on phase space. Also, in refs. [15,16] this classification was obtained by looking at the universal covering space of Q.

fifth representation ρ is of dimension two, and therefore there exists a quantization of this system with internal symmetry [15]. The flat $\mathscr{U}(2)$ connection A on the bundle $S^3\times_\rho\mathbb{C}^2$ associated with this quantization pulls back (via the universal covering projection $S^3\to S^3/Q_8$) to the connection $\tilde{A}=iV^{-1}dV$ on the product bundle $S^3\times\mathbb{C}^2$. Here $V:S^3\to\mathscr{SU}(2)$ is defined by

$$V(\alpha,\beta,\gamma)=\exp(i\alpha\sigma_3/2)\exp(i\beta\sigma_2/2)\exp(i\gamma\sigma_3/2), \tag{5}$$

where α, β and γ are the Euler angles and σ_2 and σ_3 the Pauli matrices. A local expression for A can be obtained by projecting $\tilde{A}$ onto S^3/Q_8.

Another example of interest is that of n identical particles moving on a path-connected manifold M. The configuration space for this system is [4]

$$Q_n(M)=(M^n-\Delta)/S_n, \tag{6}$$

where M^n represents the n-fold cartesian product of M, Δ is the subcomplex of M^n on which two or more particles occupy the same position, and S_n is the permutation group on n symbols with the obvious action on $M^n-\Delta$. This action is clearly free. If $\dim M\geqslant 3$ and $\pi_1(M)=\{e\}$, then it is straightforward to show that $\pi_1(M^n-\Delta)\cong\pi_1(M^n)=\{e\}$ and therefore $\pi_1(Q_n(M))\cong S_n$. The representations of S_n are well studied and have a rich structure [19]. The number of IUR's of S_n is equal to the number of partitions of the integer n, denoted by $p(n)$. Several values of $p(n)$ are given in table 1 [19]. The number of IUR's grows rapidly as n increases. For any n there are only two one-dimensional representations; namely the trivial one, and the representation sending all even permutations to +1 and all odd permutations to −1. For $n\neq 4$, all other IUR's have dimension at least $n-1$.

Table 1

n	$p(n)$
2	2
3	3
4	5
5	7
10	42
20	627
50	204 266
100	190 569 292

(S_4 has a two-dimensional representation). Note that the quantization associated with a given IUR corresponds to a choice of statistics for the n identical particles. The two scalar quantizations correspond to Bose and Fermi statistics, while the others represent a generalization of these [20]. So we see that there are $p(n)$ choices of statistics for n identical particles moving on a simply-connected manifold of three or more dimensions.

If $\dim M=2$, the situation is much more complex. For example, take $M=\mathbb{R}^2$. Here $\pi_1(\mathbb{R}^{2n}-\Delta)\neq\{e\}$ and $\pi_1(Q_n(\mathbb{R}^2))$ is an infinite group known as the n-string Artin braid group [21], which is nonabelian for $n\geqslant 3$. $\mathrm{Hom}(\pi_1(Q_n(\mathbb{R}^2)),\ \mathscr{U}(1))\cong\mathscr{U}(1)$, so the one-dimensional IUR's are labelled by a continuous parameter. In the corresponding scalar quantum theories, this parameter smoothly interpolates between quantizations with Bose and Fermi statistics. The new statistics are often called θ-statistics [7] and seem to be relevant in theoretical interpretations of the fractional quantum Hall effect [8]. The quantum theories associated with higher-dimensional representations of the n-string Artin braid group display the nonscalar generalizations of θ-statistics. One can also consider the possible quantum representations of n identical particles moving on an *arbitrary* path-connected manifold M. This involves a study of the generalized braid groups [21] and their IUR's [#6].

