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A group theoretic treatment of the geometric phase

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We define a new unitary operator in the Hilbert space of a quantum system which parallel transports the state of the system along an arbitrary curve in the projective Hilbert space. This operator is geometrical even for an open curve in the sense that it depends uniquely only on the curve and is independent of the Hamiltonian. Using this, when the curve is closed, the geometric phases discovered by Pancharatnam, Berry and Aharonov–Anandan are obtained.

The geometric phase was first discovered by Pancharatnam [1] for the cyclic evolution of a photon polarization state due to a sequence of filtering measurements. Berry [2] showed that during an adiabatic cyclic evolution of an energy eigenstate of any quantum system there is a phase factor which is independent of the rate of evolution. This phase can be measured experimentally. The geometric meaning of this adiabatic geometric phase was pointed out by Simon [3] as parallel transport in a line bundle corresponding to the eigenstate of a particular energy eigenvalue. It has been generalized to Hamiltonians with degenerate eigenvalues [4]. By looking at an arbitrary cyclic evolution of a state, a geometric part of the phase was isolated by Aharonov and Anandan [5]. They also pointed out its geometric meaning as the phase factor obtained in parallel transporting a state around a closed curve on the projective Hilbert space $\mathscr{P}$, which is the set of rays of the Hilbert space $\mathscr{H}$, with respect to a connection arising from the inner product in $\mathscr{H}$. If $\mathscr{H}=\mathrm{C}^n$, then $\mathscr{P}$ is CP^{n-1}, e.g. for $n=2$, $\mathscr{P}=\mathrm{CP}^1$ is the two-dimensional sphere S^2. The Berry phase may be seen to be the adiabatic limit of this more general geometric phase, which has also been measured experimentally [6].

In this note we obtain the geometric phase using an operator which has a geometric meaning even for a non-cyclic evolution. This result is useful also because it is of group theoretic nature and does not explicitly require parallel transport in the natural vector bundle over $\mathscr{P}$ as done previously [5]. In this respect it is analogous to the treatment of the geometric phase due to Anandan and Stodolsky [7].

To establish a physical meaning of the result that we shall obtain, consider a filtering measurement, studied by Pancharatnam, by which we mean that a quantum system which is initially in a superposition of two orthogonal states passes through a filter which absorbs one of the states but lets through the other

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state so that it has a definite phase relation with the original state. For example, the quantum system may be a photon in a superposition of two orthogonal polarization states one of which is absorbed by a polarizer while the other passes through. It was shown [1,8,9] that the final state when a Pancharatnam filtering measurement is made on a given initial state of an arbitrary quantum system may be obtained by parallel transporting the initial state along the shortest geodesic joining the two points in $\mathscr{P}$ representing the two states. The geodesic here is determined by the Fubini–Study metric [10] in $\mathscr{P}$. Hence, the Pancharatnam phase factor is the holonomy transformation obtained by parallel transporting the state around the geodesic polygon defined by the sequence of states obtained by the measurements.

Consider now a particular filtering measurement which results in the parallel transport along a given geodesic segment of this polygon of a given initial state $\xi\in\mathscr{H}$ to a final state which is proportional to $\eta\in\mathscr{H}$. We assume that $\xi^\dagger\xi=1$, $\eta^\dagger\eta=1$. Then the rays to which ξ and η belong to, which are the initial and final points of this geodesic segment, may be represented by the pure state density matrices $\xi\xi^\dagger$ and $\eta\eta^\dagger$. But there are infinite number of operators in $\mathscr{H}$ which obtain the above parallel transport. We ask now which operator $U(\eta,\xi)$ satisfies the following properties:

(i) $U(\eta,\xi)$ parallel transports ξ along the shortest geodesic joining the points $\xi\xi^\dagger$ and $\eta\eta^\dagger$ in $\mathscr{P}$.

(ii) $U(\eta,\xi)$ parallel transports a vector ζ orthogonal to ξ in the subspace spanned by ξ and η to a vector χ that is orthogonal to η along the shortest geodesic in $\mathscr{P}$ joining $\zeta\zeta^\dagger$ and $\chi\chi^\dagger$.

(iii) $U(\eta,\xi)$ leaves unchanged all vectors orthogonal to the ξ, η subspace.

