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The Structure of Quantum Dynamical Semigroups*

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Abstract

Generalizing classical stochastic processes to quantum systems we get the notion of dynamical mappings of density operators. The completely continuous completely positive mappings are coextensive with the projections of isometries of an extended system. The Kossakowski form of the quantum semigroup generators are derived and shown to be coextensive with the isometric generators of an extended system to the second order and projected to the primary system. The possibility of contractive semigroups in systems with analytic density operators is also studied.

1. Classical Stochastic Processes

For a system with N distinct configurations, the probabilistic states x are given by N-vectors with an L^1 norm and non-negative components

$$\left\{x \mid x_j \geq 0\,, 1 \leq j \leq N\,; \sum_1^N x_j = 1\right\}. \tag{1}$$

The stochastic dynamics of such a system is given by a (linear) map

$$x \to x\,(t) = A\,(t)\,x \tag{2}$$

where the stochastic matrix $A\,(t)$ satisfies the relations

$$\left\{A \mid A_{jk} \geq 0\,; \sum_j A_{jk} = 1\right\}. \tag{3}$$

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These maps constitute a convex set:

$$A = \cos^2\theta A_1 + \sin^2\theta A_2 \tag{4}$$

is an allowed map if A_1, A_2 are allowed maps. *By a direct arithmetical decomposition* we can see that the permutations are among the generating (extremal) elements of the set of stochastic matrices. (They generate "doubly stochastic" matrices.) But there are others.

Each stochastic matrix which is not extremal is a contraction map in the sense that there exists no inverse map from probability vectors to probability vectors valid for all vectors.

Since the trace is preserved there are one or more asymptotic probability vectors when the stochastic map is repeatedly applied. So one eigenvalue is 1. The other eigenvalues can be real or complex with the real part being always less than unity. In real space spanned by the probability vectors there may not be any eigenvector other than those for the eigenvalue unity.

For $N = \infty$ we have the possibility of stochastic matrices of the form

$$A_{j,k} = \delta_{j,k+1} \tag{5}$$

and polynomials in them. The adjoint of A has a null space of dimension one.

2 Quantum Stochastic Process

For quantum systems with N-dimensional Euclidean space as the bare space, the probability vectors x are replaced by density operators ρ with the following properties [16]:

$$\{\rho \mid \rho \geq 0\,;\ \rho^\dagger = \rho;\ \mathrm{tr}\,\rho = 1\}\ . \tag{6}$$

The dynamical maps [18] assign (linearly) to each ρ another density operator according to

$$\rho \rightarrow A\,(t)\,\rho = \rho' \tag{7}$$

$$\rho'_{rs} = \sum A_{rs,r's'}\ \delta_{r's'} \tag{8}$$

where A is an $N^2 \times N^2$ matrix. The constraints on A are:

$$A_{sr,s'r'} = (A_{rs,r's'})^*$$

$$\sum_r A_{rr,r's'} = \delta_{r's'}$$

$$\sum_{rsr's'} A_{rs,r's'} x_r^* x_s y_{r'} y_{s'}^* \geq 0 \text{ for all } x, y. \tag{9}$$

While the last condition assures the positivity of ρ' for all ρ it is weaker than the requirement of complete positivity

$$\sum_{rsr's'} A_{rs,r's'} \zeta_{rr'} \zeta_{ss'}^* \geq 0 \ . \tag{10}$$

By defining

$$B_{rr',ss'} = A_{rs,s'r'} \quad , \quad B^\dagger = B \tag{11}$$

complete positivity would ensure that $B \geq 0$. Not all positive maps are completely positive: the simplest example is

$$\rho \to \rho' = \rho^T.$$

For $N = 2$, this is accomplished by

$$\rho \to \rho' = \frac{1}{2}\{\sigma_1 \rho \sigma_1 + \sigma_3 \rho \sigma_3 + 1 \rho 1 - \sigma_2 \rho \sigma_2\}$$

which has eigenvalues $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$. For arbitrary finite N we can diagonalize B, and if B is completely positive we could write it in the form

$$B_{rr',ss'} = \sum_\alpha B_{rr'}(\alpha) B_{ss'}^\dagger(\alpha) \tag{12}$$

and the map as

$$\rho \to A\rho = \rho' = \sum_\alpha B(\alpha) \rho B^\dagger(\alpha). \tag{13}$$