We now turn our attention to infinite dimensional systems [#7]. Let us first consider ordinary Einstein gravity and assume that the space–time manifold has the form $\Sigma\times\mathbb{R}$ where Σ is a path-connected, compact, three-dimensional manifold with fixed topology. The configuration space for this system can be chosen as the orbit space [5]

$$R/D\equiv\mathrm{Riem}(\Sigma)/\mathrm{Diff}_F(\Sigma), \tag{7}$$

[#6] (Scalar) statistics on $M=S^2$ were first studied in refs. [8,9]. For a complete characterization of the scalar quantizations of n identical particles moving on an arbitrary manifold of dimension $\geqslant 3$, as well as *all* closed two-manifolds, see ref. [10]. For more on nonscalar statistics, see ref. [22]. Applications of braid group representations to conformal field theories can be found in ref. [23].

[#7] In the infinite-dimensional systems below we will treat the quantum state vectors as sections of a complex vector bundle over the classical configuration space. This is strictly valid only in the semiclassical approximation [1].

where $\mathrm{Riem}(\Sigma)$ is the affine space of all riemannian metrics on the space manifold Σ, and $\mathrm{Diff_F}(\Sigma)$ is the group of diffeomorphisms of Σ which leave a point and a frame at that point fixed. $\mathrm{Diff_F}(\Sigma)$ acts freely on Riem (Σ) and we can write [5]

$$\pi_1(\mathrm{R/D}) \cong \pi_0(\mathrm{Diff_F}(\Sigma)). \tag{8}$$

For many choices of Σ, this group has been shown to be nonabelian [24]. An interesting example is $\Sigma = \mathrm{S}^3/\mathrm{I}^*$ (the Poincaré sphere) where I^* is the binary icosahedral group [23] which acts freely on S^3. In this case $\pi_1(\mathrm{R/D}) \cong \mathrm{I}^*$ [24] and since I^* is a perfect group it has no non-trivial one-dimensional IUR's [2]. However, even though there exists a unique scalar quantization of this system, I^* *does* possess higher-dimensional IUR's and therefore there are other quantizations. A two-dimensional IUR of I^* can be found in ref. [25].

A second class of field theories which can have $\pi_1(\mathrm{Q})$ nonabelian are the generalized nonlinear sigma models. The fixed-time fields in these models can be considered as mappings from the space manifold Σ into an arbitrary manifold M. Here Σ is assumed to be path-connected, compact and of arbitrary dimension. The above maps can be either all free or all base-point preserving #8. The corresponding configuration spaces are M^Σ and M^Σ_* respectively, where $\mathrm{M}^\Sigma(\mathrm{M}^\Sigma_*)$ denotes the set of all free (base-point preserving) maps from Σ into M, with the compact-open topology #9. These two spaces are related through the following fibration [18]:

$$\begin{array}{l} \mathrm{M}^\Sigma_* \overset{i}{\hookrightarrow} \mathrm{M}^\Sigma \\ \qquad\ \downarrow p\ , \\ \qquad\ \mathrm{M} \end{array} \tag{9}$$

#8 Base-point preserving maps arise naturally, for example, if the true space-manifold has been (one-point) compactified to Σ by finite energy boundary conditions; or if M is a coset space G/H and the system has a global G invariance [1]. We also mention that the formalism in this paper is strictly valid only for systems which possess a global lagrangian description. In particular we assume that the above generalized sigma models have no Wess–Zumino terms in their action.

#9 There are other topologies besides the compact-open topology [18] to put on such sets of mappings. However, theorems of Palais [26] state that for a large range of choices, the resulting spaces will have the same homotopy type. Also, the maps in M^Σ or M^Σ_* can be taken to be all C^0, or C^1, etc., without affecting the homotopy type of the space [1,26].

where i is the inclusion map and p is evaluation at the basepoint of Σ.

