

Hamilton's Theory of Turns Generalized to $\text{Sp}(2, R)$

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We present a generalization of Hamilton's geometric theory of turns, originally invented for $\text{SU}(2)$, to the noncompact group $\text{Sp}(2, R)$ relevant in a variety of physical applications.

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Hamilton's method of turns¹ gives a remarkably vivid pictorial description of the elements and structure of the group $\text{SU}(2)$. It associates "turns" (equivalence classes of directed great circle arcs on a unit sphere) with elements of the group in a natural manner, and introduces a noncommutative geometrical "addition" or composition rule for them which reproduces the composition law of the group. The resulting geometrical picture for $\text{SU}(2)$ is analogous to that for the Abelian translation group by free vectors with the "parallelogram" law of addition. Thus, the Pancharatnam angle,² which constitutes an early example of the Berry phase³ as generalized by Aharonov and Anandan⁴ for nonadiabatic situations, is the "sum" of the turns corresponding to the $\text{SU}(2)$ polarization transformations taking the polarization vector over a closed circuit on the unit Poincaré sphere.⁵ It is unfortunate that this elegant work of Hamilton is not as widely known as it deserves to be, though a recent lucid and comprehensive account of it has been provided by Biedenharn and Louck.⁶

The relevance of $\text{SU}(2)$ in diverse quantum-mechanical problems needs no emphasis. A closely related noncompact simple Lie group, equally important in a variety of physical applications, is the group of real linear canonical transformations $\text{Sp}(2, R) = \text{SL}(2, R)$, isomorphic to $\text{SU}(1, 1)$. It is the twofold covering group or spinor group of the three-dimensional Lorentz group $\text{SO}(2, 1)$. We show in this Letter that Hamilton's ideas can be generalized to $\text{Sp}(2, R)$ in a useful manner. To stress that we are now dealing with a noncompact group, we use the term "screw" in place of "turn".

Fundamental to Hamilton's geometric representation is the fact that for any two unit vectors \mathbf{n}, \mathbf{n}' on the unit sphere S^2

$$u(\mathbf{n}, \mathbf{n}') = \mathbf{n} \cdot \mathbf{n}' - i \mathbf{n} \times \mathbf{n}' \cdot \boldsymbol{\sigma} \in \text{SU}(2). \quad (1)$$

Here $\boldsymbol{\sigma}$ are the Pauli matrices. It is easy to see that any $u \in \text{SU}(2)$ can be written as $u(\mathbf{n}, \mathbf{n}')$ for suitable choice of \mathbf{n}, \mathbf{n}' . The ordered pair $(\mathbf{n}, \mathbf{n}')$ can be represented by the directed great circle arc (simply arc, hereafter) of length $\leq \pi$ with tail at \mathbf{n} and head at \mathbf{n}' . Since $u(\mathbf{n}, \mathbf{n}')$ is unchanged if \mathbf{n} and \mathbf{n}' are subjected to the same $\text{SO}(2)$ rotation about $\mathbf{n} \times \mathbf{n}'$ as axis, we see that all arcs obtained by sliding a given arc over its great circle represent one and the same $\text{SU}(2)$ element. Such an equivalence class

of arcs is called a turn, and we have a one-to-one correspondence between turns and elements of $\text{SU}(2)$. From Eq. (1) we have

$$u(\mathbf{n}, \mathbf{n}')^{-1} = u(\mathbf{n}', \mathbf{n}), \quad (2a)$$

$$u(\mathbf{n}', \mathbf{n}'')u(\mathbf{n}, \mathbf{n}') = u(\mathbf{n}, \mathbf{n}''). \quad (2b)$$

From (2a) we see that inverses correspond to reversed turns, and (2b) gives the "addition" rule for turns: Given two $\text{SU}(2)$ elements u_1, u_2 , use the $\text{SO}(2)$ freedom of sliding arcs to choose the two representative arcs such that the head of the u_1 arc coincides with the tail of the u_2 arc. Then the turn from the free tail to the free head corresponds to the $\text{SU}(2)$ product $u_2 u_1$.

