PARTIALLY COHERENT BEAMS AND A GENERALIZED *ABCD*-LAW

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The beam width and the angular spread of a partially coherent beam, and their transformation as the beam passes through Sp(2, R) first order optical systems are studied using the method of generalized rays. A generalized *abcd*-law which governs this transformation is derived. Kogelnik's *abcd*-law for coherent gaussian beams and its later generalization to gaussian Schell-model beams are shown to be special cases of this law.

1. Introduction

The notion of generalized rays [1] was originally introduced in an attempt to clarify the relationship between electrodynamics and the radiative transfer theory [2]. Subsequently, this notion has developed into a technique which is of considerable advantage in handling beam propagation problems [3–7].

The generalized rays give an exact (not a short wavelength limit) ray picture of wave optics including the associated interference and diffraction phenomena [1]. Thus, it is not surprising that these rays have attributes unfamiliar to the rays of the phenomenological radiative transfer theory [8]. This is reminescent of the classical-looking phase space description of quantum mechanics using, say, the Wigner distribution function wherein, for example, the phase space density need not be pointwise nonnegative in general [9]. For the purpose of the present paper, the most attractive feature of generalized rays is that they transform in an extremely simple way under paraxial free propagation and under action of first order systems [3,4].

Various aspects of first order systems have been studied by several authors [10–13]. The first order systems correspond to the group Sp(2, R) = SL(2, R) in the axially symmetric case, and to Sp(4, R) in the more general anisotropic case [10,11]. In geometrical ray optics a first order system is described by a numerical symplectic matrix belonging to the group Sp(2, R) or Sp(4, R) as the case may be, the matrix itself acting on the position and direction of the ray arranged as a column [14]. In wave optics the first order system acts through the generalized Huyghens integral [15] which forms the metaplectic representation of the symplectic group [11]. In this connection it is useful to note that the action of first order systems in wave (ray) optics is formally identical to the evolution under quadratic hamiltonians in quantum (classical) mechanics. Free paraxial propagation and action of aberration-free thin lenses are examples of first order systems. Elsewhere [10] we have shown that every Sp(2, R) first order system, including the inverse of free propagation, can

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be synthesised with atmost three thin lenses separated by free propagation sections.

A problem of much practical interest is the transformation of (coherent) gaussian laser beams as they pass through first order systems. This problem was first studied by Kogelnik and the solution finds a compact expression in the *abcd*-law [16].

Gaussian Schell-model (GSM) beams are generalization of coherent gaussian beams. These beams have become an important class of model beams in recent studies on the radiometry of partially coherent sources [17]. We have analysed, using generalized rays, the behaviour of these beams under the action of first order systems [4]. This analysis led to a convenient geometrical picture wherein GSM beams are represented by time-like vectors with positive time component in a fictitious 2+1 dimensional Minkowski space with the first order systems acting as Lorentz transformations in this space. A generalization of Kogelnik's *abcd*-law to the partially coherent GSM beams was derived as a simple consequence of this geometric picture [#1]. Further generalization to anisotropic gaussian Schell-model (AGSM) beams has also been derived [6,19].

In the present paper we analyse the beam width and the angular spread of a *general* partially coherent beam, and their transformation under the action of Sp(2, *R*) first order systems using the method of generalized rays. A generalized *abcd*-law which governs this transformation is derived. Kogelnik's original *abcd*-law [16] and its generalization in ref. [4] to GSM beams are shown to be special cases of this law.

We will assume that the radiation field under consideration is described by a stationary ensemble [20]. This assumption is satisfied in most practical situations. As a consequence of stationarity, the field at two different frequencies do not interfere (they are mutually uncorrelated). Hence the entire analysis can be done, without loss of generality, for one frequency at a time. It is understood that the analysis in the following is for one such arbitrary frequency, but the frequency ω and the associated wave number k will be suppressed.

[#1] Subsequently we have rederived the *abcd*-law for gaussian Schell-model beams on the basis of Wolf's new theory in ref. [18].

2. Beam width, angular spread and first order systems

We will assume that the beam is propagating about the z-direction. Then $\boldsymbol{x}=(x, y)$ is the position variable in any transverse plane z=constant. We will concentrate on the field distribution in various transverse planes. It is advantageous to interpret the propagation process itself as a linear map which transform $\psi_{\text{in}}(\boldsymbol{x})$, the field distribution in the plane $z=z_{\text{in}}$ to $\psi_{\text{out}}(\boldsymbol{x})$, the field distribution in the plane $z=z_{\text{out}}$.