The set of maps constitute a convex set. Since any finite dimensional convex set is generated by its extremal elements it would be useful to search for the extremal elements. However this is quite complicated and has been solved [10] only for $N = 2$. However for completely positive maps the problem is completely solved [19]. The B matrices are of rank $r \leq N$ for extremal element and $r \leq N^2$ for generic elements and can be fully enumerated [6].

3. Quantum Stochastic Maps as Projections of Isometries

The generic B matrix, and hence the dynamical map, is the contraction of a unitary evolution of a larger system whose states are the tensor product of ρ with R a density operator for another system with $r \times r$ states [19]. We consider a unitary evolution U for the product $\rho \times R$:

$$\rho \times R \rightarrow U\rho \times RU^{\dagger} \ . \tag{14}$$

If we now take the trace with respect to the system with $r \times r$ states we get

$$\rho \rightarrow \rho' = \mathrm{tr}\,(U\rho \times RU^{\dagger}) \ . \tag{15}$$

Write U as a partitioned matrix $U_{\alpha\beta}$ with

$$U_{\alpha 1} = B_{\alpha} \ , \tag{16}$$

this restricted evolution gives the completely positive map given above [14]. In passing we note that for a positive but not completely positive map the B matrix has eigenvalues of both signs and would therefore have a structure

$$B_{rr',ss'} = \sum_{\alpha} B_{rr'}(\alpha)\, B^{*}_{ss'}(\alpha) - \sum_{\beta} C_{rr'}(\beta)\, C^{*}_{ss'}(\beta) \tag{17}$$

so that the extended space has an indefinite metric. While any unitary matrix U with $U_{\alpha 1} = B(\alpha)$ would give a positive (and completely positive) map restrictions would have to be placed on the psuedo-unitary matrix (generalizing U) to guarantee a positive map.

For the generic case $1 \le \alpha \le r \le N^2$ but it can be shown that for extremal maps $1 \le \alpha \le r \le N$. This follows from the condition that for a completely positive map to be extremal

$$B(\alpha) B^{\dagger}(\beta) \quad , \quad 1 \le \alpha, \beta \le r \tag{18}$$

should all be linearly independent; and this is impossible if $r > N$. The requirement of linear independence follows from the observation that if

$$\sum_{\alpha\beta} c_{\alpha\beta} B(\alpha) B^{\dagger}(\beta) = 0 \ . \tag{19}$$

By a linear unitary transformation we could rewrite it in the form

$$\sum_{\alpha} \mu_{\alpha} B'(\alpha) B'^{\dagger}(\beta) = 0 \ . \tag{20}$$

By suitable scaling we can make each μ_{α} smaller than unity. But then if we choose

$$B^{\mathrm{I}}(\alpha) = (1 + \mu_{\alpha})^{1/2} . B(\alpha) \quad , \quad B^{\mathrm{II}}(\alpha) = (1 - \mu_{\alpha})^{1/2} B(\alpha) \tag{21}$$

then $\Sigma_{\alpha} B^{\mathrm{I}}(\alpha) \rho B^{\mathrm{I}\dagger}(\alpha)$ and $\Sigma_{\alpha} B^{\mathrm{II}}(\alpha) \rho B^{\mathrm{II}\dagger}(\alpha)$ are both dynamical maps and the original map is a convex mean:

$$\sum_{\alpha} B^{\mathrm{I}}(\alpha) \rho B^{\mathrm{I}\dagger}(\alpha) = \frac{1}{2}\left(\sum_{\alpha} B^{\mathrm{I}}(\alpha) \rho B^{\mathrm{I}\dagger}(\alpha)\right) + \frac{1}{2}\left(\sum_{\alpha} B^{\mathrm{II}}(\alpha) \rho B^{\mathrm{II}\dagger}(\alpha)\right). \tag{22}$$

The converse is also true: if a dynamical map is extremal $B(\alpha) B^{\dagger}(\beta)$ are linearly independent.