Since $U(\eta,\xi)$ maps a given orthornormal basis into another definite orthonormal basis, it is a unitary operator uniquely determined by ξ and η. We now show that $U(\eta,\xi)$ is the rotation

$$U(\eta,\xi)=\frac{2\eta\eta^\dagger\xi\xi^\dagger-\eta\eta^\dagger-\xi\xi^\dagger+P(\eta,\xi)}{\sqrt{\xi^\dagger\eta\eta^\dagger\xi}}+1-P(\eta,\xi)\,, \tag{1}$$

where $P(\eta,\xi)$ is the projection matrix to the ξ, η complex plane, i.e. it projects a general complex vector to the ξ, η plane. Explicitly,

$$P(\eta,\xi)=\frac{(\xi\xi^\dagger-\eta\eta^\dagger)(\xi\xi^\dagger-\eta\eta^\dagger)}{1-\xi^\dagger\eta\eta^\dagger\xi}\,, \tag{2}$$

The property (iii) follows from the fact that $U(\eta,\xi)$, defined in eq. (1), depends only on η and ξ. To prove (ii) and (iii), we shall use a theorem in ref. [9] stated below eq. (3.1). Let

$$\xi'=U(\eta,\xi)\xi=\frac{\eta^\dagger\xi\eta}{\sqrt{\xi^\dagger\eta\eta^\dagger\xi}}\,, \tag{3}$$

then $(\xi')^\dagger\xi$ is positive. Hence, property (i) follows as a special case of the abovementioned theorem. To prove (ii), suppose $\eta=a\xi+b\zeta$, where ξ and ζ are orthonormal and $|a|^2+|b|^2=1$. Then, on using eq. (1), $\zeta^\dagger U(\eta,\xi)\zeta=|a|$, which is positive, provided ξ, η are non-orthogonal. The abovementioned theorem then implies property (ii). (In the degenerate case of ξ and η being orthogonal, there are infinite number of "shortest" geodesics joining the corresponding points on $\mathscr{P}$ and $U(\eta,\xi)$ cannot be defined uniquely.) It is easy to verify that

$$U(\eta,\xi)U(\xi,\eta)=1\,, \tag{4}$$

$$U(\eta,\xi)\xi\xi^\dagger U(\xi,\eta)=\eta\eta^\dagger\,. \tag{5}$$

In the Hilbert subspace spanned by ξ and η, $\xi\xi^\dagger$ and $\zeta\zeta^\dagger$ are opposite points of the corresponding projective space which is a sphere S^2. $U(\eta,\xi)$ corresponds to rotations along great circles which move these two points to two other opposite points, $\eta\eta^\dagger$ and $\chi\chi^\dagger$ of this sphere.

So $U(\eta,\xi)$, given by (1), parallel transports two vectors of an orthonormal basis and leaves the others invariant, which are also special cases of parallel transport. In this sense, $U(\eta,\xi)$ may be regarded as a special case of the operator that parallel transports each vector of an orthonormal basis, which was introduced by Anandan and Stodolsky [7]. It follows from their arguments [7] that $U(\eta,\xi)\in SU(n)$. This can also be verified directly by writing eq. (1) in a basis in which, without loss of generality, $\xi=(1,0,0,...,0)$ and $\eta=(\cos\theta,\sin\theta,0,...,0)$, and observing that $\det U(\eta,\xi)=+1$. But the group of operators used by Anandan–Stodolsky and Anandan [7], who study only continuous Schrödinger evolution and not Pancharatnam measurements, is determined by a class of Hamiltonians corresponding to all possible values of a set of parameters. And this could be a proper subgroup of SU(n). There need not be an element of their group which transforms between any two normalized ξ, $\eta\in\mathscr{H}$. But the group generated by the set of operators of the form (1), for all possible pairs (η, ξ), can be shown to be SU(n).

If we have a cyclic evolution of η in the sense of Pancharatnam, then by multiplying the sequence of the operators (1) associated with the sides of the corresponding geodesic polygon in $\mathscr{P}$, we obtain an operator $W\in\mathrm{SU}(n)$ uniquely determined by this polygon. By acting on the initial η by W, we would obtain the geometric phase factor as the eigenvalue.

Another physical meaning can be given for the operator (1) as follows. Suppose there is a continuous Schrödinger evolution from ξ to η such that the expectation value of the Hamiltonian $\langle H\rangle=0$, but the uncertainty of energy $\Delta E(t)$ for this evolution is such that

$$\frac{1}{\hbar}\int \Delta E\,\mathrm{d}t$$

is the minimum of all possible evolutions from $\xi\xi^\dagger$ to $\eta\eta^\dagger$. Then the evolution is along the shortest geodesic joining $\xi\xi^\dagger$ and $\eta\eta^\dagger$ [11]. Therefore, in this case,

$$\eta=U(\eta,\xi)\xi\,.$$

An example of geodesic motion is provided by the motion of the polarization state of light when it passes through a quarter-wave plate. The polarization states of a photon form a two-dimensional Hilbert space whose projective space is the Poincaré sphere. The two circularly polarized states, being orthogonal, may be taken as the north and south poles of this sphere. Then the linearly polarized states form the equator. Suppose light, originally in a linearly polarized state, enters a suitably oriented quarter-wave plate and leaves it circularly polarized, the motion of the polarization state inside this plate is along the geodesic of the Poincaré sphere that starts from a point on the equator and ends at one of the poles.

