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Perturbation theory on generalized quantum mechanical systems

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We present a general formalism for doing the perturbation theory in the complex energy plane, where the notion of the generalized quantum mechanical systems is used. This formalism is applied to the Friedrichs–Lee model. It reproduces the results of the exact solution, where the spectrum of the generalized quantum mechanical system consists of a discrete complex energy pole and a continuum spectrum (which passes below this discrete pole) in the complex energy plane. We also investigate the role of the "complex delta" function in the description of a resonance state. The unboundedness of the spectrum appears to be the very ingredient needed to give rise to a pure exponential decay.

1. Introduction

Recently Petrosky, Prigogine and Tasaki [1] investigated the Friedrichs–Lee model [2] based on perturbation theory. They found that the spectrum of the system may be described as a sum of a discrete resonance state and a continuum spectrum along the positive real energy. This conclusion differs from our earlier work [3–5] based on the analytic continuation of the exact solution of the model. There we found that accompanying the discrete resonance state, the continuum spectrum must be defined along a contour in the complex energy plane which passes below the resonance pole. It is curious, what is the origin of this difference?

A closer look reveals two important ingredients in their analysis, first is the use of perturbation theory and second the expansion of the wave function in powers of the imaginary part of the resonance energy. In our work in this paper we avoid any divergence difficulty by doing perturbation theory in the complex energy plane. One may also want to further explore whether there are features in the theory which are more clearly revealed in the perturbation approach.

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Perturbation theory in the complex energy plane is to our knowledge novel. The application of the perturbation theory in the complex energy plane could be an effective means to study the resonance properties of the system.

Earlier, based on solvable models we have investigated various quantum mechanical systems in the complex energy plane by working with the "generalized quantum mechanical systems". To supplement our solvable model work on such generalized systems, we now proceed to consider the perturbation theory of these systems.

The usual textbook approach to perturbation theory in quantum mechanics is the Rayleigh–Schrödinger perturbation method [6] where the wave functions and the eigenvalues are expressed in powers of the coupling constant g,

$$\Psi(g) = \psi_0 + g\psi^{(1)} + \cdots + g^n \psi^{(n)} + \cdots , \tag{1.1}$$

$$\lambda(g) = \lambda_0 + g\lambda_1 + \cdots + g^n \lambda_n + \cdots . \tag{1.2}$$

Then from the eigenvalue equation

$$(H_0 + gV)\psi(g) = \lambda(g)\, \psi(g) \tag{1.3}$$

one obtains the corresponding equation for each order of g^n. Despite the textbook exposition of the method, there seem to be essential complications just beyond the second order. Further, the method appears to be very cumbersome in higher orders. In this work we will follow an alternative method based on the Green's function approach [7], used in field theory discussions. In section 2, in the framework of generalized quantum system, we will discuss the wave operator and the corresponding eigenvalues in perturbation theory. The results of the perturbation calculations for the generalized quantum system as applied to Friedrichs–Lee model are presented in section 3.

We do perturbation expansion on the inverse of the Green's function for the Friedrich–Lee model and in the zeroth order of the perturbation expansion the discrete spectrum is the V particle. Consider a typical situation where the bare mass of the V particle is above the threshold of the continuum channel. As the interaction is switched on, the discrete state moves away from the real axis on to the second Riemann sheet. One may deform the contour to expose this resonance pole. It eventually leads to a *generalized spectrum* consisting of this resonance state together with the continuum spectrum *defined along some contour* Γ. The latter are essential components for the complete specification of the generalized spectrum.

In section 4, we investigate the related question of the role of the "complex delta" function in the description of a resonance state. We observe that the pure exponential decay may be obtained when one passes from a lower bounded weight function to its *analytic extension* with support on the full real line $\mathbb{R}$ rather than the semi-infinite real line $\mathbb{R}^+$. It appears that the unboundedness of the spectrum, which violates the general property of the quantum system and also of the generalized quantum system, is the very ingredient needed to give rise to a *pure exponential decay*. The main thrust of the present work is summarized in section 5.