Now we consider the group $\text{Sp}(2, R)$ for which we will construct geometrical objects in a $(2+1)$ -dimensional Minkowski space $M_{2,1}$ with metric and Levi-Civita symbol conventions

$$\eta_{ab} = \text{diag}(-1, +1, +1), \quad \epsilon_{012} = 1. \quad (3)$$

In place of the Pauli matrices we choose the set ρ_a defined by

$$\rho_0 = \sigma_2, \quad \rho_1 = i\sigma_3, \quad \rho_2 = i\sigma_1, \quad (4)$$

so that any $A \in \text{Sp}(2, R)$ can be uniquely written, using a scalar λ and a Lorentz vector $\boldsymbol{\mu} \in M_{2,1}$, as

$$A = \lambda + i\boldsymbol{\mu} \cdot \boldsymbol{\rho} = \lambda + i\mu^a \rho_a, \quad (5a)$$

$$\lambda^2 - \boldsymbol{\mu} \cdot \boldsymbol{\mu} = 1, \quad A^T \sigma_2 A = \sigma_2. \quad (5b)$$

The $\text{Sp}(2, R) \rightarrow \text{SO}(2, 1)$ homomorphism associates with A a Lorentz transformation "about $\boldsymbol{\mu}$ as axis." This leads to a classification of $\text{Sp}(2, R)$ elements which is important for the theory of screws. We shall say $A \in \text{Sp}(2, R)$ is of type t , l , or s according as the vector $\boldsymbol{\mu}$ is timelike, lightlike, or spacelike. Since $\boldsymbol{\mu} \cdot \boldsymbol{\mu} = \lambda^2 - 1$, these correspond, respectively, to $|\lambda| < 1$, $|\lambda| = 1$, and $|\lambda| > 1$. The identity element and its negative value, not covered in the above, correspond to vanishing $\boldsymbol{\mu}$. Every element retains its type under conjugation by any element of $\text{Sp}(2, R)$. We have here a classification of finite-group elements going beyond a similar classification of Lie algebra elements.⁷ This is related to the known fact that $\text{Sp}(2, R)$ is not of exponential type: The range of the exponential map from $\text{sp}(2, R)$ into $\text{Sp}(2, R)$

consists of all elements of type t , type l with $\lambda=1$, type s with $\lambda > 1$, and the two elements $\pm \mathbb{1}$.

The role of S^2 in the $SU(2)$ discussions is now played by the unit single-sheeted spacelike hyperboloid Σ in $M_{2,1}$:

$$\Sigma = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{x} = 1\} \subset M_{2,1}. \quad (6)$$

On Σ the analogs of great circles are constructed in the following way. For given $A \in \text{Sp}(2, R)$ with axis $\boldsymbol{\mu}$, we define $\mathcal{C}(\boldsymbol{\mu})$ to be the intersection of Σ with the plane in $M_{2,1}$ (Lorentz) orthogonal to $\boldsymbol{\mu}$ and passing through the origin:

$$\mathcal{C}(\boldsymbol{\mu}) = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{x} = 1, \boldsymbol{\mu} \cdot \mathbf{x} = 0\}. \quad (7)$$

The nature of $\mathcal{C}(\boldsymbol{\mu})$ depends on that of $\boldsymbol{\mu}$: For $\boldsymbol{\mu}$ of type t , l , and s , $\mathcal{C}(\boldsymbol{\mu})$ is, respectively, an ellipse, two parallel straight lines (generators of Σ), and the two branches of a hyperbola.

As a first step towards generalizing the $SU(2)$ construction of $u(\mathbf{n}, \mathbf{n}')$ to $\text{Sp}(2, R)$, note that for any two unit vectors $\mathbf{x}, \mathbf{y} \in \Sigma$,

$$A(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} + i\mathbf{x} \times \mathbf{y} \cdot \boldsymbol{\rho} \in \text{Sp}(2, R). \quad (8)$$

A detailed analysis⁸ shows that the converse is also true: Given any $A \in \text{Sp}(2, R)$ we can choose \mathbf{x}, \mathbf{y} on the corresponding $\mathcal{C}(\boldsymbol{\mu})$ such that $\lambda = \mathbf{x} \cdot \mathbf{y}$, $\boldsymbol{\mu} = \mathbf{x} \times \mathbf{y}$, so that A equals $A(\mathbf{x}, \mathbf{y})$. In fact we can choose \mathbf{x} or \mathbf{y} as we wish on $\mathcal{C}(\boldsymbol{\mu})$, the other being then uniquely determined. We define a screw for $\text{Sp}(2, R)$ as an equivalence class of ordered points (\mathbf{x}, \mathbf{y}) on a $\mathcal{C}(\boldsymbol{\mu})$, the equivalence being with respect to motion along $\mathcal{C}(\boldsymbol{\mu})$ induced by $SO(2, 1)$ transformation about $\boldsymbol{\mu}$ as axis, and with respect to inversion about the origin. Depending on the nature of $\boldsymbol{\mu}$, a screw is of type t , l , or s . The special elements $\pm \mathbb{1} \in \text{Sp}(2, R)$ correspond to the equivalence classes $\mathbf{y} = \pm \mathbf{x}$. Clearly, there is a one-to-one correspondence between elements of $\text{Sp}(2, R)$ and screws.

