# IDENTICAL PARTICLES, EXOTIC STATISTICS AND BRAID GROUPS

Tom D. IMBO , Chandni SHAH IMBO [b] and E.C.G. SUDARSHAN [a]

[a] *Center for Particle Theory, University of Texas at Austin, Austin, TX 78712, USA*

[b] *Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA*

Received 24 September 1989

The inequivalent quantizations of a system of $n$ identical particles on a manifold M, dim M $\geqslant$ 2, are in 1–1 correspondence with irreducible unitary representations of the braid group $B_n(M)$. The notion of the statistics of the particles is made precise. We give various examples where all the possible statistics for the system are determined, and find instances where the particles obey statistics different from the well-studied Bose, Fermi para- and $\theta$-statistics.

For the past four decades theoretical physicists have had a love–hate relationship with the idea of particles obeying exotic (i.e., non-Bose and non-Fermi) statistics. The best known of these are the so-called parastatistics [1] that can occur in particle mechanics and field theory. These statistics are associated with higher dimensional representations of the permutation groups, just as Bose and Fermi statistics are associated with the one-dimensional representations. More recently, there has been much interest in the fractional (or $\theta$-) statistics that can occur in some two-dimensional systems [2,3]; they are labelled by an angle $\theta$ and smoothly interpolate between the Bose ($\theta=0$) and Fermi ($\theta=\pi$) cases. Fractional statistics play a role in several theoretical investigations in 2D condensed-matter physics (e.g., the fractional quantum Hall effect [4] and high-$T_c$ superconductivity [5]). The purpose of this Letter is to demonstrate the existence of exotic statistics other than those mentioned above, and to develop a procedure for their classification.

In particle mechanics, the above exotic possibilities are a consequence of the non-trivial topology of the relevant configuration space. When quantizing a classical system with configuration space [#1] Q, the standard procedure is to construct the fixed time quantum mechanical state vectors as functions from Q into the complex numbers $\mathbb{C}$. However, more generally we may choose them as sections of a $\mathbb{C}^N$-bundle over Q, $N\geqslant 1$. The classical limit of a quantum theory built as above on a bundle $B$ will differ from the original classical system by the introduction of an external gauge potential; namely, the natural $U(N)$ connection on $B$. In order to classify the inequivalent quantizations of a *fixed* classical system, we require this connection to be flat so as not to change the classical equations of motion. On each such bundle, the holonomy of the flat connection provides an $N$-dimensional unitary representation of the fundamental group $\pi_1(Q)$. Conversely, given any such representation $\rho$, one can construct a complex vector bundle whose holonomy realizes $\rho$ [6]. If $\rho$ is reducible, then the corresponding bundle $B_\rho$ breaks up into a Whitney sum of bundles $\{B_{\rho_i}\}$, where the $\rho_i$ are the irreducible components of $\rho$. A similar decomposition of the Hilbert space of sections of $B_\rho$ occurs. If we let $\mathscr{R}(\pi_1(Q))$ denote the set of all (equivalence classes of) finite-dimensional irreducible unitary representations (IUR's) of $\pi_1(Q)$, then the quantum theories associated with the irreducible bundles $B_\alpha$, $\alpha\in\mathscr{R}$, represent the "prime quantizations" of the original system [#2]. $\mathscr{R}$ always contains at least one ele-

[#1] We assume that the system is not interacting with any external field. We also take $Q$ to be path connected.

[#2] These are the only quantizations of genuine interest since the Hilbert space of any other quantization is just the direct sum of the Hilbert spaces of various prime quantizations. Throughout this work a "quantization" will always mean a "prime quantization".

ment, namely the trivial IUR, and the associated quantum theory has ordinary complex-valued functions as state vectors. However, in general $\mathscr{R}$ will contain more than one element, showing the essential "kinematical ambiguity" in quantizing a classical system [#3]. The quantizations corresponding to $N=1$, the so-called scalar quantum theories, are labelled by the character group $\Omega$ of $\pi_1(Q)$; $\Omega=\mathrm{Hom}(\pi_1(Q), U(1))\cong \mathrm{Hom}(H_1(Q), U(1))\cong H^1(Q, U(1))$ [8]. The quantum theories associated with irreducible $\mathbb{C}^N$-bundles, $N>1$, possess an "internal symmetry" of topological origin associated with the entire system [9].