2. Perturbation theory and the generalized quantum system

2.1. The wave operator

Consider a quantum system with the Hamiltonian

$$H = H_0 + V\,, \tag{2.1}$$

where the free Hamiltonian satisfies

$$H_0\psi_0 = z\psi_0\,, \tag{2.2}$$

and the full Hamiltonian satisfies

$$H\psi = (H_0 + V)\psi = z\psi\,. \tag{2.3}$$

Introduce the *unnormalized* wave operator which transforms the free particle wave function ψ_0 to the *unnormalized* wave function ψ_{un},

$$\psi_{\mathrm{un}} = \Omega_{\mathrm{un}}\psi_0\,. \tag{2.4}$$

Denoting the difference by

$$\Delta\psi = \psi_{\mathrm{un}} - \psi_0 \tag{2.5}$$

the eigenvalue equation of (2.3) becomes

$$(H_0 + V)\psi_0 + H\,\Delta\psi = z\psi_0 + z\,\Delta\psi\,, \quad \text{or} \quad \Delta\psi = \frac{1}{z-H}V\psi_0\,. \tag{2.6}$$

Thus

$$\psi_{\mathrm{un}} = \psi_0 + \Delta\psi = \left(1 + \frac{1}{z-H}V\right)\psi_0 = \frac{1}{1-G_0V}\psi_0\,, \tag{2.7}$$

where the free Green's function is

$$G_0(z) = \frac{1}{z - H_0} . \tag{2.8}$$

The *unnormalized* wave operator is given by

$$\Omega_{\mathrm{un}} = \frac{1}{1 - G_0 V} = \sum_{n=0}^{\infty} (G_0 V)^n . \tag{2.9}$$

So far we have not paid attention to the renormalization of the wave function. In particular, for the renormalized wave function ψ, there is the corresponding *renormalized* wave operator Ω, with the relation

$$\psi = \Omega \psi_0 . \tag{2.10}$$

In the complex energy plane, the scalar product is defined to be the inner product between the wave function and its dual, where the dual wave function is defined by

$$\tilde{\psi}(z) = \psi^{\dagger}(z^*) , \qquad \tilde{\psi}(z) = (\widetilde{\Omega \psi_0}) = \tilde{\psi}_0 \tilde{\Omega} . \tag{2.11}$$

So the scalar product is

$$\langle \tilde{\psi}_{\mathrm{un}} | \psi_{\mathrm{un}} \rangle = \langle \tilde{\psi}_0 | \tilde{\Omega}_{\mathrm{un}} \Omega_{\mathrm{un}} \psi_0 \rangle \equiv \langle \tilde{\psi}_0 D^2 \psi_0 \rangle \tag{2.12}$$

or the renormalized wave function:

$$\psi = \Omega \psi_0 = \Omega_{\mathrm{un}} D^{-1} \psi_0 , \qquad \Omega = \Omega_{\mathrm{un}} D^{-1} . \tag{2.13}$$

In a multichannel case, it is convenient to define the free wave function in the basis where for the ith free wave function, only the ith element is non vanishing. With this choice of the basis vectors eqs. (2.12) and (2.13) can be generalized to

$$\langle \tilde{\psi}_{oi} D^2 \psi_{0j} \rangle = D^2_{ij} , \tag{2.14}$$

and

$$\psi_j = (\Omega_{\mathrm{un}})_{ji} D^{-1}_{ik} \psi_{0k} , \quad \text{with } \Omega_{jk} = (\Omega_{\mathrm{un}})_{ji} D^{-1}_{ik} . \tag{2.15}$$

Note that D^2 and by choice D^{-1} commute with H_0. Hence eqs. (2.3) and (2.4) are unaffected by the renormalization. This construction yields the perturbed wave functions to all orders.

2.2. The eigenvalues

Our analysis so far does not tell us the location of the singularities of the wave operator Ω except the generic result that they are at the spectra of H. As mentioned earlier we shall follow the conventional approach in field theory and work with the perturbation expansion of the Green's functions. The full Green's function is given by

$$G(z) = \frac{1}{z - H} = \frac{1}{1 - G_0 V} G_0 . \tag{2.16}$$

The spectra are obtained from the singularities of $G(z)$ which are, apart from the spectrum of $G_0(z)$, the zeros of the denominator, that is the place where $G_0(z)\, V = 1$. It is possible in exceptional cases, for part of the spectrum of G_0 to be cancelled by the zeros of $1/(1 - G_0 V)$; that is by the poles of $1 - G_0 V$. Note that since the left-hand-side is a matrix and the right-hand-side is a number we are really talking about the eigenvalue of $G_0(z)\, V$.