4. Stochastic Maps for Infinite Dimensional Systems

Let S be an infinite dimensional system which has a countable basis of states. The statistical states are represented by infinite density matrices ρ belonging to the trace class and obeying positivity and normalization. The dynamical maps are given by

$$\rho_{rs} \to \sum_{r's'}^{\infty} A_{rs,r's'} \rho_{r's'} \equiv \sum_{r's'} B_{rr',ss'} \rho_{r's'} \tag{23}$$

with

$$\sum_{r}^{\infty} B_{rr',s'r} = \delta_{r's'} \; ; \; B_{ss',r'r} = (B_{rr',s's})^{*} \quad . \tag{24}$$

The B has a "diagonal" form as a Stieltje's integral

$$B_{rr',s's} = \int d\alpha \, B_{rr'}(\alpha) \, B^{*}_{s's}(\alpha) \, \varepsilon(\alpha) \; ; \; \varepsilon^{2}(\alpha) = 1 \quad . \tag{25}$$

The map may now be rewritten

$$\rho \to \int d\alpha \, \varepsilon(\alpha) \, B(\alpha) \rho \, B^{\dagger}(\alpha) \; ; \quad \int d\alpha \, B^{\dagger}(\alpha) \, B(\alpha) = \mathbf{1} \tag{26}$$

If all $\varepsilon(\alpha) = 1$, then the map is *completely positive.* These are the more physical maps as we shall see shortly.

The simplest completely positive map is given by

$$\rho \to V \rho V^{\dagger} \; , \; V^{\dagger} V = \mathbf{1} \quad . \tag{27}$$

This map is generated by an *isometry* which may or may not be unitary. If $V V^{\dagger} \neq 1$ then it must be of the form

$$V V^{\dagger} = \mathbf{1} - \Delta \; , \quad \Delta \geq 0 \tag{28}$$

where Δ is the deficiency subspace of V. If $\Delta = 0$, then we have the unitary map. Note that these are invariant under change of basis:

$$V \to U_1 V U_2^{\dagger} \quad ; \quad \Delta \to U_1 \Delta U_1^{\dagger} ;$$

$$V V^{\dagger} \to \mathbf{1} - \Delta' \; ; \; V^{\dagger} V \to \mathbf{1} \quad . \tag{29}$$

The elementary isometry is:

$$\Lambda_{n,n'} = \delta_{n+1,n'} . \tag{30}$$

This is isometric but has a *deficiency* subspace of dimension 1. The mth-power of Λ would have a deficiency subspace of dimension m. These maps are all *contracting.*

When the Stieltje's integral in (26) reduces to a (countable) sum of terms

$$\rho \to \sum_{\alpha} B(\alpha) \rho B^{\dagger}(\alpha) \tag{31}$$

with the index α running over a finite or infinite set of values positivity is assured by the map; and the normalization requires

$$\sum_{\alpha} B^{\dagger}(\alpha) B(\alpha) = 1 \quad . \tag{32}$$

These completely continuous isometric maps can be viewed as the projection of a (possibly contracting) isometry of an extended system [19]. Consider a space $\mathcal{K}$ which is spanned by the Kronecker product of the states of S in $\mathcal{H}$ and a "reservoir" system R with states in $\mathcal{H}'$. If the state in $\mathcal{H}'$ is denoted by σ then the generic map above can be seen to be a (possibly contracting) isometry in $\mathcal{K}$.

$$\rho \times \sigma \to V(\rho \times \sigma) V^{\dagger} \; ; \; V^{\dagger} V = 1 \; .$$