More generally, consider an evolution $\xi(t)$ around an arbitrary loop $l(t)=\xi(t)\xi^\dagger(t)$ in an arbitrary projective Hilbert space $\mathscr{P}$ such that

$$\xi(0)\xi^\dagger(0)=\xi(T)\xi^\dagger(T)\,. \tag{6}$$

This is a cyclic evolution and it could be obtained by a suitable time dependent Hamiltonian H. We assume further that $\langle H\rangle=0$, which implies that $\xi(t)$ is being parallel transported along this curve [5]. Eq. (6) implies

$$\xi(T)=W\xi(0)=\mathrm{e}^{\mathrm{i}\varphi}\xi(0)\,, \tag{7}$$

where W is an element of the little group which changes $\xi(0)$ only by a phase. By approximating l by a polygon with an indefinitely large number of sides, we shall obtain such a W by multiplying our operators U for the sides of this polygon. Then by the earlier argument, $W\in\mathrm{SU}(n)$. Hence, from (7) the set of such operators W for all possible cyclic evolutions beginning and ending at a fixed point on $\mathscr{P}$ is a subgroup of $\mathrm{S}(\mathrm{U}(n-1)\times\mathrm{U}(1))$. This is unlike the holonomy transformations studied by Anandan and Stodolsky [7] which belong to $\mathrm{S}(\mathrm{U}(1)\times...\times\mathrm{U}(1)$ (n times)).

To obtain an explicit expression for U associated with an arbitrary loop $l(t)$, let us take two infinitesimally close points $l(t_1)$ and $l(t_2)$ on the loop. Let

$$l(t_1)=(\eta-\Delta)(\eta-\Delta)^\dagger\,,$$

$$l(t_2)=(\eta+\Delta)(\eta+\Delta)^\dagger\,. \tag{8}$$

We consider the sequence of transformations defined by

$$w(\eta,\xi,\Delta)=U(\xi,\eta+\Delta)U(\eta+\Delta,\eta-\Delta)U(\eta-\Delta,\xi)\,. \tag{9}$$

This corresponds to a rotation from $\xi\xi^\dagger$ to $(\eta+\Delta)(\eta+\Delta)^\dagger$, a rotation from $(\eta+\Delta)(\eta+\Delta)^\dagger$ to $(\eta-\Delta)(\eta-\Delta)^\dagger$ followed by a rotation back to $\xi\xi^\dagger$. Hence, it is a special c case of the spherical triangle used in computing the Pancharatnam phase [1] for a cyclic evolution in which a photon passes through a sequence of four polarization states, the last of which is the same as the first one up to a phase. To obtain W we can divide the loop $l(t)$ into infinitesimal parts and take the product

$$W(l(t))=\prod_i w(\eta_i,\xi,\Delta_i)\,. \tag{10}$$

This is analogous to obtaining the holonomy transformations for an arbitrary loop by multiplying a sequence of holonomy transformations for infinitesimal triangles [12].

It is easier to work out $w(\eta,\xi,\Delta)$ by the following method rather than by direct multiplication. From (1) we find, up to $\mathrm{O}(\Delta)$,

$$U(\eta+\Delta,\eta-\Delta)$$
$$=1+2(\eta+\Delta)(\eta+\Delta)^{\dagger}(\eta-\Delta)(\eta-\Delta)^{\dagger}$$
$$-(\eta+\Delta)(\eta+\Delta)^{\dagger}-(\eta-\Delta)(\eta-\Delta)^{\dagger}\,. \tag{11}$$

Using the fact

$$U(\xi,\eta+\Delta)(\eta+\Delta)(\eta+\Delta)^{\dagger}=\xi\xi^{\dagger}U(\xi,\eta+\Delta)\,, \tag{12}$$

we can rewrite $w(\eta,\xi,\Delta)$ as

$$w(\eta,\xi,\Delta)=V(\xi,\eta+\Delta,\eta-\Delta)$$
$$+2\xi\xi^{\dagger}V(\xi,\eta+\Delta,\eta-\Delta)\xi\xi^{\dagger}$$
$$-\xi\xi^{\dagger}V(\xi,\eta+\Delta,\eta-\Delta)$$
$$-V(\xi,\eta+\Delta,\eta-\Delta)\xi\xi^{\dagger}\,, \tag{13}$$

where $V(\xi,\eta+\Delta,\eta-\Delta)=U(\xi,\eta+\Delta)U(\eta-\Delta,\xi)$.