3. Perturbation solution of the Friedrichs–Lee model

The Hamiltonian for the Friedrichs–Lee model and the corresponding equation for the free wave functions and that for the total eigenfunctions in the lowest sectors are given by

$$H_0 = \begin{pmatrix} m_0 & 0 \\ 0 & \omega\delta(\omega - \omega') \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & f(\omega') \\ \tilde{f}(\omega) & 0 \end{pmatrix},$$

$$H_0 \psi_0 = z\psi_0 \quad \text{and} \quad H\psi = (H_0 + V)\psi = z\psi , \tag{3.1}$$

where the variable ω is defined to be along some contour Γ in the complex energy plane, and $\tilde{f}(\omega) = f^*(\omega^*)$.

3.1. The unnormalized wave operator

In evaluating the wave operator, it is convenient to regroup the perturbation series in terms of those which are even power in $G_0 V$ and those odd in $G_0 V$, i.e.

$$\Omega_{\text{un}} = \sum_{n=0}^{\infty} (G_0 V)^n = \Omega_{\text{un}}^{\text{even}} + \Omega_{\text{un}}^{\text{odd}} , \tag{3.2}$$

where

$$\Omega_{\text{un}}^{\text{even}} = 1 + (G_0 V)^2 + (G_0 V)^4 + \cdots \quad \text{and} \quad \Omega_{\text{un}}^{\text{odd}} = G_0 V \Omega_{\text{un}}^{\text{even}} .$$

We proceed to evaluate the even part,

$$G_0V=\begin{pmatrix}\dfrac{1}{\lambda-m_0} & 0\\ 0 & \dfrac{\delta(\omega-\omega')}{\lambda-\omega}\end{pmatrix}\begin{pmatrix}0 & f(\omega'')\\ \tilde{f}(\omega') & 0\end{pmatrix}=\begin{pmatrix}0 & \dfrac{f(\omega'')}{\lambda-m_0}\\ \dfrac{\tilde{f}(\omega)}{\lambda-\omega} & 0\end{pmatrix}.$$

Note that the isolated zero of $(1-G_0V)$ occurs at $\lambda-M$ for

$$\int\frac{f(\omega'')\,\tilde{f}(\omega'')\,\mathrm{d}\omega''}{(\lambda-m_0)(\lambda-\omega'')}=1\,,$$

$$(G_0V)^2=\begin{pmatrix}\dfrac{1}{\lambda-m_0}\langle\cdots\rangle & 0\\ 0 & \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{f(\omega')}{(\lambda-m_0)}\end{pmatrix},$$

$$\text{with } \langle\cdots\rangle=\int\mathrm{d}\omega\,\frac{\tilde{f}(\omega)\,f(\omega)}{\lambda-\omega}\,,$$

$$(G_0V)^4=\begin{pmatrix}\left(\dfrac{\langle\cdots\rangle}{\lambda-m_0}\right)^2 & 0\\ 0 & \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{\langle\cdots\rangle}{\lambda-m_0}\dfrac{f(\omega')}{\lambda-m_0}\end{pmatrix}.$$

The even part is given by

$$\Omega_{\mathrm{un}}^{\mathrm{even}}=\begin{pmatrix}1+\dfrac{\langle\cdots\rangle}{\lambda-m_0}+\left(\dfrac{\langle\cdots\rangle}{\lambda-m_0}\right)^2+\cdots & 0\\ 0 & \delta(\omega'-\omega)+\dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{f(\omega')}{\alpha(\lambda)}\end{pmatrix}. \tag{3.3}$$

The odd part is

$$\Omega_{\mathrm{un}}^{\mathrm{odd}}=G_0V\Omega_{\mathrm{un}}^{\mathrm{even}}=\begin{pmatrix}0 & \dfrac{f(\omega')}{\lambda-m_0}+\dfrac{\langle\cdots\rangle}{\lambda-m_0}\dfrac{f(\omega')}{\alpha(\lambda)}\\ \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{\lambda-m_0}{\alpha(\lambda)} & 0\end{pmatrix} \tag{3.4}$$

$$=\begin{pmatrix}0 & \dfrac{f(\omega')}{\alpha(\lambda)}\\ \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{\lambda-m_0}{\alpha(\lambda)} & 0\end{pmatrix}. \tag{3.5}$$