To determine the inequivalent quantizations of a given system one must identify the configuration space Q, calculate $\pi_1(Q)$, and then construct $\mathscr{R}(\pi_1(Q))$. We now carry out this program for the system of $n$ identical particles moving on a smooth, path-connected manifold M (without boundary) of dimension $d\geqslant 2$ [#4]. For $n=1$, Q is simply M and the IUR's of $\pi_1(M)$ label the inequivalent quantizations. If $n\geqslant 1$ and the particles are *distinguishable*, then $Q=M^n$, the $n$-fold cartesian product of M with itself. However, when the particles are *identical* we must identify any two points of $M^n$ which differ only by a permutation of the particle labels. The configuration space could then be the orbit space of $M^n$ under this action by $S_n$, the permutation group on $n$ symbols. We denote this space by $M^n/S_n$, called the $n$-fold symmetric product of M. There are two problems with the choice $Q=M^n/S_n$. First, the $S_n$ action on $M^n$ has fixed points and therefore Q is not, in general, a smooth manifold; hence ordinary techniques of quantization utilizing the tangent bundle of Q cannot be applied. Second, even if a consistent quantization procedure can be found, one can demonstrate that only theories with Bose statistics will be obtained since we have included points of coincidence of two or more particles in our configuration space [#5]. One may remedy both of the above problems by removing from $M^n$ the subcomplex $\Delta$ consisting of all points where two or more particle coordinates coincide. Now $S_n$ acts freely (i.e., without fixed points) on $M^n-\Delta$ and the orbit space $(M^n-\Delta)/S_n\equiv Q_n(M)$ is a smooth manifold. We choose this manifold as our configuration space. The group $\pi_1(Q_n(M))\equiv B_n(M)$ is called the $n$-string braid group of M [11]. (Note that $B_1(M)\cong\pi_1(M)$.) The set $\mathscr{R}(B_n(M))$ of IUR's of $B_n(M)$ labels the inequivalent quantizations. Speaking vaguely for a moment, the different quantizations are related to the different possible "statistics" for the $n$ identical particles ($n\geqslant 2$), but one must be careful not to overcount. There is, in general, a quantization ambiguity already present for $n=1$ (and therefore having nothing to do with statistics) which will manifest itself again in $\mathscr{R}(B_n(M))$ for any $n$. To get the set which labels the different statistics, one must take $\mathscr{R}(B_n(M))$ and "mod out" by $\mathscr{R}(B_1(M))$ in an appropriate way.

An element of $B_n(M)$ can be thought of as a homotopy class of paths in $M^n-\Delta$ whose (fixed) initial and final points are related by a permutation of the particle coordinates. It is straightforward to identify a set of such paths which generate $B_n(M)$. (In what follows we will speak of an element of $M^n-\Delta$ as an ordered set of $n$ distinct points in one copy of M. We also identify a path with its homotopy class.) Fix $n$ distinct points $m_1, m_2, ..., m_n$ in M which together will represent the initial point of our paths. First, consider a path which takes a given point, say $m_1$, around a noncontractible loop in M and fixes the points $m_2$ through $m_n$. Call the set of all such paths $\mathscr{L}$. Next, let D be an open $d$-disk in M containing $m_1$, ..., $m_n$. For each $i<n$ consider a path $\sigma_i$ which interchanges $m_i$ and $m_{i+1}$ in D, not enclosing any of the other $m_j$ which all remain fixed. Denote the set of all $\sigma_i$, $1\leqslant i\leqslant n-1$, by $\mathscr{P}$. $\mathscr{L}$ and $\mathscr{P}$ generate $B_n(M)$. For example $\sigma_1^{-1}\circ l\circ\sigma_1$, $l\in\mathscr{L}$, is (homotopic to) a path which takes $m_2$ around a loop in M, fixing and avoiding all other particles, while the path $l\circ\sigma_1$ interchanges $m_1$ and $m_2$ in a nonsimply connected region of M, etc. A nice property of this set of generators is that it decouples into those pertaining to the loop topology of M (namely $\mathscr{L}$) and those associated with permutations alone ($\mathscr{P}$). The relations among these generators may be very complicated and M-dependent. In particular for $d=2$ they can mix the $\mathscr{L}$ and $\mathscr{P}$ generators in a highly nontrivial way (see below).