$w(\eta,\xi,\Delta)$ is a unitary matrix of η, ξ, Δ and their adjoints. The form of (13) implies that it must be of the form

$$w(\eta,\xi,\Delta)=1+\mathrm{i}a\xi\xi^{\dagger}+b(1-\xi\xi^{\dagger})\eta\eta^{\dagger}(1-\xi\xi^{\dagger})$$
$$+c(1-\xi\xi^{\dagger})\eta\Delta^{\dagger}[1-P(\eta,\xi)]$$
$$-c^{*}[1-P(\eta,\xi)]\Delta\eta^{+}(1-\xi\xi^{\dagger})\,. \tag{14}$$

This implies that $w(\eta,\xi,\Delta)$ is an element of the little group of ξ. In (14) a, b, c, are constants depending on ξ, η and Δ. Here a and b are of $o(\Delta)$ whereas c is of $O(1)$. To get these constants, it is easy to verify

$$\xi^{\dagger}w(\eta,\xi,\Delta)\xi=1+\mathrm{i}a \tag{15}$$

and

$$\eta^{\dagger}(1-\xi\xi^{\dagger})w(\eta,\xi,\Delta)(1-\xi\xi^{\dagger})\eta$$
$$=(1-\eta^{\dagger}\xi\xi^{\dagger}\eta)+6(1-\eta^{\dagger}\xi\xi^{\dagger}\eta)^{2}\,. \tag{16}$$

Computation of the left hand sides of (15) and (16) becomes simple using (3) and

$$U(\xi,\eta)\xi=\frac{2\xi^{\dagger}\eta\eta^{\dagger}\xi\xi-\eta^{\dagger}\xi\eta}{\sqrt{\xi^{\dagger}\eta\eta^{\dagger}\xi}}\,. \tag{17}$$

It follows

$$a=-\mathrm{i}\,\frac{\xi^{\dagger}\eta\eta^{\dagger}\xi(\Delta^{\dagger}\eta-\eta^{\dagger}\Delta)+\eta^{\dagger}\xi\xi^{\dagger}\Delta-\xi^{\dagger}\eta\Delta^{\dagger}\xi}{\xi^{\dagger}\eta\eta^{\dagger}\xi}\,, \tag{18}$$

$$b=-\frac{a}{1-\eta^{\dagger}\xi\xi^{\dagger}\eta}\,. \tag{19}$$

To compute c, consider a state vector ϵ such that

$$\eta^{\dagger}\epsilon=\xi^{\dagger}\epsilon=0\,. \tag{20}$$

Then

$$\eta^{\dagger}w(\eta,\xi,\Delta)\epsilon=c(1-\eta^{\dagger}\xi\xi^{\dagger}\eta)\Delta^{\dagger}\epsilon\,, \tag{21}$$

which gives

$$c=\frac{2\sqrt{\eta^{\dagger}\xi\xi^{\dagger}\eta}}{1-\eta^{\dagger}\xi\xi^{\dagger}\eta}\,. \tag{22}$$

From eq. (18), $\bar{a}=a$ so that a is real. Hence, from (15), the phase acquired by ξ under the action of $w(\eta,\xi,\Delta)$ is a, which is given by (18). Hence *a is the geometric phase.*

The first non-trivial example is $n=2$. Then $\mathscr{P}$ is a sphere. Consider the infinitesimal geodesic triangle on $\mathscr{P}$ corresponding to the vectors ξ, $\eta-\Delta$ and $\eta+\Delta$ defined by

$$\xi=(1,0)\,,\quad \eta=(\cos\tfrac{1}{2}\theta,\sin\tfrac{1}{2}\theta)\,,$$

$$\eta+\Delta=(\exp(\tfrac{1}{2}\mathrm{i}\,\mathrm{d}\phi)\cos\tfrac{1}{2}\theta,\exp(-\tfrac{1}{2}\mathrm{i}\,\mathrm{d}\phi)\sin\tfrac{1}{2}\theta)\,.$$

Then $\Delta=\frac{1}{2}\mathrm{i}\,\mathrm{d}\phi(\cos\frac{1}{2}\theta,-\sin\frac{1}{2}\theta)$. The geometric phase from (18) for this triangle is

$$a=2\,\mathrm{d}\phi\,\sin^{2}(\tfrac{1}{2}\theta)\,,$$

which is half the solid angle subtended by this geodesic triangle at the center of the sphere. Now given an arbitrary cyclic evolution described by a closed curve C on $\mathscr{P}$, we can fill it with infinitesimal triangles like the above ones, neglecting the defect area which is of second order in small quantities. Therefore the geometric phase for C is the sum of the contributions of the infinitesimal triangles. So, we conclude that, when $n=2$, for an arbitrary cyclic evolution C, the geometric phase is half the solid angle subtended by C at the center of the sphere which represents $\mathscr{P}$. It should be noted that this result has been obtained by purely group theoretical arguments and not by integrating the connection due to Aharonov and Anandan [5] around C.