Next we turn to the question of connecting \mathbf{x} and \mathbf{y} by a directed arc along $\mathcal{C}(\boldsymbol{\mu})$. Here the interplay between our t , l , or s classifications and the nonexponential nature of $\text{Sp}(2, R)$ shows up in an interesting way. In the t case $\mathcal{C}(\boldsymbol{\mu})$ is a connected curve (ellipse) and we readily have a connected arc from \mathbf{x} to \mathbf{y} along $\mathcal{C}(\boldsymbol{\mu})$. In the l and s cases, $\mathcal{C}(\boldsymbol{\mu})$ is made of two pieces; and in these cases if λ is negative (i.e., A is not in the range of the exponential map), then \mathbf{x} and \mathbf{y} are definitely on different branches of $\mathcal{C}(\boldsymbol{\mu})$.⁸ This difficulty can be simply handled by making use of the fact that if A is not in the range of the exponential map, then $-\mathbb{1}A$ is, and can be definitely represented by a connected arc on $\mathcal{C}(\boldsymbol{\mu})$ from \mathbf{x} to $-\mathbf{y}$ or from $-\mathbf{x}$ to \mathbf{y} . Since $-\mathbb{1}$ commutes with all elements of $\text{Sp}(2, R)$ it can be treated as a "flag" in the composition of screws (see below) corresponding to multiplication of $\text{Sp}(2, R)$ elements. Thus, in all cases a screw is a pair consisting of an equivalence class of directed connected arcs on a $\mathcal{C}(\boldsymbol{\mu})$ and a flag which assumes the values $\pm \mathbb{1}$.

To complete the theory of screws we derive the geometrical rule for their composition to reproduce the $\text{Sp}(2, R)$ multiplication. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Sigma$, we have from (8)

$$A(\mathbf{x}, \mathbf{y})^{-1} = A(\mathbf{y}, \mathbf{x}), \quad (9a)$$

$$A(\mathbf{y}, \mathbf{z})A(\mathbf{x}, \mathbf{y}) = A(\mathbf{x}, \mathbf{z}). \quad (9b)$$

Equation (9a) shows that inverses correspond to reversed screws, and (9b) contains the geometric rule for composition of screws. Given two elements $A, B \in \text{Sp}(2, R)$, if the corresponding $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ intersect at \mathbf{y} , say, on Σ , then we can choose \mathbf{x} on $\mathcal{C}(\boldsymbol{\mu}_A)$ and \mathbf{z} on $\mathcal{C}(\boldsymbol{\mu}_B)$ such that $A = A(\mathbf{x}, \mathbf{y})$ and $B = A(\mathbf{y}, \mathbf{z})$. Then (9b) implies that the screw from \mathbf{x} to \mathbf{z} corresponds to the $\text{Sp}(2, R)$ product BA . But unlike the great circles on S^2 , $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ may not always intersect. In fact $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ will intersect if and only if the vector $\boldsymbol{\mu}_A \times \boldsymbol{\mu}_B$ is of type s . Of the six possible kinematical situations for the pair $\boldsymbol{\mu}_A, \boldsymbol{\mu}_B$, in the four cases tt, tl, ts , and ll , $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ definitely intersect so that (9b) immediately gives the geometric addition rule for screws. In the other two cases the situation is indefinite: Depending on the specific vectors $\boldsymbol{\mu}_A$ and $\boldsymbol{\mu}_B$, $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ may or may not intersect.

Fortunately, the following (rather remarkable) result⁸ comes to our rescue: Any element $B \in \text{Sp}(2, R)$ can be written (in many ways) as the product $B = B''B'$ where both $B'', B' \in \text{Sp}(2, R)$ are of type t . With this decomposition theorem, which expresses an interesting structural property of the group $\text{Sp}(2, R)$, the product BA of any two elements can be handled geometrically as an operation on screws, requiring at most two applications of (9b). If A, B belong to one of the four cases tt, tl, ts , or ll , we "slide" the representative arcs on $\mathcal{C}(\boldsymbol{\mu}_A)$ and $\mathcal{C}(\boldsymbol{\mu}_B)$ till the "head" of the A screw and the "tail" of the B screw coincide at $\mathbf{y} \in \mathcal{C}(\boldsymbol{\mu}_A) \cap \mathcal{C}(\boldsymbol{\mu}_B)$. Then a single use of (9b) gives for BA the screw from the tail of the A screw to the head of the B screw. If A, B belong to either the ls or ss case, and $\mathcal{C}(\boldsymbol{\mu}_A) \cap \mathcal{C}(\boldsymbol{\mu}_B) = \emptyset$, we use the t - t decomposition theorem to write $BA = B''B'A$ with both B'' and B' being of type t . The screws for B' and A can then be composed geometrically using (9b) to give the screw for $B'A$; this can then be composed with the screw for B'' , using (9b) again, to give the screw for BA .