[#3] We will ignore possible ambiguities of dynamical origin. The above classification can be found in ref. [7].

[#4] Quantizations on nonflat bundles may yield new types of "statistics" other than these considered below. See, e.g., ref. [10]. However, such statistics no longer have a purely kinematical definition.

[#5] A proof and further discussion of this will be given in a forthcoming paper by the authors using the ideas in this work.

Let $\Sigma_n(M)$ be the subgroup of $B_n(M)$ generated by $\mathscr{P}$. It is clear that the statistics of the $n$ identical particles on M provided by an IUR $\rho$ of $B_n(M)$ is determined by $\rho \downarrow \Sigma_n(M)$, the restriction of $\rho$ to $\Sigma_n(M)$. ($\rho \downarrow \Sigma_n$ is, in general, reducible.) We propose the following definition:

*Definition*. Two IUR's $\rho_1$ and $\rho_2$ of $B_n(M)$ are *statistically equivalent* (written $\rho_1 \sim \rho_2$) if for some positive integers $s$ and $t$

$$\mathbb{1}_s \otimes (\rho_1 \downarrow \Sigma_n) \simeq \mathbb{1}_t \otimes (\rho_2 \downarrow \Sigma_n) .$$

(Here the symbol "$\simeq$" means equivalence as representations, $\otimes$ denotes the inner tensor product, and $\mathbb{1}_s$ and $\mathbb{1}_t$ are the trivial representations of $\Sigma_n$ of dimensions $s$ and $t$ respectively.) The presence of $\mathbb{1}_s$ and $\mathbb{1}_t$ in the above equality accounts for differences which only pertain to the distinct dimensionalities of $\rho_1$ and $\rho_2$. It is easy to check that "$\sim$" is an equivalence relation on $\mathscr{R}(B_n(M))$. Therefore, $\mathscr{R}(B_n(M))$ breaks up into equivalence classes, each containing only IUR's whose corresponding quantizations yield the same statistics for the $n$ identical particles. If M is simply connected then $\Sigma_N(M) = B_n(M)$ and distinct quantizations give distinct statistics as expected. Our definition provides a natural generalization to the case $\pi_1(M) \neq \{e\}$.

Since the $S_n$ action on $M^n - \Delta$ is free, we have the following vibration [11]:

$$\begin{array}{c} S_n \hookrightarrow M^n - \Delta \\ \downarrow \\ Q_n(M). \end{array} \qquad (1)$$

The long exact homotype sequence [12] of this vibration yields the following short exact sequence for $B_n(M)$:

$$\{e\} \to \pi_1(M^n - \Delta) \xrightarrow{\alpha} B_n(M) \xrightarrow[\beta]{} S_n \to \{e\} . \qquad (2)$$

The generators $\mathscr{L}$ of $B_n(M)$ are in the kernel of the epimorphism $\beta$, while the generators $\sigma_i$ in $\mathscr{P}$ map onto the corresponding transpositions in $S_n$. Thus $\beta \downarrow \Sigma_n$ is an epimorphism from $\Sigma_n(M)$ onto $S_n$. Given an IUR $\rho$ of $S_n$, one can "lift" it to an IUR $\tilde{\rho}$ of $B_n(M)$, i.e., $\tilde{\rho}(b) \equiv \rho(\beta(b))$ for all $b \in B_n(M)$. Clearly $\tilde{\rho} \downarrow \Sigma_n$ is the lift of $\rho$ to $\Sigma_n(M)$. So there are at least as many distinct choices of statistics for the $n$ particles as there are IUR's of $S_n$. The statistics so obtained correspond to the parastatistics mentioned earlier. (Here we consider Bose and Fermi statistics as special cases of parastatistics.) In general, there will be many other possibilities.