For stationary beams, it is convenient to represent the field distribution in a transverse plane by the cross-spectral density $\Gamma_z(\boldsymbol{x}_1, \boldsymbol{x}_2)$. For brevity, we will often suppress the subscript z and write $\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in place of $\Gamma_z(\boldsymbol{x}_1, \boldsymbol{x}_2)$. If the field is coherent, then

$$\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2) = \psi(\boldsymbol{x}_1)\, \psi(\boldsymbol{x}_2)^* \tag{2.1}$$

for some function $\psi(\boldsymbol{x})$. That is, Γ is a projection operator except for a multiplicative constant $\text{tr}(\Gamma) = \int \mathrm{d}^2x\, \Gamma(\boldsymbol{x}, \boldsymbol{x})$. In the more general case it follows from Wolf's new theory [21] that Γ can be written as a convex combination (linear combination with positive coefficients) of such projection operators. We have

$$\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_n \lambda_n\, \varphi_n(\boldsymbol{x}_1)\, \varphi_n(\boldsymbol{x}_2)^* , \tag{2.2}$$

where $\varphi_n(\boldsymbol{x})$ and λ_n are the eigenfunctions and eigenvalues of Γ.

Given the cross-spectral density, the density of generalized rays is given by the Wolf function [4] $W(\boldsymbol{x}, \boldsymbol{s})$ which is nothing but the Wigner-Moyal transform [22] of Γ:

$$W(\boldsymbol{x}, \boldsymbol{s}) = [k^2/(2\pi)^2] \int \mathrm{d}^2x' \times \Gamma(\boldsymbol{x}-\boldsymbol{x}'/2, \boldsymbol{x}+\boldsymbol{x}'/2) \exp(\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{x}') . \tag{2.3}$$

$W(\boldsymbol{x}, \boldsymbol{s})$ gives the strength of the generalized pencil with position $(\boldsymbol{x}, z)$ and direction $(\boldsymbol{s}, s_3 = (1-\boldsymbol{s}\cdot\boldsymbol{s})^{1/2})$ in the transverse plane under consideration [4]. The Wolf function is real as a consequence of the hermiticity of $\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2)$, and has all the information contained in it. This follows by noting that the transformation (2.3) is invertible:

$$\Gamma(\boldsymbol{x}_1,\boldsymbol{x}_2)=\int d^2s\, W\left(\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2},\boldsymbol{s}\right)$$
$$\times\exp[-ik\boldsymbol{s}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)]\ . \qquad (2.4)$$

The total irradiance in the given transverse plane is

$$A=\int d^2x\,\Gamma(\boldsymbol{x},\boldsymbol{x})=\iint d^2x\, d^2s\, W(\boldsymbol{x},\boldsymbol{s})\ . \qquad (2.5)$$

For any quantity $f(\boldsymbol{x})$ it is easily seen that the average value $\langle f(\boldsymbol{x})\rangle$ is

$$\langle f(\boldsymbol{x})\rangle=A^{-1}\int d^2x f(\boldsymbol{x})\,\Gamma(\boldsymbol{x},\boldsymbol{x})$$
$$=A^{-1}\iint d^2x\, d^2s f(\boldsymbol{x})\, W(\boldsymbol{x},\boldsymbol{s})\ . \qquad (2.6)$$

To concentrate on the essentials of the problem under consideration, we will restrict attention henceforth to beam fields which are invariant under rotation about the z-axis (the beam axis). For such fields it is clear that $W(\boldsymbol{x},\boldsymbol{s})$ depends on $\boldsymbol{x}$ and $\boldsymbol{s}$ only through the rotationally invariant combinations $\boldsymbol{x}^2=\boldsymbol{x}\cdot\boldsymbol{x}$, $\boldsymbol{s}^2=\boldsymbol{s}\cdot\boldsymbol{s}$ and $\boldsymbol{x}\cdot\boldsymbol{s}$. As one consequence of this we have from (2.5) $\langle\boldsymbol{x}\rangle=0$, and hence the square of the beam width is given by the second moment $\langle\boldsymbol{x}^2\rangle$ of $\boldsymbol{x}$:

$$\langle\boldsymbol{x}^2\rangle=A^{-1}\iint d^2x\, d^2s\, \boldsymbol{x}^2\, W(\boldsymbol{x},\boldsymbol{s})\ . \qquad (2.7)$$