This completes our generalization of Hamilton's theory to the noncompact group $\text{Sp}(2, R)$. The role of great circles on S^2 for $SU(2)$ is now played by $\mathcal{C}(\boldsymbol{\mu})$'s which are the intersections of planes through the origin with Σ . Two difficulties were encountered: The first one was related to the nonexponential nature of $\text{Sp}(2, R)$, and the second to the fact that, unlike two great circles, two $\mathcal{C}(\boldsymbol{\mu})$'s are not guaranteed to intersect. The former was overcome with the notion of a flag and the latter using the t - t decomposition theorem.

To conclude we outline some applications. The basic building blocks of first-order optics⁹ are $F(d)$, free propagation through a distance d , and $L(g)$, thin lens of

power g . Both are of type l and are represented by the $\text{Sp}(2, \mathcal{R})$ matrices with $\lambda_d = 1$, $\mu_d = d(1, 0, -1)$, and $\lambda_g = 1$, $\mu_g = g(1, 0, 1)$, respectively. $\mathcal{C}(\mu_d)$ consists of the pair of straight lines $(a, \pm 1, -a)$, $a \in \mathcal{R}$, on Σ and $\mathcal{C}(\mu_g)$ consists of $(a, \pm 1, a)$. In Fig. 1, $DB = BE$ is the screw¹⁰ for $F(1)$ and $FB = BC$ is the screw for $L(1)$. An important first-order system is the Fourier transformer \mathcal{F} with $\text{Sp}(2, \mathcal{R})$ matrix $i\rho_0$. Its screw is the arc AB on the waist circle. Now consider the $\text{Sp}(2, \mathcal{R})$ product $F(1)L(1)F(1)$ which represents the focal-plane-to-focal-plane transformation produced by a thin lens of unit power. To compute this product, "add" the screw DB for $F(1)$ to the lens screw BC for $L(1)$ to obtain the screw DC for $L(1)F(1)$. Slide DC along its $\mathcal{C}(\mu)$ to AD and then add it to DB to obtain AB , the screw for $F(1)L(1)F(1)$. But AB is the Fourier-transformer screw. Thus, we have a simple pictorial representation of the fact that the focal-plane-to-focal-plane transformation is indeed a Fourier transformation. Another way of realizing \mathcal{F} becomes obvious from Fig. 1. Add FB to BE to obtain FE , the screw for $F(1)L(1)$. Slide it to AF and add to it FB to obtain AB as the screw for $L(1)F(1)L(1)$, showing that two lenses of unit power separated by unit distance produce immediately after the second lens the Fourier transform of the field distribution immediately before the first lens.

Our classification of the $\text{Sp}(2, \mathcal{R})$ elements is of

relevance to periodically focusing systems of which a laser resonator is an example.¹¹ To find the ray-transfer matrix for n periods we have to add n replicas of the screw for one period. Clearly, a screw and its n -fold sum are on the same $\mathcal{C}(\mu)$. Thus, we have a bounded system if and only if $\mathcal{C}(\mu)$ is closed; i.e., if it is of type t the usual stability condition $|\text{tr} A| < 2$ is the same as $|\lambda| < 1$.

In an earlier paper⁹ Gaussian pure states were described geometrically as points on the positive branch of the timelike unit hyperboloid Ω in $\mathcal{M}_{2,1}$, but the composition of $\text{Sp}(2, \mathcal{R})$ systems acting on these states was handled algebraically. Now we have a geometrical description also of the systems and their composition. Given a screw and an arbitrary state P on Ω , construct the plane containing the screw (and the origin). Let the plane through P parallel to this plane intersect Ω along the curve γ . It is clear that, under the $\text{SO}(2, 1)$ transformation produced by the screw, P moves on γ . As one consequence we see that P will be an eigenstate of the screw if and only if γ was a point, i.e., if the latter plane was a tangent plane to Ω . It readily follows that for every type- t element of $\text{Sp}(2, \mathcal{R})$, and only for this type, there is a Gaussian eigenstate, and that every Gaussian pure state is the eigenstate of a one-parameter subgroup of $\text{Sp}(2, \mathcal{R})$ systems.