The codimension of $\Delta$ in $M^n$ is $d$. Therefore if $d \geqslant 3$, standard general position arguments give [13] $\pi_1(M^n - \Delta) \cong \pi_1(M)^n$. Hence by eq. (2), $B_n(M) \cong S_n$ if $\pi_1(M) = \{e\}$, and only parastatistics are possible. In particular this is true for $M = \mathbb{R}^d$, $d \geqslant 3$. Now let $f: \mathbb{R}^d \to M$ be a local coordinate chart. By the naturality [12] of the long exact homotopy sequence of eq. (1) we obtain the following commutative diagram ($d \geqslant 3$):

$$\begin{array}{ccccccccc} \{e\} & \longrightarrow & \pi_1(\mathbb{R}^{dn}) & \longrightarrow & B_n(\mathbb{R}^d) & \xrightarrow{\gamma} & S_n & \longrightarrow & \{e\} \\ & & \downarrow f^n_* & & \downarrow g & & \downarrow h & & \\ \{e\} & \longrightarrow & \pi_1(M)^n & \xrightarrow{\alpha} & B_n(M) & \xrightarrow{\beta} & S_n & \longrightarrow & \{e\} , \end{array}$$

where $h$ is the identity map and $\gamma$ is an isomorphism since $\pi_1(\mathbb{R}^{dn}) = \{e\}$. The map $\tau = g \circ \gamma^{-1} \circ h^{-1}$ is a splitting homomorphism for the short exact sequence for $B_N(M)$, that is, $\beta \circ \tau$ is the identity map on $S_n$. Hence $B_n(M)$ is a semidirect product of $\pi_1(M)^n$ with $S_n$

$$B_n(M) = \pi_1(M)^n \rtimes_\mu S_n .$$

The defining map $\mu: S_n \to \mathrm{Aut}(\pi_1(M)^n)$ is given by ($\pi \in S_n$, $l_i \in \pi_1(M)$, $1 \leqslant i \leqslant n$)

$$\mu(\pi)(l_1, ..., l_n) = (l_{\pi(1)}, ... l_{\pi(n)}) .$$

This semidirect product is called the *wreath product* [14] of $\pi_1(M)$ with $S_n$ and denoted by $\pi_1(M) \wr S_n$. So $B_n(M) = \pi_1(M) \wr S_n$ for $d \geqslant 3$. The image of $\tau$ in $B_n(M)$ is $\Sigma_n(M)$, that is, $\Sigma_n(M) \cong S_n$. However this does not imply that we only obtain parastatistics, for given an IUR $\rho$ of $B_n(M)$, $\pi \downarrow \Sigma_n$ may contain inequivalent irreducible components. Also, the quantization corresponding to an IUR $\rho$ is not just a "direct sum" of many parastatistical quantizations, one for each irreducible component of $\rho \downarrow \Sigma_n$, since $\rho$ is irreducible. However since $\Sigma_n(M)$ is just $S_n$, we call the associated statistics *generalized parastatistics*. An important consequence of the above is that the only types of statistics associated with scalar quantizations (which we call *scalar statistics*) of $n$ identical particles on M, dim $M \geqslant 3$, are Bose and Fermi.

As an example consider two identical particles moving on real projective three-space $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$. Since $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$ we have $B_2(\mathbb{RP}^3) = \mathbb{Z}_2 \wr \mathbb{Z}_2$ which is the dihedral group $D_8$ of order 8. This group is generated by two elements $l$ and $\sigma$ subject to the relations $l^2 = \sigma^2 = e$ and $(l\sigma)^2 = (\sigma l)^2$. We have $\Sigma_2(\mathbb{RP}^3) \cong \mathbb{Z}_2$. There are five IUR's of $D_8$ four have dimension 1 and one has dimension 2. They are given by

$$\rho_1(l) = 1 , \quad \rho_1(\sigma) = 1 , \quad \rho_3(l) = -1 , \quad \rho_3(\sigma) = 1 ,$$

$$\rho_2(l) = 1 , \quad \rho_2(\sigma) = -1 ,$$

$$\rho_4(l) = -1 , \quad \rho_4(\sigma) = -1 ,$$

$$\rho_5(l) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_5(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3)$$

$\rho_1$ and $\rho_3$ give Bose statistics for the two particles, $\rho_2$ and $\rho_4$ yield Fermi statistics and $\rho_5$ provides a new type of "half Bose–half Fermi" exotic statistics. Note that

$$\rho_5(l)\,\rho_5(\sigma)\,\rho_5(l) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which shows the coupling of $\mathscr{L}$ and $\mathscr{P}$. Although there are five inequivalent quantizations of the systems, there are only three inequivalent statistics. We call the statistics associated with $\rho_5$ *ambistatistics*.