Using (3.3) and (3.5), (3.2) becomes

$$\Omega_{\mathrm{un}} = \begin{pmatrix} \dfrac{\lambda - m_0}{\alpha(\lambda)} & \dfrac{f(\omega)}{\alpha(\lambda)} \\ \dfrac{\tilde{f}(\omega)}{\lambda - \omega}\dfrac{\lambda - m_0}{\alpha(\lambda)} & \delta(\omega - \omega') + \dfrac{\tilde{f}(\omega)}{\lambda - \omega}\dfrac{f(\omega')}{\alpha(\lambda)} \end{pmatrix}. \tag{3.6}$$

For definiteness, we consider the wave operator, with the eigenvalue of the discrete state labelled by M, where $\alpha(M) = 0$. For the resonance pole, M is a complex number. It gives

$$\Omega_{\mathrm{un}} = \begin{pmatrix} \dfrac{\lambda - m_0}{(\lambda - M)\alpha'} & \dfrac{f(\omega')}{\alpha(\lambda)} \\ \dfrac{\lambda - m_0}{(\lambda - M)\alpha'}\dfrac{\tilde{f}(\omega)}{\lambda - \omega} & \delta(\omega - \omega') + \dfrac{\tilde{f}(\omega)}{\lambda - \omega}\dfrac{f(\omega')}{\alpha(\lambda)} \end{pmatrix}. \tag{3.7}$$

For the discrete solution, we have used: $\alpha(\lambda) \approx (\lambda - M)\alpha'$, with the slope $\alpha' = (\mathrm{d}\alpha/\mathrm{d}\lambda)|_{\lambda = M}$. The pole of G_0 at $\lambda = m_0$ is cancelled by a zero of $(1 - G_0 V)^{-1}$ at $\lambda = m_0$.

3.2. Renormalized wave operator

We proceed now to evaluate the appropriate matrix to incorporate the effect of renormalization,

$$D^2 = \tilde{\Omega}_{\mathrm{un}}\Omega_{\mathrm{un}} \equiv \begin{pmatrix} B_1 & 0 \\ 0 & B_{\lambda'\lambda} \end{pmatrix}. \tag{3.8}$$

We leave it to the reader to verify that the off diagonal elements do vanish. The (1,1) element is given by

$$\begin{aligned} B_1 &= \left(\frac{\lambda - m_0}{\tilde{\alpha}(\lambda)}, \frac{\lambda - m_0}{\tilde{\alpha}(\lambda)}\frac{\tilde{f}(\omega)}{M - \omega}\right)\begin{pmatrix} \dfrac{\lambda - m_0}{\alpha(\lambda)} \\ \dfrac{\lambda - m_0}{\alpha(\lambda)}\dfrac{f(\omega)}{M - \omega} \end{pmatrix} \\ &= \frac{(\lambda - m_0)^2}{\tilde{\alpha}(\lambda)\,\alpha(\lambda)}\left(1 + \int \mathrm{d}\omega\,\frac{\tilde{f}(\omega)\,f(\omega)}{(M - \omega)^2}\right) = \frac{1}{\alpha'}\left(\frac{\lambda - m_0}{\lambda - M}\right)^2, \end{aligned} \tag{3.9}$$

where

$$\alpha' = 1 + \int \mathrm{d}\omega\,\frac{\tilde{f}(\omega)f(\omega)}{(M - \omega)^2}$$

was used. And the diagonal elements for the continuum are given by

$$
\begin{aligned}
B_{\lambda'\lambda} &= \tilde{D}_{\lambda'}(\omega)\, D_\lambda(\omega) \\
&= \left(\frac{\tilde{f}(\lambda')}{\tilde{\alpha}(\lambda')},\, \delta(\lambda'-\omega) + \frac{f(\omega)\,\tilde{f}(\lambda')}{(\lambda'-\omega)\alpha(\lambda')}\right) \begin{pmatrix} \dfrac{f(\lambda)}{\alpha(\lambda)} \\ \delta(\lambda-\omega) + \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{f(\lambda)}{\alpha(\lambda)} \end{pmatrix} \\
&= \frac{\tilde{f}(\lambda')\, f(\lambda)}{\tilde{\alpha}(\lambda')\,\alpha(\lambda)} + \delta(\lambda'-\lambda) + \frac{f(\lambda)\,\tilde{f}(\lambda')}{(\lambda'-\lambda)\alpha(\lambda')} + \frac{\tilde{f}(\lambda')\, f(\lambda)}{(\lambda-\lambda')\alpha(\lambda)} \\
&\quad + \frac{\tilde{f}(\lambda')}{\tilde{\alpha}(\lambda')}\frac{-1}{\lambda-\lambda'}\left(\frac{\tilde{\alpha}(\lambda')}{\lambda-\lambda'} - \frac{\alpha(\lambda)}{\lambda-\lambda'} + 1\right)\frac{f(\lambda)}{\alpha(\lambda)} = \delta(\lambda'-\lambda)\,.
\end{aligned} \tag{3.10}
$$