Given an arbitrary screw we can always slide it so that its tail falls on the waist as shown by AB in Fig. 2. Now

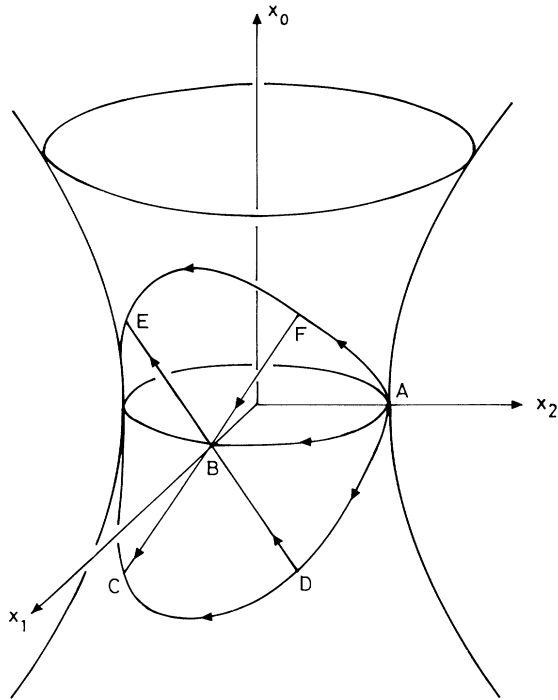


FIG. 1. Synthesis of Fourier transformer. The coordinates of the marked points are $A = (0, 0, 1)$, $B = (0, 1, 0)$, $C = (-\frac{1}{2}, 1, -\frac{1}{2})$, $D = (-\frac{1}{2}, 1, \frac{1}{2})$, $E = (\frac{1}{2}, 1, -\frac{1}{2})$, and $F = (\frac{1}{2}, 1, \frac{1}{2})$.

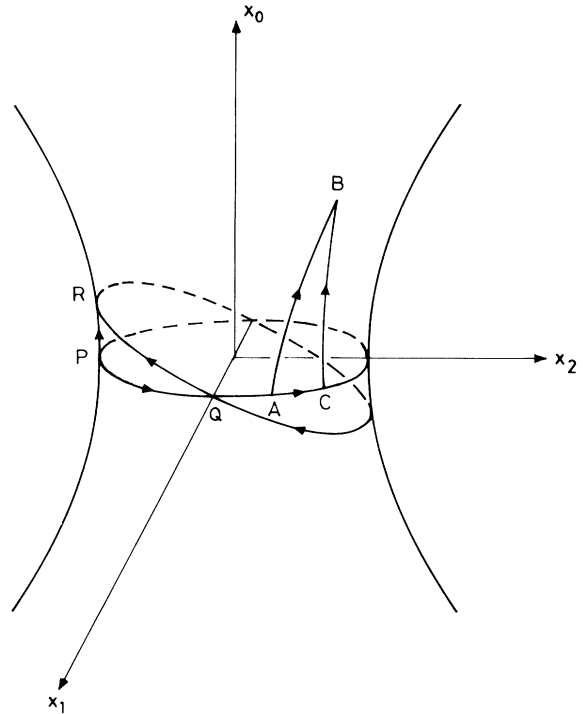


FIG. 2. Decomposition of an arbitrary screw into a waist screw and a vertical screw, and a scheme for squeezing based on the t - t decomposition theorem.

construct the vertical screw CB lying in the plane containing the x_0 axis, so that the given AB is the sum of the waist screw AC and the vertical screw CB . Since waist screws are $SO(2)$ rotations and vertical screws are boosts, it follows that every $Sp(2, R)$ transformation is uniquely a rotation followed by a boost.

Finally, we consider the problem of squeezing.¹² Evolution under the free Hamiltonian H_0 corresponds to a waist screw. The nonlinear interaction results in an s -type generator of the form

$$H_1 = \alpha(xp + px)/2 = i\alpha(a^2 - a^{\dagger 2})/2$$

in the Hamiltonian. If the nonlinearity is not strong enough to make this term dominate H_0 , the total Hamiltonian $H_0 + H_1$ will be a generator of type t and the state will squeeze and unsqueeze during every period resulting in a periodic evolution. Our t - t decomposition theorem suggests a scheme for producing squeezing even with small H_1 . Let the Hamiltonian be periodic with H_0 acting for a fraction v and $H_0 + H_1$ for the other fraction $(1-v)$ of the period. With $v=0.75$, these two type- t evolutions are given by the screws PQ and QR in Fig. 2. The screw for one full period is PR which is a vertical screw representing monotone squeezing. Details of these and other applications will be presented elsewhere.

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¹⁰For brevity sometimes a representative arc itself will be called the screw.

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