Comment on "Gauge Invariance in Chern-Simons Theory on a Torus"

In a recent paper Hosotani[1] claims to have proved the quantization of the Chern-Simons term on a torus. Because his result applies not only to the non-Abelian theory but also to the Abelian case (which had previously defied attempts at quantization), it is important that the various steps in his proof be subjected to careful scrutiny. This Comment reports the result of one such examination and the detection of a crucial flaw in his argument.

Hosotani bases his analysis on the photonless gauge theory[2] which he attempts to solve on a torus. His assertion is that the field equations for A^μ imply

$$A_0=\frac{e}{\mu}\int dy\, D(x-y)\nabla\times J\,,$$

$$A_i=\frac{1}{eL_i}\theta_i(t)+\frac{\epsilon_{ij}x_jQ}{2\mu L_1L_2} \qquad (1)$$

$$+\frac{e}{\mu}\int dy\, D(x-y)\epsilon_{ij}\partial_j\left[J^0(y)-\frac{Q}{L_1L_2}\right],$$

where the θ's are time-dependent quantities which are crucial to the quantization argument. The function $D(x)$ is claimed to satisfy $\nabla^2D=\delta(x)$ and $\int D(x)=0$, but can be seen, in fact, to be a solution of the equation $\nabla^2D=\delta(x)-1/L_1L_2$.

Unfortunately, (1) is not an acceptable solution. This result follows immediately from a consideration of the Lagrangian equation

$$\mu\epsilon_{ij}(\partial_0A_j-\partial_jA_0)=eJ_i\,,$$

and the substitution of the expressions (1) for the term in parentheses. A contradiction is obtained unless A_0 is modified by the inclusion of a term $(1/e)\sum x_i\dot{\theta}_i/L_i$. Furthermore, the condition that the Hamiltonian generate the time development of ψ requires the inclusion of a term in A_0 which is an integral over J. This is necessary in order to cancel a term in the commutator of ψ with the part of A_i which is proportional to $\epsilon_{ij}x_jQ$. The final result is given by

$$A_0=\partial_0\Lambda+(e/\mu)\int dy\, J\times\nabla_y\mathcal{D}(x,y)\,, \qquad (2)$$

$$A_i=\partial_i\Lambda+(e/\mu)\epsilon_{ij}\nabla_j\int\mathcal{D}(x,y)J^0(y)\,,$$

where

$$\Lambda=(1/e)\sum x_i\theta_i/L_i$$

and

$$\mathcal{D}(x,y)\equiv D(x-y)+\frac{x^2+y^2}{4L_1L_2}=\mathcal{D}(y,x)\,.$$

The function $\mathcal{D}(x,y)$ is verified to satisfy

$$\nabla^2\mathcal{D}(x,y)=\delta(x-y)\,.$$

This modification of the form of A_0 has a remarkable effect on the argument in Ref. 1 claiming that θ_1 and θ_2 are canonically conjugate. In particular, substitution of (2) into the Lagrangian yields a precise cancellation between the terms quadratic in θ and thus a total absence of any basis for the quantization of e^2/μ which has been argued by Hosotani. Evidently, the Λ term in A^μ can be eliminated by a gauge transformation leaving one with a solution which is identical to that of Ref. 2 except for the introduction of the inverse Laplacian appropriate to a torus. Finally, it may be remarked that the extension of the above to the other cases considered in Ref. 1 is immediate.

The question of whether the coefficient k in the Chern-Simons term need be quantized for the torus and other manifolds has been studied by several authors. In particular, Polychronakos[3] has categorically concluded that it is not necessary with a similar view having been expressed by Lee.[4] Jackiw has called our attention to the possibility[5] of rewriting the theory in an equivalent form with only the gauge-invariant variable $f^\mu=\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta$ [see the comments following Eq. (17) in Ref. 5]. Also Elitzur *et al.*[6] have shown that for quantized k there is a close connection with rational conformal theories. While these general studies are worthwhile, they do not alter the above demonstration of the inconsistency of Hosotani's "solutions."

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