For the same reason $\langle\boldsymbol{s}\rangle=0$, and a measure of the square of the angular spread is given by

$$\langle\boldsymbol{s}^2\rangle=A^{-1}\iint d^2x\, d^2s\, \boldsymbol{s}^2\, W(\boldsymbol{x},\boldsymbol{s})\ . \qquad (2.8)$$

We will need to consider in the sequel also the average value of $\boldsymbol{x}\cdot\boldsymbol{s}$:

$$\langle\boldsymbol{x}\cdot\boldsymbol{s}\rangle=\langle\boldsymbol{s}\cdot\boldsymbol{x}\rangle$$
$$=A^{-1}\iint d^2x\, d^2s\, \boldsymbol{x}\cdot\boldsymbol{s}\, W(\boldsymbol{x},\boldsymbol{s})\ . \qquad (2.9)$$

This quantity represents the position–direction correlation as shown by Davis and Heller [23].

It is useful to combine $\boldsymbol{x}$ and $\boldsymbol{s}$ into a two-element column vector

$$Q=\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{pmatrix}, \qquad (2.10)$$

so that the Wolf function $W(\boldsymbol{x},\boldsymbol{s})$ can be written as $W(Q)$. Also we will arrange $\langle\boldsymbol{x}^2\rangle$, $\langle\boldsymbol{s}^2\rangle$ and $\langle\boldsymbol{x}\cdot\boldsymbol{s}\rangle$ into a 2×2 real symmetric variance matrix V:

$$V=\begin{pmatrix}\langle\boldsymbol{x}^2\rangle & \langle\boldsymbol{x}\cdot\boldsymbol{s}\rangle\\ \langle\boldsymbol{s}\cdot\boldsymbol{x}\rangle & \langle\boldsymbol{s}^2\rangle\end{pmatrix}=\langle QQ^{\mathrm{T}}\rangle\ . \qquad (2.11)$$

It is clear that V is a positive definite matrix. With the aid of V, the three equations (2.7) – (2.9) can be combined into a single compact equation:

$$V=A^{-1}\int d^4Q\,(QQ^{\mathrm{T}})\,W(Q)\ . \qquad (2.12)$$

This way of writing these equations will turn out to be useful later on.

We turn our attention now to the action of first order systems on arbitrary partially coherent beams. Every first order system is represented by an associated 2×2 real ray transfer matrix $S\in\mathrm{Sp}(2,R)$. We recall that a 2×2 real matrix S belongs to $\mathrm{Sp}(2,R)$ if and only if

$$SKS^{\mathrm{T}}=K,\qquad K=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \qquad (2.13)$$

This condition is equivalent to

$$S=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\qquad ad-bc=1\ . \qquad (2.14)$$

Thus $\mathrm{Sp}(2,R)$ is identical to $\mathrm{SL}(2,R)$, though $\mathrm{Sp}(2n,R)$ is a proper subgroup of $\mathrm{SL}(2n,R)$ for $n\geqslant2$. Free paraxial propagation through distance D, action of thin lens with focal length f, and linear magnifier with magnification strength m are examples of first order systems and have, respectively, the following $\mathrm{Sp}(2,R)$ matrices representing them:

$$S(D)=\begin{pmatrix}1 & D\\ 0 & 1\end{pmatrix},\qquad S(f)=\begin{pmatrix}1 & 0\\ -f^{-1} & 1\end{pmatrix},$$
$$S(m)=\begin{pmatrix}m & 0\\ 0 & m^{-1}\end{pmatrix}. \qquad (2.15)$$