Now take the projection of this density matrix in $\mathcal{K}$ to a density matrix in $\mathcal{H}$ by taking partial traces in $\mathcal{H}'$:

$$\mathrm{tr}_{\mathcal{H}'} (V(\rho \times \sigma) V^{\dagger}) = \sum_{\alpha, \beta} V_{\alpha, \beta} \rho \, \sigma_{\alpha\beta} (V_{\alpha, \beta})^{\dagger} \tag{33}$$

where we have partitioned V with respect to the space $\mathcal{H}'$ with $V_{\alpha, \beta}$ being matrices in $\mathcal{H}$. If $\sigma_{\alpha\beta}$ is diagonal with the diagonal matrix elements $\sigma_1 = 1, \sigma_n = 0, n \neq 1$ then we could reexpress this map in the form

$$\rho \to \mathrm{tr}_{\mathcal{H}'} \, (V(\rho \times \sigma) V^{\dagger}) = \sum_{\alpha} V(\alpha) \rho V^{\dagger}(\alpha) \tag{34}$$

with

$$V(\alpha) = V_{\alpha, 1} \; ; \; \sum_{\alpha} V^{\dagger}(\alpha) V(\alpha) = 1 \, . \tag{35}$$

So we have obtained the (possibly contracting) map by restriction of the isometry from $\mathcal{K}$ to $\mathcal{H}$. Conversely, if we have such a map we could also construct an extended system $\mathcal{K} \sim \mathcal{H} \times \mathcal{H}'$ with the dimension of the reservoir R being equal to the range of the index set α.

The requirement of complete positivity and complete continuity for $B_{rr',ss'}$ can be relaxed if we are willing to relax the conditions on $\mathcal{H}'$. He would label the system R states in $\mathcal{H}'$ by continuous parameters α, β and write

$$\rho \to \mathrm{tr}_{\mathcal{H}'} \sum_{\beta\beta'} \left\{ V_{\alpha,\beta} \rho \, (V_{\alpha,\beta'})^{\dagger} \sigma_{\beta\beta'} \right\} \tag{36}$$

to obtain

$$\rho \to \int d\alpha \; B(\alpha) \rho B^{\dagger}(\alpha) \; . \tag{37}$$

If the space $\mathcal{H}'$ admitted an indefinite metric we need to write, in place of the above equation,

$$\rho \to \int d\alpha \, B(\alpha) \rho B^{\dagger}(\alpha) \, \varepsilon(\alpha) \; ; \; \varepsilon^{2}(\alpha) = 1 \; . \tag{38}$$

The inverse construction also obtains. Note that if $\mathcal{H}'$ is not separable and / or has indefinite metric so is $\mathcal{K}$.

The elementary contracting map Λ can be expressed in the form

$$\Lambda = \exp(i \pi L)$$

with

$$L_{n,n'} = \delta_{n,n'} + \frac{1 - \delta_{nn'}}{\pi i (n - n')} \; .$$

Given this expression we could compute Λ^{m} ; and any other such generator of the form

$$ULU^{\dagger} \; , \; UU^{\dagger} = U^{\dagger} U = 1$$

for operator $U \Lambda U^{\dagger}$.

5. Quantum Semigroups

Given a Hamiltonian, the temporal evolution is generated by the exponential of the Hamiltonian. If the Hamiltonian is selfadjoint the effective isometry will be a unitary transformation.

$$\rho \to U \rho U^{\dagger} \, ; UU^{\dagger} = U^{\dagger} U = \mathbf{1} \ .$$

$$U = \exp(-itH) \, ; \dot{\rho} = [\rho \, , H] \ . \tag{39}$$

Now let us consider an extended system $R \times S$ with a unitary time development:

$$\rho \times \sigma \to V(\rho \times \sigma) V^{\dagger}$$

and its projection to the subspace $\mathcal{H}$ by partial tracing. Using the partitioning discussed in the last section we can write

$$\rho \to \mathrm{tr}_{\mathcal{H}'} (V(\rho \times \sigma) V^{\dagger}) = V_{\alpha\beta} \rho \, \sigma_{\beta\beta'} (V_{\alpha\beta'})^{\dagger}$$

