Geometrical and Algebraic Aspects of Nonlinear Field Theory
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Topology and Quantum Internal Symmetries in Nonlinear Field Theories

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Abstract

The inequivalent quantization over a given configuration space Q can be associated with the representations of the fundamental group $\pi_1(Q)$. Nonabelian finitely generated groups are considered as candidates for $\pi_1(Q)$ for several field theories and the possibility of nonscalar quantizations examined.

Introduction

Given a dynamical system with configuration space Q the space of states may be chosen as functions from Q into the complex numbers C. We have become aware in recent times that not only can we use multiple valued "scalar state vectors" but we could consider N-component "vector state vectors" taking values in C^N. When $q \in Q$ is taken around a generic loop l in Q [1]

$$\psi_n(q) \longrightarrow \sum_{m=1}^{N} V_{nm}([l])\psi_m(q)$$

where $V([l])$ is an $N \times N$ unitary matrix depending on the homotopy class $[l]$ of the loop l. Consequently

$$V([l_1])V([l_2]) = V([l_1 l_2])$$

and

$$\sum_{n=1}^{N} \psi_n^\dagger(q)\psi_n(q)$$

is invariant.

Thus $V([l])$ furnish an N-dimensional unitary representation of the fundamental homotopy group $\pi_1(Q)$. Clearly superposition holds only for vectors which belong to the same representation. Consequently the dynamics must be formulated in any one such irreducible representation and the decomposition of the state space into such superselection sectors furnishing irreducible unitary representations (IUR) of the fundamental group is an essential step. If $\mathcal{R}(\pi_1(Q))$ denote the set of all finite-dimensional IUR's of $\pi_1(Q)$ the quantum theories defined by the sectors $\mathcal{A}_\alpha$ of the state space, $\alpha \in \mathcal{R}(\pi_1(Q))$ represent the "prime" quantizations of the original system. $\mathcal{R}$ contains the trivial IUR, but in general $\mathcal{R}$ will contain other elements revealing the essential "kinematic ambiguity" in choosing the quantum theory.

The one-dimensional IUR's may be called the scalar quantizations and are labelled by the set

$$\Omega = \mathrm{Hom}\,(\pi_1(Q), U(I)) \cong \mathrm{Hom}\,(H_1(Q), U(I))$$

where $H_1(Q)$ is the first (integral) homology group of Q.

Please note that the discussion here is at the dynamical level. All the operators are inert with regard to the fundamental group; and this extends to the dynamical evolution and its expression in terms of the propagator. This is borne out by the explicit calculation of path integral in multiply-connected spaces by Morandi and Menossi [2].

On the other hand if the statespace belongs to a representation Ω of the fundamental group with character χ this character may be incorporated into the propagator. Such a symmetry adapted propagator used by Laidlaw and de Witt-Morette [2] annihilates the initial states with other symmetries while the Morandi-Menossi type symmetry inert propagates the initial states irrespective of its symmetry type. Related comments apply to nonscalar realizations.

The many-dimensional IUR's are non-scalar and correspond to an internal symmetry of topological origin.

On each $\mathcal{H}_\alpha$, $\alpha \in \mathcal{R}$ we look for a complete set of operators [3]. The position operator $\hat{q}$, the finite displacement operator $\hat{T}^\sigma_{qq'}$, from q to q' along path σ and finally the internal symmetry operators $\tau(\gamma)$, $\gamma \in \pi_1(Q)$

$$\psi_n(q\gamma) = \sum_{m=1}^{N} \tau_{nm}(\gamma)\psi_m(q)$$

Position operator: $\hat{q}$ = multiplication by q

Displacement $\hat{T}^\sigma_{qq'} = Pe^{i\int_\sigma D}$ $\quad D = -i1\frac{\delta}{\delta q} - A(q)$, where $A(q)$ is the flat connection which always exists [8]

$$\hat{T}^l_{qq} = U(l) \longrightarrow \tau([l])$$

Some Theorems on Groups

A group which is in its own commutator subgroups is called perfect.

Theorem 1 [3] The scalar quantization is unique if and only if $\pi_1(Q)$ is a perfect group. A group with no nontrivial finite dimensional IUR is a U-inert group; if it has no finite dimensional nonscalar IUR it is a U-scalar group.

Theorem 2 [4] A finitely generated group is U-inert if and only if has no nontrivial finite quotient groups.

Theorem 3 [4] A finitely generated group is U-scalar if and only if it has no finite nonabelian quotient groups.

We wish to see if U-inert and U-scalar groups can arise as the fundamental group suitable smooth manifold. Compact 2-manifolds were completely classified; none of them have a nonabelian U-scalar fundamental group [5]. For 3-manifolds it is a longstanding conjecture that their fundamental groups are residually finite and this in turn implies that their fundamental groups cannot be nonabelian U-scalar [5].