For $n=3$, i.e. SU(3), define

$$\xi\xi^{\dagger}=\tfrac{1}{3}(1+\tfrac{1}{2}n^{a}\lambda_{a}) \tag{23}$$

and

$$\eta\eta^{\dagger}=\tfrac{1}{3}(1+\tfrac{1}{2}m^{a}\lambda_{a})\,, \tag{24}$$

where λ_a are generators of SU(3). Suppose $\Delta=\dot{\eta}\,\mathrm{d}t$, where the overdot denotes time derivative. Then, it can be shown that

$$a=\frac{f_{abc}m^a\dot{m}^b n^c}{(\frac{2}{3}+n^d m_d)}, \tag{25}$$

where f_{abc} are the structure constants of SU(3). If $m(t)$ is given then by integrating (25) with respect to time, the geometric phase for an arbitrary cyclic evolution can be obtained [13].

As already mentioned, U belongs to the intersection of SU(n) and U($n-1$)×U(1) which is isomorphic to SU($n-1$)×U(1). Since the U(1) part of this element is obtained from each of the slices of the triangles, it is proportional to the area of the surface enclosed by the loop. The overall phase of a state is arbitrary. The overall geometric phase for a closed path is however measurable as already shown in the literature [6]. For $n=2$ this is the geometric phase measured by Simon, Sudarshan and Kimble [6].

The SU(n) phase defined in this Letter is meaningful even for open paths. This is unlike the geometric phase which is defined for closed paths, and could be associated with open paths only by implicitly closing the path by a geodesic joining the end points. Eq. (18), as far as we know, provides for the first time the geometric phase for an arbitrary infinitesimal triangle in $\mathscr{P}$. Also, it is interesting that this was done by a group theoretical argument and not by integrating the curvature along the surface.

References

[1] S. Pancharatnam, Proc. Indian Acad. Sci. A 44 (1956) 247.
[2] M.V. Berry, Proc. R. Soc. A 392 (1984) 45.
[3] B. Simon, Phys. Rev. Lett. 51 (1983) 2167;
E. Kiritsis, Commun. Math. Phys. 111 (1987) 417.
[4] F. Wilczek and A. Zee, Phys. Rev. Lett. 52 (1984) 2111;
C.A. Mead, Phys. Rev. Lett. 59 (1987) 171.
[5] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58 (1987) 1593.
[6] R. Simon, E.C.G. Sudarshan and J. Kimble, Phys. Rev. Lett. 61 (1988) 19;
D. Suter, K.T. Mueller and A. Pines, Phys. Rev. Lett. 60 (1988) 1218;
R.Y. Chiao, A. Antaramian, K.M. Ganga, H. Jiao, S.R. Wilkenson and H. Nathen, Phys. Rev. Lett. 60 (1988) 1214.
[7] J. Anandan and L. Stodolsky, Phys. Rev. D 35 (1987) 2597;
J. Anandan, Phys. Lett. A 129 (1988) 201;
R. Simon, N. Mukunda and E.C.G. Sudarshan, Phys. Rev. Lett. 62 (1989) 1331.
[8] M.V. Berry, J. Mod. Opt. 34 (1987) 1401;
J. Samuel and R. Bhandari, Phys. Rev. Lett. 60 (1988) 2339.
[9] J. Anandan and Y. Aharonov, Phys. Rev. D 38 (1988) 1863.
[10] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. II, p. 160;
D.N. Page, Phys. Rev. A 36 (1987) 3479.
[11] J. Anandan and Y. Aharonov, Phys. Rev. Lett. 65 (1990) 1697;
J. Anandan, Phys. Lett. A 147 (1990) 3.
[12] J. Anandan, in: Conf. on Differential geometric methods in theoretical physics, Trieste, July 1981, eds. G. Denardo and H.D. Doebner (World Scientific, Singapore, 1983) p. 211.
[13] C. Bouchiat and G.W. Gibbons, J. Phys. (Paris) 49 (1988) 187.