A first order system S acts on the Wolf function in a simple way [4]:

$$W(Q)\to W'(Q)=W(S^{-1}Q)\ . \qquad (2.16)$$

The output Wolf function at the phase space point Q is the input Wolf function at the point $S^{-1}Q$. In other words, the generalized rays in a first order system follow the same trajectories as the rays of geometrical optics:

$$Q\to Q'=SQ\ . \qquad (2.17)$$

Since det $S=1$, it follows from (2.5) that the total irradiance A is invariant for every beam under the action of first order systems:

$$A \to A' = \int d^4Q\, W'(Q) = \int d^4Q\, W(S^{-1}Q) = A\,. \qquad (2.18)$$

We wish to find the transformation properties of the beam width and the angular spread under the action of first order systems. This is most easily done through consideration of the transformation properties of the variance matrix V. From (2.12), (2.16) and (2.18) we have

$$V \to V' = A^{-1} \int d^4Q\, (QQ^{\mathrm{T}})\, W(S^{-1}Q) = A^{-1} \int d^4Q\, S(QQ^{\mathrm{T}})\, S^{\mathrm{T}} W(Q)\,, \qquad (2.19)$$

where the fact that det $S=1$ was used. Thus

$$V' = SVS^{\mathrm{T}}\,. \qquad (2.20)$$

The effect of a first order system is to take the variance matrix V into a new variance $V'=SVS^{\mathrm{T}}$. Since V is symmetric positive definite, so is also V'.

In the following section we will use this transformation law for the variance matrix to derive a generalized *abcd*-law. It is appropriate to make, however, the following observations here:

We were originally interested in the transformation properties of the beam width and the angular width. But now we find that the propagation law for these quantities necessarily involves also the off-diagonal correlation term $V_{12}=V_{21}=\langle \boldsymbol{x}\cdot\boldsymbol{s}\rangle$. This is analogous to the familiar situation where (at optical frequencies in particular) one's interest is in the diagonal elements of the two-point function which corresponds to the intensity distribution. Yet, the propagation law for intensity involves the full two-point function, including its off-diagonal elements [20].

It is instructive to see how the various elements of V can be measured using a profile detector which can measure only the beam width V_{11}. First, note that scaled Fourier transformers $S_{\mathrm{F}}(c)$ [it is easier to construct optical Fourier transformers with $c\neq 1$] are Sp(2, R) systems with ray transfer matrix

$$S_{\mathrm{F}}(c) = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}. \qquad (2.21)$$

Now pass the beam through $S_{\mathrm{F}}(c)$. The transformed variance matrix can be computed from (2.20), and we have

$$V'_{11} = c^2 V_{22}\,. \qquad (2.22)$$

Thus V_{22} can be computed by measuring the new beam width V'_{11}. On further free propagation through a distance D, the beam width square becomes, on using (2.15) and (2.20)

$$V''_{11} = c^{-2} D^2 V_{11} + c^2 V_{22} - 2DV_{12}\,. \qquad (2.23)$$

Since V_{11} and V_{22} are known, V_{12} can be computed once V''_{11} is measured.

We conclude this section by exhibiting a universal hyperbolic variation of the beam width under free propagation. Let $V(z_0)$ be the variance matrix in the initial transverse plane z_0. The variance matrix $V(z)$ in any other transverse plane obtains when $S(D)$ in (2.15) with $D=z-z_0$ is used in (2.20). In particular we have for the square of the beam width

$$V_{11}(z) = V_{11}(z_0) - V_{12}(z_0)^2/V_{22}(z_0) + V_{22}(z_0)\,[z - z_0 + V_{12}(z_0)/V_{22}(z_0)]^2\,. \qquad (2.24)$$

We find that the plot of the beam width $[V_{11}(z)]^{1/2}$ as a function of z is a hyperbola for *all* partially coherent beams. Clearly, the waist where the beam width becomes minimum occurs at

$$(z)_{\mathrm{waist}} = z_0 - V_{12}(z_0)/V_{22}(z_0)\,, \qquad (2.25)$$

the beam width at the waist itself being given by the expression $[V_{11}(z_0) - V_{12}(z_0)^2/V_{22}(z_0)]^{1/2}$. The Rayleigh range, the distance from the waist at which the square of the beam width is double its value at the waist, is given by

$$z_{\mathrm{R}} = \{[V_{11}(z_0) - V_{12}(z_0)^2/V_{22}(z_0)]/V_{22}(z_0)\}\,. \qquad (2.26)$$

And, finally, the far-zone where the beam exhibits the van Cittert-Zernike scaling behaviour is given by

$$|z - (z)_{\mathrm{waist}}| \gg z_{\mathrm{R}}\,. \qquad (2.27)$$

This is a generalized Fraunhofer far-zone criterion [3,24] applicable to every partially coherent beam.