(Each $g \in G$, $g \neq e$, there exists a normal subgroup N of G, $g \notin N$ such that G/N is finite, G is residually finite).

For dim $M \geq 4$ the situation is vastely different.

Theorem 4 [4] There exists, for every $n \geq 4$, a compact orientable n-manifold with $\pi_1(M)$ being any finitely presented group.

The first non-trivial U-inert groups in the mathematical literature seem to be the Higman groups H_n, $n \geq 4$.

They are defined by [6]

$$H_n = < a_1, a_2, \ldots, a_n | \begin{array}{r} a_1^{-1}a_2a_1 = a_2^2 \\ a_2^{-1}a_3a_2 = a_3^2 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_n^{-1}a_1a_n = a_1^2 \end{array} \rangle$$

The discussion of U-inert groups naturally brings in the question of whether the fundamental groups, can be permitted to have infinite dimensional IUR's. We can show using a theorem of Wehrfitz [7] that:

1. Every nontrivial U-inert group has an infinite dimensional complex representation.

2. Every nonabelian finitely generated U-scalar group has at least one complex infinite dimensional representation.

Unfortunately we cannot strenghten this result to "unitary" in place of complex.

Pure Gauge Theories

For a pure gauge theory with compact space manifold Σ of any dimension d and structure group G defined with respect to a principal G-bundle P over Σ take the affine space $\mathcal{A}$ of all connection one-forms over P. Let $\mathcal{G}_*$ be the group of automorphisms of P which fix Σ and the fibre above a chosen base point of Σ. Then $\mathcal{A}/\mathcal{G}_*$ is the infinite dimensional coordinate manifold. Take P to be the trivial bundle $\Sigma \times G$. Then $\mathcal{G}_*$ is G_*^Σ, the set of all basepoint preserving maps from Σ into G with the identity as basepoint of G. So chosen $\mathcal{G}_*$ acts freely on $\mathcal{A}$.

Then

$$\pi_1(\mathcal{A}/\mathcal{G}_*) = \pi_0(G_*^\Sigma)$$

For $\Sigma = S^3$ with a simple nonabelian Lie group G

$$\pi_1(\mathcal{A}/\mathcal{G}_*) = \pi_3(G) \cong Z$$

Hence

$$\Omega = \mathrm{Hom}\,(\pi_1(\mathcal{A}/\mathcal{G}_*), U(1)) \cong U(1)$$

The corresponding realization is labelled by an angle θ, wellknown from $Q(1)$, but seen here already at the kinematic level. The results are unaltered if we add ordinary (non-Higgs) matter fields to the gauge fields.

For general choice of Σ and G, Imbo [10] has shown using Posnikov techniques that

(1) $\pi_1(\mathcal{A}/\mathcal{G}_*)$ is solvable.

(2) Derived length of $\pi_1(\mathcal{A}/\mathcal{G}_*) \leq \left\{ \begin{array}{c} \text{number of nontrivial} \\ \text{homotopy groups of } G \\ \text{from dim 1 to dim } \Sigma \end{array} \right.$

It then follows that $\pi_1(\mathcal{A}/\mathcal{G}_*)$ can be

$$\text{abelian} \longleftrightarrow U\text{-scalar}$$

$$\text{trivial} \longleftrightarrow U\text{-inert}.$$

If dim $\Sigma = 1$, $\pi_1(\mathcal{A}/\mathcal{G}_*)$ is abelian; dim $\Sigma = 2$ since $\pi_2(G) = e$ for any Lie group G, π_1 is still abelian. For dim $\Sigma = 3$ Isham had shown that $\pi_1(\mathcal{A}/\mathcal{G}_*)$ is abelian for all compact simple Lie groups. Despite these disappointments, for sufficiently large dim Σ one should be able to construct a gauge theory with a nonabelian fundamental group.

Gravity Theories

For gravity theories Σ is a connected compact 3-manifold. The configuration space may be chosen as [9]

$$\mathrm{Riem}(\Sigma)/\mathrm{Diff}_F(\Sigma) = \mathrm{R/D}$$

and consequently

$$\pi_1(\mathrm{R/D}) \cong \pi_0(\mathrm{Diff}_F(\Sigma))$$

There are choices of Σ for which π_1 is abelian or nonabelian. An example of particular interest in the present context is the Poincaré sphere

$$\Sigma = S^3/I^*$$

The binary icosahedral group I^* acts freely on S^3. Moreover [12]

$$\pi_1(\mathrm{R/D}) = I^* \qquad ; \qquad |I^*| = 120$$

I^* is perfect and so has no nontrivial one-dimensional representation. But it does have several many-dimensional representations including one of dimension 2.