M^Σ and M^Σ_* are not, in general, path-connected indicating the possible existence of solitons in these models [1]. In order to discuss the above topological effects, we must choose a path-component on which to quantize. In what follows, we will choose the path-component of M^Σ or M^Σ_* to be the one containing the constant map c which sends all of Σ to the basepoint m_0 of M. (Note that $c \in \mathrm{M}^\Sigma_* \subseteq \mathrm{M}^\Sigma$.) Many of our results are strictly valid only for this component. It has recently been shown [11] that $\pi_1(\mathrm{M}^\Sigma_*, c)$ is a solvable group of derived length $\leqslant [\log_2(n)+1]$, where n is the number of non-trivial homotopy groups of the path-component of M containing m_0, from dimension 2 through $d+1$ where $d = \dim \Sigma$. Recall that a group has derived length 0 if and only if it is trivial and derived length 1 if and only if it is non-trivial and abelian [3]. We therefore see there is a possibility that generalized sigma models with nonabelian $\pi_1(\mathrm{M}^\Sigma_*, c)$ exist #10. However, constructing explicit examples is difficult. For more on this see ref. [11].

The situation is much better if we consider M^Σ. The long exact homotopy sequence [18] of the fibration in eq. (9) gives

$$\begin{aligned} \dots &\to \pi_2(\mathrm{M}, m_0) \to \pi_1(\mathrm{M}^\Sigma_*, c) \xrightarrow{i_*} \pi_1(\mathrm{M}^\Sigma, c) \\ &\xrightarrow{p_*} \pi_1(\mathrm{M}, m_0) \to \pi_0(\mathrm{M}^\Sigma_*, c) \to \dots . \end{aligned} \tag{10}$$

We see that sufficient conditions for $\pi_1(\mathrm{M}^\Sigma, c)$ to be nonabelian are that M^Σ_* is path-connected and $\pi_1(\mathrm{M}, m_0)$ nonabelian. These conditions are not difficult to satisfy. Also, it is known [27] for $\Sigma = \mathrm{S}^d$, $d \geqslant 1$, that if $\pi_1(\mathrm{M}, m_0)$ is nonabelian, so is $\pi_1(\mathrm{M}^{\mathrm{S}^d}, c)$, regardless of $\pi_0(\mathrm{M}^{\mathrm{S}^d}_*)$. Moreover, if (M, m_0) is $(d+1)$-simple #11, then [27]

$$\pi_1(\mathrm{M}^{\mathrm{S}^d}, c) \cong \pi_{d+1}(\mathrm{M}, m_0) \times \pi_1(\mathrm{M}, m_0). \tag{11}$$

These possibilities are especially interesting for $d=1$ since properties of such $(1+1)$-dimensional generalized sigma models may have relevance to closed bosonic string theory. We also note that the classical

#10 The above result does not allow this possibility if $\dim \Sigma = 1$ since then n is at most 1. In this case $\Sigma = \mathrm{S}^1$ and $\pi_1(\mathrm{M}^\Sigma_*, c) \cong \pi_2(\mathrm{M}, m_0)$ which is, of course, abelian.

#11 A pointed space (X, x_0) is called p-simple if the natural action of $\pi_1(\mathrm{X}, x_0)$ on $\pi_p(\mathrm{X}, x_0)$ is trivial [18]. Of course any space with $\pi_p(\mathrm{X}, x_0) = \{e\}$ is p-simple.

hamiltonian for a generalized sigma model may possess a symmetry group S of transformations of the fixed-time fields. The correct configuration space on which to construct quantum theories (assuming there is no anomalous breaking of S) is then the orbit space X/S where $X=M^{\Sigma}$ or M^{Σ}_{*}. If S acts freely on X and the projection map $p: X \to X/S$ is a fibration, then it is easy to show [11] (from the associated long exact homotopy sequence and the fact that $\pi_1(S)$ is abelian since S is a topological group [18]) that if $\pi_1(X, x_0)$ is nonabelian, then so is $\pi_1(X/S, p(x_0))$, except possibly in the case when $\pi_1(X, x_0)$ has derived length 2.