$$= \sum_{\alpha \beta \beta'} V_{\alpha\beta} \rho \, (V_{\alpha\beta'})^{\dagger} \sigma_{\beta\beta'} \tag{40}$$

which, with the choice $\sigma_{\beta\beta'} = \delta_{\beta 1} \delta_{\beta' 1}$ gives

$$\rho \to \sum_{\alpha} V_{\alpha 1} \rho \, (V_{\alpha 1})^{\dagger} \ . \tag{41}$$

If V is generated by a Hamiltonian $H_{\alpha\beta}$, then, to second order in t^2

$$V_{\alpha 1} = \delta_{\alpha 1} \mathbf{1} - itH_{\alpha 1} + \frac{(it)^2}{2!} \sum_{\beta} H_{\alpha\beta} \, H_{\beta 1} + \ldots$$

$$(V_{\alpha 1})^{\dagger} = \delta_{\alpha 1} \, \mathbf{1} + it (H_{\alpha 1})^{\dagger} + \frac{(it)^2}{2!} \sum_{\beta} (H_{\beta 1})^{\dagger} (H_{\alpha\beta})^{\dagger} + \ldots \ .$$

Therefore

$$\rho \to \rho - it H_{11}\rho + it\rho\,(H_{11})^{\dagger}$$

$$+ \frac{(it)^2}{2!} H_{1\beta} H_{\beta 1}\rho + \frac{(it)^2}{2!}\rho\,(H_{\beta 1})^{\dagger}\,(H_{1\beta})^{\dagger} - (it)^2 \sum_{\alpha} H_{\alpha 1}\rho\,(H^{\dagger}_{\alpha 1}) + \ldots$$

$$= \rho - it\,[h,\rho] + \frac{(it)^2}{2!}[h,[h,\rho]] - t\left(L^{\dagger}_{\alpha} L_{\alpha}\rho + \rho\, L\alpha^{\dagger} L_{\alpha} - 2 L^{\dagger}_{\alpha}\rho\, L_{\alpha}\right) + \ldots \tag{42}$$

where we have used

$$h = H_{11} \qquad L_{\alpha} = t^{1/2} H_{\alpha 1}\ . \tag{43}$$

If the limit of L_{α} as $t \to 0$ is a useful quantity then we have the Kossakowski form [13] for the generator of the semigroup:

$$\dot{\rho} = -i\,[h,\rho] + \sum_{\alpha} [L^{\dagger}_{\alpha},\rho\,]\,L_{\alpha} + L^{\dagger}_{\alpha}\,[\rho, L_{\alpha}]. \tag{44}$$

Thus the projection of a (possibly contracting) isometry of an extended system back to the primitive system yields a quantum semigroup of temporal evolution.

The Kossakowski form for the semigroup of evolutions can be derived from first principles [11, 13]. Let p_j denote the set of N mutually orthogonal one-dimensional self-adjoint projections in N dimensions:

$$p_j p_k = \delta_{jk} p_k \ , \ p^{\dagger}_j = p_j \ ; \ \sum_{j=1}^{N} p_j = 1$$

Then the conditions on the stochastic generator $\mathcal{L}$ in

$$\dot{\rho} = \mathcal{L}\rho$$

are

$$\mathrm{tr}\,[p_j\,(\mathcal{L}\,p_k)] \geq 0 \quad j \neq k \;;$$

$$\sum_j \mathrm{tr}\,[p_j\,(\,\mathcal{L}\,p_k)] = 0 \;;\; 1 \leq j\,,k \leq N \;. \tag{45}$$