Generalized Sigma Models

The generalized nonlinear sigma models can have $\pi_1(Q)$ nonabelian. The fields are mappings from Σ of dim d into an arbitrary manifold M. These are not all path connected and so a distinguished component is chosen to be the one containing the constant map $\Sigma \longrightarrow m_0 \in M$. If we choose the configuration space to be M_*^Σ (basepoint fixed), then Imbo has recently shown [10]:

$\pi_1(M_*^\Sigma, c)$ is solvable of lenght $\leq$
number of nontrivial homotopy groups of M
from dim 2 to dim $d+1$

For $\pi_1(M^\Sigma, c)$ the situation is different; it need not be solvable. In particular $\Sigma = S^d$, $d \geq 1$ provided M is $(d+1)$-simple

$$\pi_1(M^{S^d}, c) \simeq \pi_{d+1}(M) \times \pi_1(M)$$

So that if $\pi_{d+1}(M) = e$, then

$$\pi_1(M^{S^d}, c) \cong \pi_1(M)$$

Since by theorem 4 we can construct a manifold with $\pi_1(M)$ U-scalar or U-inert and $\pi_{d+1}(M) = e$ the corresponding sigma model can be made to have unique quantization even though (M^{S^d}, c) is multiply connected. For $d = 1$ these may have bearing on closed bosonic string theory.

Discussion

The exploration in this paper has been on the inequivalent quantum descriptions of a physical system with a given configuration space Q. There are as many distinct quantum systems at the kinematical level as the IUR's of $\pi_1(Q)$. The theories with IUR's with degree greater than one possess an internal quantum symmetry of topological origin. We saw that nontrivial $\pi_1(Q)$ may exist which nevertheless admits only scalar or even only unique quantizations. Various quantum field theories were examined to see where scalar and vector quantizations were possible and several possibilities were identified. Since we do not have much confidence or experience in nonscalar quantum theories it may be desirable to first get familiar with them by studying simple quantum systems with finite number of degrees of freedom which admit of nonscalar quantization with $Q = S^3/Q_8$ where Q^8 is the quaternion group. A flat connection is obtained by projecting

$$\tilde{A} = iV^{-1}dV; \qquad V(\alpha, \beta, \gamma) = e^{i\alpha\sigma_{3/2}} e^{i\beta\sigma_{2/2}} e^{i\gamma\sigma^{3/2}}$$

onto S^3/Q_8. The space of state vectors corresponding to the 2-dimensional IUR can then be constructed. Similarly for 3 particles moving on R^2 or S^2 the nonscalar IUR's can be constructed and studied. The physical significance of nonscalar quantum mechanics deserves more detailed study.

Appendix: General Questions of Statistics

1. Scalar Statistics

$$\Omega_n(M) = \mathrm{Hom}(\pi_1(Q_n(M)), U(1)) \cong \mathrm{Hom}(H_1(Q_n(M)), U(1))$$

For dim $M \geq 3$ or M closed 2-manifold not S^2, then

$$H_1(Q_n(M)) \cong H_1(M) \oplus \mathrm{Z}_2; \qquad n \geq 2$$

$$\Omega_n(M) \cong \Omega_1 \oplus \mathrm{Z}_2; \qquad n \geq 2$$

$$\Omega_n(M)/\Omega_1(M) \cong \mathrm{Z}_2. \qquad \text{Bose or Fermi.}$$

$$M = S^2 \quad , \quad H_1(Q_n(S^2)) \cong \Omega_n(S^2) \cong \mathrm{Z}_{2n-2}$$

$$M = R^2 \quad , \quad H_1(Q_n(R^2)) \cong U(1).$$

2. Nonscalar Statistics

$$\dim\ M \geq 3 \quad , \quad \pi_1(M^n - \Delta) = \{e\}$$

so

$$\pi_1((M^n - \Delta/S_n) = S_n \equiv B_n(M)$$

n	2	5	10	20	50
$p(n)$	2	7	42	627	204,266

$$\dim\ M = 2 \quad B_n(R^2) \qquad n-\text{string Artin braid group.}$$

$$B_n(R^2) = <\sigma_1, \ldots, \sigma_{n-1}| \begin{array}{ll} \sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}; & 1 \leq r \leq n-2 \\ \sigma_r\sigma_s = \sigma_s\sigma_r \quad |r-s| \geq 2 & \end{array} \Big\rangle$$

$$B_2(R^2) = Z_2$$

$$B_3(R^2) = < a, b | a^3 = b^2 >$$

$$a = e^{2i\phi} \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \qquad b = e^{3i\phi} \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \qquad \omega^3 = 1$$

$$a = e^{2i\phi} \begin{pmatrix} 1 & 0 & O \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad b_{jk} = e^{3i\phi}(-\delta_{jk} + 2n_j n_k)$$

and a host of other realizations of increasing dimensions.

$$B_3(S^2) = < c, d | c^3 = d^2 \quad ; \quad d = cdc > \quad ; \quad |B_3(S^2)| = 12$$

$$c = \lambda^2 \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad d = \lambda^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$B_n(S^2)$, $B_n(R^2)$ for $n \geq 4$ are infinite and difficult to deal with.

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