For $d=2$ the situation is much more complex, leading to a richer spectrum of stastistics. The first complication is that the codimension of $\Delta$ in $M^n$ is 2 and hence $\pi_1(M^n - \Delta) \ncong \pi_1(M)^n$ in general. For $M = \mathbb{R}^2$ and $n \geqslant 2$, $\pi_1(\mathbb{R}^{2n} - \Delta)$ is not trivial and $B_n(\mathbb{R}^2) = \Sigma_n(\mathbb{R}^2)$ is an infinite group which is nonabelian for $n \geqslant 3$ $(B_2(\mathbb{R}^2) \cong \mathbb{Z})$. For all $n \geqslant 2$, $H_1(Q_n(\mathbb{R}^2)) \cong B_n(\mathbb{R}^2)_{ab} \cong \mathbb{Z}$, and since $\mathrm{Hom}(\mathbb{Z}, U(1)) \cong U(1)$ the scalar quantizations and statistics of the system are labelled by an angle $\theta$ [3] [#6]. These are the fractional statistics discussed previously. We already see that in two dimensions, unlike the situation in higher dimensions, the representation of $\Sigma_n(M)$ defining the statistics need not be equivalent to a representation over $\mathbb{R}$. Also, for an arbitrary two-manifold M, the extension in eq. (2) will not be split in general. The case $M = S^2$ is well studied [15], $B_2(S^2) \cong \mathbb{Z}_2$ and there are only Bose and Fermi statistics for the two-particle system. $B_3(S^2)$ is the metacyclic group of order 12 which has six IUR's, four of dimension a and two of dimension 2, yielding six distinct statistics [16]. If $n \geqslant 4$, $B_n(S^2)$ is infinite and nonabelian. For all $n \geqslant 2$, $B_n(S^2)_{ab} \cong \mathbb{Z}_{2n-2}$ and the number of scalar quantizations and statistics grows with $n$ [17]. For a further discussion of the higher dimensional IUR's of $B_n(\mathbb{R}^2)$ and $B_n(S^2)$, see ref. [16].

Explicit presentations of the braid groups $B_n(M)$ of all other closed two-manifolds are known [13,18,19]. These groups are all infinite and nonabelian except for $B_2(\mathbb{RP}^2)$ which is generated by $l$ and $\sigma$ with relations $l^2 = \sigma^2 = (\sigma l^{-1})^4$; $\Sigma_2(\mathbb{RP}^2) \cong \mathbb{Z}_4$. $B_2(\mathbb{RP}^2)$ is the dicylic group of order 16 and has seven IUR's, four of dimension 1 and three of dimension 2. They are the "same" as those for $B_2(\mathbb{RP}^3)$ (see eq. (3)) except there are two additional IUR's given by

$$\rho_6(l) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho_6(\sigma) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\rho_7(l) = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho_7(\sigma) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\rho_6$ and $\rho_7$ are statistically equivalent, yielding a type of "fractional" ambistatistics. Thus the scalar statistics are only Bose and Fermi, but there are two types of exotic nonscalar statistics. More generally for *all* closed two-manifolds $M \neq S^2$ and any $n \geqslant 2$, the only scalar statistics are Bose and Fermi since $B_n(M)_{ab} \cong H_1(M) \oplus \mathbb{Z}_2$ and the image of $\Sigma_n(M)$ under the abelianization map is the $\mathbb{Z}_2$ direct factor. This can be demonstrated by explicitly abelianizing the presentations in ref. [19]. Thus these manifolds allow the same scalar statistics as all manifolds of higher dimension. Note the peculiarity of the $M = S^2$ case where $B_n(S^2)_{ab}$ depends on $n$. We do not know if this is the only manifold which displays this behavior. However it can be shown [20] that for any open two-manifold M, $B_n(M)_{ab} \cong B_{n+1}(M)_{ab}$ for $n \geqslant 4$. In a future communication [#7] we will study exotic statistics on $M = S^1$.

[#6] We have used $G_{ab}$ to denote the abelianization of the group G; i.e., $G_{ab}$ is the quotient $G/[G, G]$ where $[G, G]$ is the commutator subgroup of G. For any path-connected space X, $\pi_1(X)_{ab} \cong H_1(C)$.

For footnote see next page.