3. Generalized *abcd*-law

We have noted that the symmetric variance matrix V is positive definite. Let det $V=\Omega^2$, where we choose $\Omega>0$. From (2.13) and (2.19) it follows that

$$V'+\mathrm{i}\Omega K=S(V+\mathrm{i}\Omega K)S^{\mathrm{T}} . \tag{3.1}$$

Since det $V=$ det V', we see that $V+\mathrm{i}\Omega K$ and also $V'+\mathrm{i}\Omega K$ are singular hermitian matrices. Hence each of them has only one non zero eigenvalue. Thus, we can find complex vectors ξ and ξ' such that

$$V+\mathrm{i}\Omega K=\xi\xi^{\dagger}, \qquad V'+\mathrm{i}\Omega K=\xi'\xi'^{\dagger} . \tag{3.2}$$

In fact, writing out (3.2) in component form, one solves for ξ and ξ' by inspection:

$$\xi=\mathrm{e}^{\mathrm{i}\theta}\sqrt{V_{22}}\begin{pmatrix} q \\ 1\end{pmatrix}, \quad q=\frac{V_{12}+\mathrm{i}\Omega}{V_{22}},$$

$$\xi'=\mathrm{e}^{\mathrm{i}\theta'}\sqrt{V'_{22}}\begin{pmatrix} q' \\ 1\end{pmatrix}, \quad q'=\frac{V'_{12}+\mathrm{i}\Omega}{V'_{22}} . \tag{3.3}$$

Here θ, θ' are real arbitrary. Note that the imaginary part of $q=\xi_1/\xi_2$ as also of $q'=\xi'_1/\xi'_2$ is always positive. With the aid of (3.2) we can rewrite (3.1) as

$$\xi'\xi'^{\dagger}=S\xi\xi^{\dagger}S^{\mathrm{T}} . \tag{3.4}$$

Since S is real, this equation implies

$$\xi'=S\xi\mathrm{e}^{\mathrm{i}\varphi} , \tag{3.5}$$

where φ is real arbitrary. That is, using (2.14) for S,

$$\xi'_1=(a\xi_1+b\xi_2)\mathrm{e}^{\mathrm{i}\varphi} ,$$

$$\xi'_2=(c\xi_1+d\xi_2)\mathrm{e}^{\mathrm{i}\varphi} . \tag{3.6}$$

Now, one can eliminate φ in (3.6) and write

$$q'=(aq+b)/(cq+d) . \tag{3.7}$$

This is our generalized *abcd*-law: Arrange the elements of the variance matrix V into a real quantity $\Omega=[\langle x^2\rangle\langle s^2\rangle-(\langle x\cdot s\rangle)^2]^{1/2}$ and a complex quantity $q=\langle x\cdot s\rangle/\langle s^2\rangle+\mathrm{i}\Omega/\langle s^2\rangle$. [Clearly, given $\Omega>0$ and q with $\mathrm{Im}(q)>0$, the elements of V can be uniquely reconstructed.] Under the action of a first order system, Ω remains invariant (det V is an invariant) and q undergoes transformation according to the generalized *abcd*-law (3.7).

We shall illustrate these results with gaussian Schell-model beams whose cross-spectral density is [4]

$$\Gamma(\boldsymbol{x}_1,\boldsymbol{x}_2)=A\frac{2}{\pi}\frac{1}{\sigma_{\mathrm{I}}^2}\exp\left[-\frac{\boldsymbol{x}_1^2+\boldsymbol{x}_2^2}{\sigma_{\mathrm{I}}^2}-\frac{1}{2}\frac{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2}{\sigma_{\mathrm{g}}^2}\right]$$
$$\times\exp[(\mathrm{i}k/2R)(\boldsymbol{x}_1^2-\boldsymbol{x}_2^2)] . \tag{3.8}$$