If $F_j\,,\,0 \leq j \leq N^2 - 1$ is a set of orthonormal traceless matrices,

$$\mathrm{tr}\left(F_j^\dagger\,F_k\right) = \delta_{jk}\;;F_0 = N^{-1/2}\mathbf{1}$$

then,

$$\sum_j F_j^\dagger\,A\,F_j = \mathrm{tr}\,(A)\,.\,\mathbf{1}\;.$$

Then there exists a set of C_{jk} such that

$$\mathcal{L}\,\rho = \sum C_{jk}\,F_j\,\rho\,F_k^\dagger \;;\; C_{jk}^* = C_{kj}\;. \tag{46}$$

On rewriting Eq. (46) in the form

$$\mathcal{L}\,\rho = -\,i\,[h,\rho] + [G,\rho] + \sum_1^{N^2-1} C_{jk}\,F_j\,\rho\,F_k^\dagger \tag{47}$$

the requirements of tracelessness of $\mathcal{L}\,\rho$ implies

$$G = -\frac{1}{2}\sum_1^{N^2-1} C_{jk}\,F_k^\dagger\,F_j\;. \tag{48}$$

The matrix C_{jk} can be diagonalized by suitable linear transformation among the F to get

$$\mathcal{L}\,\rho = -\,i\,[h,\rho] + \frac{1}{2}\sum C_j\left\{\left[F_j\,,\rho\,F_k^\dagger\right] + \left[F_j\,\rho\,,F_k^\dagger\right]\right\}\;. \tag{49}$$

If all the C_j are positive they can be absorbed into a redefinition of F_j. In such a case the complete positivity condition Eq. (47) are automatically satisfied. The proof that $C_j \geq 0$, required by complete positivity, requires more detailed computations.

6. Quantum Semigroups as Projections of Isometries

Conversely, if we are given a quantum semigroup in the form

$$\dot{\rho} = -i\,[h,\rho] + \sum_{\alpha} \left[L_{\alpha}^{\dagger},\rho\right] L_{\alpha} + \sum_{\alpha} L_{\alpha}^{\dagger}\,[\rho, L_{\alpha}] \tag{50}$$

then we can reconstruct this semigroup dynamics as the projection of an extended dynamics. Let R be a system with vector space $\mathcal{H}'$ and $\mathcal{S}$ the primitive system with vector space $\mathcal{H}$. The space $\mathcal{H}'$ is indexed by α. Now construct an extended Hamiltonian in the space $\mathcal{K} \supset \mathcal{H} \otimes \mathcal{H}'$ by:

$$H_{11} = h$$

$$H_{\alpha 1} = L_{\alpha} / (t)^{1/2}\,;\, H_{1\alpha} = L_{\alpha}^{\dagger} / (t)^{1/2}$$

$$H_{\alpha,\beta}\,,\, \alpha \neq \text{ or } \beta \neq 1 \text{ arbitrary}$$

and a density operator σ in $\mathcal{H}'$ by

$$\sigma_{11} = 1 \quad \sigma_{\alpha\beta} = 0 \qquad \alpha \neq 1\,, \beta \neq 1\ .$$

Then the evolution

$$\frac{d}{dt}\,(\rho \times \sigma) = -i\,[H\,,\rho \times \sigma] \tag{51}$$

in the space $\mathcal{K}$ has as its projection to $\mathcal{H}$ the semigroup described by Kossakowski's equation.

Thus we have shown that every semigroup of evolutions can be realized as the projection of a (possibly contracting) evolution of isometries; and vice versa. For this extended space to be a (separable, positive metric) Hilbert space it is only necessary that the semigroup has the Kassakowski form.

7. Generalized Analytic density Operators and Semigroups

Is it possible to induce a semigroup of time evolutions for a quantum system with a continuous spectrum? As long as we deal with density operators in the space $\mathcal{H}$ there is no question of semigroups since the time evolution is realised by unitary matrices. To get isometries which are not unitary we must go to a generalised space [2,3,4,20]. Such a space is provided by $\mathcal{J}$ where the energy spectrum and a dense set of vectors in $\mathcal{H}$ have counterparts in $\mathcal{J}$ with scalar products preserved.