To arrive at the last term in the second last step, the following identity was used:

$$
\int \frac{\tilde{f}(\omega)\, f(\omega)\, \mathrm{d}\omega}{(\lambda'-\omega)(\lambda-\omega)} = -\frac{1}{\lambda-\lambda'}\left(\frac{\tilde{\alpha}(\lambda')}{\lambda-\lambda'} - \frac{\alpha(\lambda)}{\lambda-\lambda'} + 1\right). \tag{3.11}
$$

Collecting the terms, we get

$$
D^2 = \begin{pmatrix} \left(\dfrac{\lambda-m_0}{\lambda-M}\right)^2 \dfrac{1}{\alpha'} & 0 \\ 0 & \delta(\lambda-\lambda') \end{pmatrix}. \tag{3.12}
$$

Notice that the renormalization, in this case, only affects the discrete state wave function, not the continuum state wave function. The D^2 matrix implies that

$$
\begin{aligned}
D_\lambda(\omega) &= \begin{pmatrix} \left(\dfrac{\lambda-m_0}{\lambda-M}\right)\dfrac{1}{\sqrt{\alpha'}} & 0 \\ 0 & \delta(\lambda-\omega) \end{pmatrix} \\
\text{and}\quad D_\lambda^{-1}(\omega) &= \begin{pmatrix} \sqrt{\alpha'}\,\dfrac{(\lambda-M)}{\lambda-m_0} & 0 \\ 0 & \delta(\lambda-\omega) \end{pmatrix}.
\end{aligned} \tag{3.13}
$$

So the renormalized wave operator is

$$
\Omega = \Omega_{\text{un}} D^{-1}
$$

$$= \begin{pmatrix} \dfrac{\lambda - m_0}{(\lambda - M)\alpha'} & \dfrac{f(\omega')}{\alpha(\lambda)} \\ \dfrac{\lambda - m_0}{(\lambda - M)\alpha'}\dfrac{\tilde{f}(\omega)}{M-\omega} & \delta(\omega-\omega') + \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{f(\omega')}{\alpha(\lambda)} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha'}\dfrac{(\lambda - M)}{\lambda - m_0} & 0 \\ 0 & \delta(\lambda - \omega') \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\sqrt{\alpha'}} & \dfrac{f(\lambda)}{\alpha(\lambda)} \\ \dfrac{1}{\sqrt{\alpha'}}\dfrac{\tilde{f}(\omega)}{M-\omega} & \delta(\lambda-\omega) + \dfrac{\tilde{f}(\omega)}{\lambda-\omega}\dfrac{f(\lambda)}{\alpha(\lambda)} \end{pmatrix} . \tag{3.14}$$

It is gratifying that the results obtained here through perturbation calculation are the same as those obtained from solving the theory exactly [3]. This is, of course, not unexpected. We have seen that by applying perturbation theory on the generalized quantum mechanical system, one arrives at the spectrum which contains explicitly the discrete "resonance" state together with the deformed contour which is a necessary component in the specification of the generalized spectrum. Since the generalized spectrum is completely equivalent to the spectrum of the original theory defined along the real axis, the generalized spectrum would imply for instance the non exponential decay character near the time $t = 0$, i.e. the presence of the Zeno region [8] in the survival probability.