As our final example we consider pure gauge theories with path-connected compact space manifold Σ (of arbitrary dimension) and structure group G. Such a theory is defined with respect to a principal G-bundle P over Σ. The configuration space for the system can be taken to be $\mathscr{A}/\mathscr{G}_*$, where $\mathscr{A}$ is the affine space of all connection one-forms on P and $\mathscr{G}_*$ is the group of automorphisms of P which fix the base manifold Σ as well as the fiber above a chosen basepoint of Σ [5]. In what follows we always take P to be the trivial bundle, $P=\Sigma\times G$. In this case we can write $\mathscr{G}_*=G^{\Sigma}_{*}$, the set of all basepoint preserving maps from Σ into G (with the identity element as basepoint of G) with the compact-open topology. $\mathscr{G}_*$ acts freely on $\mathscr{A}$ and we obtain [5]

$$\pi_1(\mathscr{A}/\mathscr{G}_*)\cong\pi_0(G^{\Sigma}_{*}). \qquad (12)$$

In gauge theories on $\Sigma=S^3$ with nonabelian simple Lie group G, we have $\pi_1(\mathscr{A}/\mathscr{G}_*)\cong\pi_3(G)\cong\mathbb{Z}$. Hence $\mathrm{Hom}(\pi_1(\mathscr{A}/\mathscr{G}_*), \mathscr{U}(1))\cong\mathscr{U}(1)$ and the corresponding quantum theories are labelled by the so called θ-angle (or vacuum angle) [1,5]. In general it can be shown [11] that $\pi_0(G^{\Sigma}_{*})$ is solvable of derived length $\leqslant[\log_2(m)+1]$, where m is the number of non-trivial homotopy groups of G, from dimension 1 through $d=\dim\Sigma$. If $\dim\Sigma=1$, then $\pi_1(\mathscr{A}/\mathscr{G}_*)$ is clearly abelian. If $\dim\Sigma=2$, then $\pi_1(\mathscr{A}/\mathscr{G}_*)$ is still abelian since $\pi_2(G)=\{e\}$ for any Lie group G [#12]. So $\dim\Sigma=3$ is the first situation where $\pi_1(\mathscr{A}/\mathscr{G}_*)$ can be nonabelian. However, in ref. [6] $\pi_0(G^{\Sigma}_{*})$ is calculated and found to be abelian for all compact, simple Lie groups G (and a few others) when $\dim\Sigma=3$, and Σ is orientable. Despite these negative results in low dimensions, it is nonetheless clear that if $\dim\Sigma$ is large enough, one should be able to, in principle, construct a gauge theory with nonabelian $\pi_1(\mathscr{A}/\mathscr{G}_*)$. For some explicit examples, see ref. [11] [#13].

[#12] More specifically, if $\dim\Sigma=1$, then $\Sigma=S^1$ and $\pi_1(\mathscr{A}/\mathscr{G}_*)\cong\pi_1(G)$. If $\dim\Sigma=2$, then $\pi_1(\mathscr{A}/\mathscr{G}_*)\cong H^2(\Sigma;\pi_1(G))$ [11].

In this paper we have studied the construction of quantum theories on a general configuration space. We have seen that there are as many inequivalent quantizations of a system with configuration space Q as there are irreducible unitary representations of $\pi_1(Q)$. In particular, if $\pi_1(Q)$ is nonabelian, then there may exist quantizations which possess an "internal symmetry" of topological origin associated with the *entire system* [#14]. Various examples of interesting systems in quantum mechanics and quantum field theory which can realize this scenario were given. A more detailed investigation of nonscalar quantum theories is underway.

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[#13] There are natural choices of the configuration spaces for gauge theories and gravity other than the ones considered here [5,28,29]. Results similar to the ones above hold for these alternative choices as well [11].

[#14] $\pi_1(Q)$ being nonabelian is a necessary but *not* sufficient condition for nonscalar quantizations to exist. For necessary *and* sufficient conditions, see ref. [30].

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