A contractive semigroup realizing time evolution [1,5] can be obtained for a class of density operators which are analytic in the "Ritz frequencies", the difference of the two continuous indices of the density operator when labelled by the continuous energy variable. If the energy labels are $0 \leq \lambda_1, \lambda_2 < \infty$ then we could use the mean energy and the Ritz frequency ν as labels.

$$\lambda = \frac{1}{2}(\lambda_1 + \lambda_2) \;\; ; \;\; \nu = \lambda_1 - \lambda_2 \; . \tag{52}$$

The time dependence is then given by

$$\rho(\lambda, \nu; t) = \exp(-i\nu t)\,\rho(\lambda, \nu; 0) \; . \tag{53}$$

This density matrix element vanishes outside $-2\lambda < \nu < 2\lambda$. However we generalize this and define the analytic *generalized* density operator elements [21].

$$R(\lambda, \nu) = \frac{1}{2\pi i} \int d\nu' \;\; \rho(\lambda, \nu') / (\nu' - \nu) \; . \tag{54}$$

Clearly $R(\lambda, \nu)$ is nonvanishing even outside the interval $-2\lambda < \nu < 2\lambda$. Now the time dependence of R is given by

$$\int d\nu\, R(\lambda, \nu; t) \;=\; \int d\nu \;\exp(-i\nu t)\, R(\lambda, \nu; 0) \tag{55}$$

$$= 0 \;,\; t < 0 \tag{56}$$

Thus we have a semigroup ; the group composition law holds for $t > 0$ but not for $t < 0$.

Further, the semigroup is contractive. To see this we define the Fourier transform of $R(\lambda, \nu)$ with respect to the Ritz frequencies ν to obtain

$$\mathfrak{R}(\lambda, \tau) = \int R(\lambda, \nu) \exp(i \nu \tau) \, d\nu \ . \tag{57}$$

Then

$$T(t) R(\lambda, \nu) = \exp(i \nu \tau) R(\lambda, \nu)$$

implies that

$$T(t) \mathfrak{R}(\lambda, \tau) = \mathfrak{R}(\lambda, \tau - t) \ . \tag{58}$$

So $\mathfrak{R}(\lambda, \tau - t)$ vanishes in the region $\tau - t < 0$, or $\tau < t$. While we have

$$T(t_1) T(t_2) \mathfrak{R}(\lambda, \tau) = \mathfrak{R}(\lambda, \tau - t_1 - t_2) = T(t_1 + t_2) \mathfrak{R}(\lambda, \tau) \ , \tag{59}$$

the effect of $T(t)$ is a contraction since the functions vanish for $0 < \tau < t$ also; hence it has no inverse defined on the whole set of analytic density operators.

Such density operators which evolve according to a semigroup is obtained by generalizing the states to be analytic: since $\mathfrak{R}(\lambda, \tau)$ vanishes for $\tau < 0$ the quantities $R(\lambda, \nu)$ are analytic in a half plane [1,5] and hence nonvanishing everywhere in ν except at isolated points. They are therefore unphysical. This applies to all semigroup developments. In particular a pure exponential decay, or a collection of exponential decays cannot obtain from a physical density operator but only from one which has all Ritz frequency components.

8. Age of Density Operators

In classical stochastic process the notion of "age" of a state has been introduced [13]. If $A(t)$ is the semigroup of time dependent stochastic maps for $t > 0$ we can have a retracing

$$x \to A(t) x \ , \ t > -\tau_0 \tag{60}$$

for a state of *age* τ_0. If we had an extremal probability vector x_0 with all but one of the elements are zero it has age 0. Any state developed from x_0 by acting with $A(\tau_0)$ has age τ_0. The age depends on x *and* the semigroup.

We could assign a similar notion to the generalised density operator elements $R(\lambda, \nu)$. Any $R(\lambda, \nu)$ for which $\mathfrak{R}(\lambda, \tau)$ vanishes for $\tau > \tau_0$ has age τ_0. The construction of a generic $R(\lambda, \nu)$ from $\rho(\lambda, \nu)$ has age 0.

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