After the completion of this work we became aware of a recent preprint [21] by Aneziris et al. demonstrating that three-dimensional geons in quantum gravity "may be neither bosons nor fermions (nor paraparticles)". This fact, among others, leads them to conclude "our usual conceptions about the statistics of particle species thus do not seem to be valid in generally covariant theories". In the examples which motivate these statements the identical geons obey what we would here call (fractional) ambistatistics. Thus these strange geon statistics *do* have an analog in particle mechanics, and our examples above may provide a simple theoretical laboratory in which to study the properties of these interesting topological excitations in quantum gravity.

It is a pleasure to thank A.K. Bousefield, Duane Dicus, John Durbin, Cameron Gordon, Gary Hamrick, and Greg Nagao for useful discussions. this work was supported in part by the US Department of Energy under grant no. DE-FG05-85ER40200.

#7 This work will also contain a detailed treatment of the IUR's of wreath products (and their applications to statistics) as well as a proof that all the inequivalent quantizations of $n$ identical particles on M obey the cluster decomposition property.

## References


[1] H.S. Green, Phys. Rev. 90 (1953) 270;
T. Okayama, Prog. Theor. Phys. 7 (1952) 517;
A.M.L. Messiah and O.W. Greenberg, Phys. Rev. 136 (1964) B248.

[2] J.M. Leinaas and J. Myrheim, Nuovo Cimento 37B (1977) 1;
G.A. Goldin, R. Menikhoff and D.H. Sharp, J. Math. Phys. 22 (1981) 1664;
F. Wilczek, Phys. Rev. Lett. 49 (1982) 957;
F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250;
R. Mackenzie and F. Wilczek, Intern. J. Mod. Phys. A 3 (1988) 2827.

[3] Y.S. Wu, Phys. Rev. Lett. 52 (1984) 2103; 53 (1984) 111.

[4] B.I. Halpern, Phys. Rev. Lett. 52 (1984) 1583;
D.A. Arovas, R. Schrieffer and F. Wilczek, Phys. Rev. Lett. 53 (1984) 722.

[5] P.B. Wiegmann, Phys. Rev. Lett. 60 (1988) 821;
R.B. Laughlin, Phys. Rev. Lett. 60 (1988) 2677.

[6] J. Milnor, Commun. Math. Helv. 32 (1957) 215.

[7] C.J. Isham, in: Relativity, groups and topology II, eds. B.S. DeWitt and R. Stora (Elsevier, Amsterdam, 1984).

[8] M.G.G. Laidlaw and C. Morette DeWitt, Phys. Rev. D 3 (1971) 1375;
B. Kostants in: Lecture Notes in Mathematics, Vol. 170 (Springer, Berlin, 1970);
T.D. Imbo and E.C.G. Sudarshan, Phys. Rev. Lett. 60 (1988) 481.

[9] E.C.G. Sudarshan, T.D. Imbo and T.R. Govindarajan, Phys. Lett. B 213 (1988) 471;
A.P. Balachandran, Syracuse University Reports No. SU-4428-361 (1987), SU-4428-373 (1988).

[10] F.H. Bloore, I. Bratley and J.M. Selig, J. Phys. A 16 (1983) 729.

[11] E. Fadell and L. Neuwirth, Math. Scand. 10 (1962) 111.

[12] E. Spanier, Algebraic topology (Springer, Berlin, 1966).

[13] J.S. Birman, Comm. Pure App. Math. 22 (1969) 41.

[14] G. James and A. Kerber, The representation theory of the symmetric group (Addison-Wesley, New York, 1981).

[15] E. Fadell and J. Van Buskirk, Duke Math. J. 29 (1962) 243.

[16] E.C.G. Sudarshan, T.D. Imbo and C. Shah Imbo, Ann. Inst. Henri Poincaré, Phys. Théor. 49 (1988) 387.

[17] J.S. Dowker, J. Phys. A 18 (1985) 3521;
D.J. Thouless and Y.S. Wu, Phys. Rev. B 31 (1985) 1191.

[18] J. Van Buskirk, Trans. Amer. Math. Soc. 122 (1966) 81.

[19] G.P. Scott, Proc. Camb. Phil. Soc. 68 (1970) 605.

[20] G. Segal, Acta Math. 143 (1979) 39.

[21] C. Aneziris, A.P. Balachandran, M. Bourdeau, S. Jo, T.R. Ramadas and R.D. Sorkin, Syracuse University Report No. SU-4228-392 (1988).