4. Lower bounded support, complex delta function and pure exponential decay

We begin with several definitions. Denote the spectral function of a quantum system by $\sigma(\omega)$. The corresponding temporal function, which is its Fourier transform is given by

$$\tilde{\sigma}(t) = \int_L^\infty d\omega\, \sigma(\omega)\, e^{-i\omega t} . \tag{4.1}$$

When the spectrum is lower bounded, L is finite. An unbounded spectrum corresponds to having $L = -\infty$. The initial temporal function, i.e. the function evaluated at the initial time, $t = 0$, is given by

$$\tilde{\sigma}(0) = \int_L^\infty d\omega\, \sigma(\omega) . \tag{4.2}$$

We proceed to consider an unstable quantum system, which is defined by including a term of the type

$$\int_0^\infty \mathrm{d}\omega\, \rho(\omega)\, \delta_c(\omega - z)\,,$$

where the "weight function", $\rho(\omega)$ is the restriction to the semi-real axis $\mathbb{R}^+$ of a function analytic in the lower halfplane. The complex delta function δ_c is the so-called Gel'fand–Shilov complex delta function [9] (a generalization of the delta function along a contour considered by Nakanishi [10] and by Sudarshan, Chiu and Gorini [3]). Its contribution to the initial temporal function is given by

$$\tilde{\sigma}_1(0) = \int_0^\infty \mathrm{d}\omega\, \rho(\omega)\, \delta_c(\omega - z_1) = \rho(z_1)\,, \tag{4.3}$$

where $\rho(z)$ is obtained as the analytic continuation of $\rho(\omega)$ to the complex point z_1. When z is real, $\delta_c(\omega - z)$ reverts back to the usual Dirac δ-function. The integral ranges from 0 to ∞, because along the negative real axis, $\rho(z) = 0$. The time translation operation on the temporal amplitude is constructed in the following manner:

$$T(\tau)\, \tilde{\sigma}(t) = \int_L^\infty \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega\tau}[\sigma(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}] = \tilde{\sigma}(t + \tau)\,. \tag{4.4}$$

If the spectral function contains a complex delta function factor, the temporal function becomes

$$\tilde{\sigma}(t) = \int_0^\infty \mathrm{d}\omega\, \rho(\omega)\, \delta_c(\omega - z_1)\, \mathrm{e}^{\mathrm{i}\omega t} = \rho(z_1)\, \mathrm{e}^{-\mathrm{i}z_1 t}\,. \tag{4.5}$$

Superficially this would seem to offer an example which says that a spectrum bounded from below could lead to a purely exponential decay. This is a misleading interpretation, however. We recall that the weight function may be analytically extended through the use of Hardy class functions defined by

$$\Psi_+(z) = \frac{-1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{d}\omega\, \frac{\rho(\omega)}{\omega - z + \mathrm{i}\epsilon}\,, \tag{4.6}$$

$$\Psi_-(z) = \frac{+1}{2\pi i} \int_0^\infty d\omega \, \frac{\rho(\omega)}{\omega - z - i\epsilon} . \tag{4.7}$$

Here $\rho(\omega)$ is expressed in terms of the sum of two analytic functions $\Psi_+(\omega)$ and $\Psi_-(\omega)$. One can also verify that for the time translation operation forward in time

$$T(t)\,\Psi_+(z) = \Psi_+(z, t) \quad \text{and} \quad T(t)\,\Psi_-(z) = 0 . \tag{4.8}$$

For the backward time translation, we have

$$T(t)\,\Psi_+(z) = 0 \quad \text{and} \quad T(t)\,\Psi_-(z) = \Psi_-(z, t) . \tag{4.9}$$

Now it is instructive to look at the integrand of eq. (4.5). For definiteness, consider the case of forward propagation. The integrand may be re-expressed in terms of its analytic extension $\rho_{\text{ext}}(z) = \Psi_+(z)$,

$$\begin{aligned} &\int_0^\infty d\omega \, \rho(\omega)\, \delta_c(\omega - z_1)\, e^{-i\omega t} \\ &\quad \to \int_{-\infty}^\infty dz \, \rho_{\text{ext}}(z) \frac{1}{2\pi i} \left(-\frac{1}{z - z_1 + i\epsilon} + \frac{1}{z - z_1 - i\epsilon} \right) e^{-izt} \\ &\quad = -\frac{1}{2\pi i} \int_{-\infty}^\infty dz \, \frac{\rho_{\text{ext}}(z)\, e^{-izt}}{z - z_1 + i\epsilon} , \end{aligned} \tag{4.10}$$

where in the last step we use the fact that for positive t the integral can be closed in the lower halfplane. Since the second term does not contain singularity in the lower half-plane the integral vanishes. Only the first term survives, which involves integrating over the unbounded spectrum, which extends from $-\infty$ to ∞. Thus for $t \geqslant 0$, we have

$$\begin{aligned} \int_0^\infty d\omega \, \rho(\omega)\, \delta_c(\omega - z_1)\, e^{-i\omega t} &= -\frac{1}{2\pi i} \int_{-\infty}^\infty dz \, \frac{\rho_{\text{ext}}(z)\, e^{-izt}}{z - z_1 + i\epsilon} \\ &= [\rho_{\text{ext}}(z_1)\, e^{-imt}]\, e^{-\Gamma t/2} . \end{aligned} \tag{4.11}$$

