Simply no hidden variables

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It is shown very simply that for any two observables represented by Hermitian operators that do not commute, there is a state for which there is no joint probability distribution for the two observables.

I. INTRODUCTION

Does quantum mechanics not allow hidden variables? If so, it is not easy to see why from the classic proofs.[1-3] They are too complicated. They assume as little as possible and prove as much as possible. To do so, they avoid full use of quantum mechanics[1,2] and/or use unfamiliar language involving a lattice of propositions[2] or partial algebra.[3] They left an impression that questions about hidden variables are only for specialists.

That has changed. Hidden variables are used to derive Bell inequalities.[4] We see the absence of these hidden variables as a basic property of nature which quantum mechanics accommodates and describes. It was tested when experiments[5] showed Bell inequalities are false. Still, arguments for these hidden variables, involving separated subsystems, are the most difficult to dismiss.

Recently, Sudarshan and Rothman[6] pointed out that calculations of correlations in quantum mechanics can be done in a way that is similar in structure to derivations of Bell inequalities. It highlights the key difference. In derivations of Bell inequalities, the distributions used to calculate correlations are supposed to be actual nonnegative probabilities. They are joint probability distributions for observables that in quantum mechanics are represented by operators that do not commute.[7] Fine[8] showed that existence of these joint probability distributions is equivalent to the Bell inequalities and equivalent to the assumptions about hidden variables used to derive the Bell inequalities. He concludes[8] that his "investigations suggest that what the different hidden variables programs have in common, and the common source of their difficulties, is the provision of joint distributions in those cases where quantum mechanics denies them." Fine also formulated a criterion for existence of joint probabilities, in terms of operator functions of the observables, and showed it implies that the observables are represented by commuting operators.[8]

Here, we show very simply that for any two observables represented by Hermitian operators that do not commute there is a state for which there is no joint probability distribution for the two observables. We regard these imagined joint probabilities as the characteristic features that define hidden variables. Thus we offer our proof as a simple way to see why there can be no hidden variables in quantum mechanics. We do not hesitate to use quantum mechanics in the proof. In fact, all we do is bring out this feature of quantum mechanics and put it in focus.

II. STATEMENT

Consider two observables represented by Hermitian operators A and B that do not commute. There must be projection operators E in the spectral decomposition of A and F in the spectral decomposition of B such that E and F do not commute. This means E projects onto a subspace for states where the quantity represented by A has a particular value, or is in a particular interval of possible values. We can think of E as representing the proposition that the observable has that value, or has a value in that interval; it is 1 if the observable has a value in the interval and 0 if it has a value outside. Similarly, F represents a proposition about the values of the observable represented by B.

Consider the idea, drawn from theories with hidden variables, that for each state there is a joint probability distribution for the values of the two quantities represented by A and B. It gives a probability for each possible pair of values or intervals. The probability distribution for the values of the quantity represented by A, for this state, is obtained from the joint probability distribution by integrating over the values of the quantity represented by B. Similarly, the probabilities for B are obtained by integrating over the values for A.[9] The joint probability distribution might be obtained from a probability distribution for many variables by integrating over all the others, some of which may be "hidden."

In particular, this means there are joint probabilities,

$$p(1,1), \quad p(1,0),$$

$$p(0,1), \quad p(0,0),$$

for the four pairs of possible values 1 or 0 for the propositions represented by E and F. For the proposition represented by E, the probabilities for the values 1 and 0 are

$$p_E(1) = p(1,1) + p(1,0) \tag{1}$$

and

$$p_E(0) = p(0,1) + p(0,0), \tag{2}$$

and for the proposition represented by F, the probabilities for the values 1 and 0 are

$$p_F(1) = p(1,1) + p(0,1) \tag{3}$$

and

$$p_F(0) = p(1,0) + p(0,0). \tag{4}$$

Because E and F do not commute, there are states for which this is not true. That is what we now show.

III. PROOF

Since E and F do not commute, there must be a state represented by a vector ψ such that

$$E\psi = 0, \tag{5}$$

$$F\psi \neq 0, \tag{6}$$

$$EF\psi \neq 0. \tag{7}$$

If not, $EF\psi$ would be 0 for every

$$\psi = (1 - E)\phi, \tag{8}$$

which means

$$EF(1 - E)\phi = 0 \tag{9}$$

for every vector ϕ, which implies E and F commute, because if

$$EF(1 - E) = 0, \tag{10}$$

then

$$EF = EFE = (EFE)^\dagger = FE. \tag{11}$$

For the state represented by ψ, we can see from (5) that $p_E(1)$ is 0. Then (1) implies $p(1,1)$ and $p(1,0)$ are both 0. But for this state it is not correct to say that $p(1,1)$ is 0.

Suppose we measure the observable represented by B. The state is represented by ψ. The probability that we find 1 for the proposition represented by F is $\|F\psi\|^2$. After the measurement, the state is represented by the vector

$$(1/\|F\psi\|)F\psi. \tag{12}$$

Suppose, then, we measure the observable represented by A. The probability that we find the value 1 for the proposition represented by E is

$$\|(1/\|F\psi\|)EF\psi\|^2 = \|EF\psi\|^2/\|F\psi\|^2. \tag{13}$$

The probability that we find the pair of values 1,1 in this way is $\|EF\psi\|^2$, which is not 0. This could not happen if it were true that $p(1,1)$ is 0. We conclude that for this state there are no joint probabilities satisfying the conditions (1)–(4).

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