Clearly A, $\sigma_{\mathrm{I}}/\sqrt{2}$, σ_{g} and R are respectively the total irradiance, beam width, coherence length and radius of curvature. Using (3.8) in (2.3), the Wolf function is computed to be

$$W(Q)=(A/\pi^2)\det(G)\exp(-Q^{\mathrm{T}}GQ) ,$$

$$G=\begin{pmatrix} 2/\sigma_{\mathrm{I}}^2+k^2\gamma^2/2R^2 & -k^2\gamma^2/2R \\ -k^2\gamma^2/2R & k^2\gamma^2/2\end{pmatrix}, \tag{3.9}$$

where the parameter γ is defined through

$$1/\gamma^2=1/\sigma_{\mathrm{g}}^2+1/\sigma_{\mathrm{I}}^2 . \tag{3.10}$$

From (3.9) it is readily seen that the variance matrix is

$$V=G^{-1}=\begin{pmatrix} \sigma_{\mathrm{I}}^2/2 & \sigma_{\mathrm{I}}^2/2R \\ \sigma_{\mathrm{I}}^2/2R & 2/k^2\gamma^2+\sigma_{\mathrm{I}}^2/2R^2\end{pmatrix}, \tag{3.11}$$

and

$$\Omega^2=\det V=\sigma_{\mathrm{I}}^2/k^2\gamma^2 . \tag{3.12}$$

We recall that the invariant Ω is related to the degree of global coherence [4]. Now $q=(V_{12}+\mathrm{i}\Omega)/V_{22}$ implies $q^{-1}=(V_{12}-\mathrm{i}\Omega)/V_{11}$. We thus have for the gaussian Schell-model beam

$$1/q=1/R-\mathrm{i}(2/k\gamma\sigma_{\mathrm{I}}) . \tag{3.13}$$

Comparing this expression with eq. (4.10) of ref. [4] we see that q in this case is indeed the complex radius of curvature of gaussian Schell-model beams. In other words, we find that our generalized *abcd*-law (3.7) specialized to gaussian-model beams yields our earlier generalization [#2] of Kogelnik's *abcd*-law to these beams.

To conclude this section, we note that in the coherent limit $\sigma_{\mathrm{g}}\to\infty$, (3.8) goes over to the coherent

[#2] Recently Turenen and Friberg [25] have considered the possibility of generalizing Kogelnik's *abcd*-law to gaussian Schell-model beams. It is our belief that these authors were unfamiliar with our earlier work in ref. [4] wherein this problem had been solved.

gaussian beam. In this limit $\gamma=\sigma_I$ as seen from (3.10), and q^{-1} becomes

$$1/q=1/R-\mathrm{i}\,(2/k\sigma_I^2)\,. \tag{3.14}$$

Thus q obeying the generalized *abcd*-law in this case is the familiar complex radius of curvature for coherent gaussian beams, and hence (3.7) becomes Kogelnik's *abcd*-law [16].

4. Concluding remarks

We have analysed the propagation of the beam width and angular spread of an arbitrary partially coherent beam through first order systems, and derived a generalized *abcd*-law which describes this propagation. A complex quantity q with $\mathrm{Im}(q)>0$ constructed from the variance matrix obeys this generalized law. For coherent gaussian beams and partially coherent gaussian Schell-mode beams we have shown that q becomes the respective complex radii of curvature, and thus our generalized law reproduces Kogelnik's *abcd*-law for coherent gaussian beams and our generalization of the latter law to the gaussian Schell-model beams in ref. [4].

Since V is a 2×2 real symmetric positive definite matrix it can be represented by a time-like vector with positive time component in a $2+1$ dimensional Minkowski space. Then the transformation (2.20) will act as a Lorentz transformation in this space. That this geometric picture leads to powerful results has been demonstrated in refs. [4,26] [#3].

In the case of gaussian Schell-model beams of which the coherent gaussian beams form a special case, the variance matrix completely fixes the Wolf function, and hence the cross-spectral density, through $G=V^{-1}$ as can be seen from (3.9). Thus, for these beams the *abcd*-law alone is sufficient to describe completely their propagation through first order systems. For an arbitrary partially coherent beam, however, the generalized *abcd*-law describes only the propagation of the second moments. It will be worthwhile to study the propagation of higher order moments through first order systems, and find ways of expressing these propagations in a convenient form as in the case of the generalized *abcd*-law. Another interesting problem in this connection is to classify partially coherent beams into families closed and irreducible under the action of first order systems.

As noted earlier, first order systems correspond to evolution under quadratic hamiltonians in the quantum mechanical context. The squeeze operator is such a hamiltonian corresponding to the linear magnifier $S(m)$ in the optical case, and the beam width–angular spread pair corresponds to fluctuations in a conjugate pair of variables in the quantum context. Thus, it becomes clear that the generalized *abcd*-law is of relevance to the study of squeezed states [27].

[#3] An approach to squeezed states similar to the present one can be found in ref. [26].

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