The last step is obtained by closing the contour of integration in the lower halfplane, which picks up the pole contribution as $z = z_1 = m - i\Gamma/2$, leading to a pure exponential decay as displayed.

We see that the delta distribution for complex z, is really defined only for

$\rho(\omega)$ which are restricted to $\mathbb{R}^+$ of *analytic* functions and the computation uses an ordinary function on the Hardy class function $\Psi_+ \equiv \rho_{\mathrm{ext}}$ which is the *analytic extension* of $\rho(\omega)$ to $-\infty < \omega < \infty$, so there is no mystery in getting the exponential decay. Also we observe that the analytic decomposition of eqs. (4.6) and (4.7) may be carried out for *any* integrable function $\rho(\omega)$, not necessarily analytic. So our result of eq. (4.11) is general.

5. Summary

We have studied the perturbation theory of Friedrichs–Lee model in the generalized space where the energy variable may be complex. We find that the conclusion which we reached earlier based on a solvable model continues to be true in perturbation theory. When doing the perturbation theory, to avoid the convergence difficulty, one needs to deform the contour to expose the resonance pole contribution, in order to be able to isolate the discrete state contribution. In this case, the continuum states must be defined along a contour which passes below the resonance pole. In other words, the corresponding continuum spectrum in its entirety cannot be along the real axis. Furthermore, by applying the Cauchy theorem the generalized spectrum obtained containing the second sheet discrete state and the continuum states along the contour, is equivalent to the Friedrichs–Lee spectrum along the real axis. In this sense we disagree with the assertion of Petrosky, Prigogine and Tasaki that within the perturbation theory, the system admits a discrete spectrum plus a continuum spectrum defined along the positive real axis. Although the use of the notion of the complex delta function gives the appearance that a pure exponential decay results from a lower bounded spectrum we explicitly display, through the use of analytic extension, that pure exponential decay property is directly associated with a spectrum which is not bounded from below.

In this paper, we have arrived at a spectral decomposition for a generalized quantum mechanical system. We define the spectrum as the singularities of the full Green's function. We find a continuum along a complex contour and possibly a discrete point spectrum which may be exposed by a proper choice of the contour.

This spectrum differs from that obtained by Petrosky, Prigogine and Tasaki where there are a discrete complex energy pole and a continuum spectrum along the positive real axis. They defined the spectrum in a nonstandard manner. Their spectral decomposition cannot be derived from the singularities of the Green's function.[#1]

[#1] We thank Professor Tasaki for clarification on this point.

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References

[1] T. Petrosky, I. Prigogine and S. Tasaki, Physica A 173 (1991) 175.
[2] K.O. Friedrich, Commun. Pure Appl. Math. 1 (1948) 361;
T.D. Lee, Phy. Rev. 95 (1954) 1329;
P.A.M. Dirac, Principles of Quantum Mechanics, 3rd ed. (Oxford Univ. Press, London, 1947).
[3] E.C.G. Sudarshan, C.B. Chiu and V. Gorini, Phys. Rev. D 18 (1978) 2914.
[4] E.C.G. Sudarshan and Charles B. Chiu, Phys. Rev. D 47 (1993) 2602.
[5] Charles B. Chiu and E.C.G. Sudarshan, Theory of the neutral kaon system, Center for Particle Physics, preprint CPP-34, 1993.
[6] L.I. Schiff, Quantum Mechanics, 3rd ed. (McGraw-Hill, New York, 1968) ch. 8.
[7] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1968) ch. 6.
[8] B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18 (1977) 756;
C. Chiu, E.C.G. Sudarshan and B. Misra, Phys. Rev. 16 (1977) 520.
[9] I.M. Gel'fand and G.E. Shilov, Generalized Functions, vols. 1 and 2 (Academic Press, New York, 1964).
[10] N. Nakanishi, Prog. Theor. Phys. 19 (1958) 607.