Comment on "Operator Algebra in Chern-Simons Theory on a Torus"

Recent Letters by Hosotani (H) [1] and Ho and Hosotani (H^2) [2] have sought to give a dynamical meaning to certain time dependent variables $\theta_i(t)$ (called "nonintegrable phases of Wilson line integrals" in H and H^2), which occur in the solution of the pure Chern-Simons theory [3] on a torus. In an earlier Comment [4] it was stated that the resulting quantization of the Chern-Simons coefficient found in H for the Abelian model was merely a consequence of errors made there in solving the field equation

$$(\kappa/2\pi)\epsilon^{\mu\nu\alpha}\,\partial_\nu a_\alpha = -J^\mu\,. \qquad (1)$$

The aim of this subsequent Comment is to indicate even more clearly the lack of dynamical content in the $\theta_i(t)$ parameters. To this end the solution obtained in Ref. [4] will be summarized (using H^2 notation), the differences with H and H^2 clearly noted, and the validity of our modifications pointed out. The solution of (1) is from Ref. [4]:

$$a_\mu = \partial_\mu\Lambda + \frac{2\pi}{\kappa}\epsilon_{\mu\nu\alpha}\nabla^\nu \int d^2y\,\mathcal{D}(x,y)J^\alpha(y)\,, \qquad (2)$$

where $\Lambda = \sum x_i\theta_i(t)/L_i$, $\nabla^\nu = (\nabla, 0)$, and

$$\mathcal{D}(x,y) = D(x-y) + \frac{x^2+y^2}{4L_1L_2} = \mathcal{D}(y,x)\,,$$

and

$$\nabla^2\mathcal{D}(x,y) = \nabla^2 D(x-y) + \frac{1}{L_1L_2} = \delta(x-y)\,.$$

The defense [5] of the solution found in Ref. [1] claimed that a certain theorem excluded terms linear in x from the solution and that Ref. [4] improperly identified a c-number q with an operator Q. In response, it should be pointed out that since only gauge independent quantities need be single valued on the torus, the theorem of Ref. [5] does not imply the absence of a term linear in x in a_0. Also Eq. (7) of H unambiguously identifies a and thus q with the charge operator.

With regard to the more recent work (H^2) there are three significant points of difference between (2) and H^2.

(i) H^2 deny $Q = (\kappa/2\pi)\Phi$ as an operator relation. However, insertion of their result for a_i into the temporal component of (1) leads to an immediate verification of that result. Note that this also suffices to establish the vanishing of $[P_i, P_j]$ and $[P_i, \mathcal{H}]$, in contrast to the claim of H^2. (The operators P_i and $\mathcal{H}$ are the momentum and Hamiltonian operators, respectively.)

(ii) H^2 do not have the term $\partial_0\Lambda$ in a_0 which gives a contribution $\sum_i x_i\dot{\theta}_i/L_i$ to that operator. This additional term is clearly compatible with the requirement that gauge independent quantities be single valued on the torus and is also necessary to cancel the corresponding $\dot{\theta}_i$ term in $\partial_0 a_i$ when one considers the spatial components of Eq. (1). It also satisfies the boundary conditions (3) of H^2, the contrary claim [5] of Hosotani notwithstanding. Note, however, that the specific $\beta_j(x)$ given in H^2 requires a different normalization for the solution [2].

(iii) There is a contribution to a_0 arising from the difference between $\mathcal{D}$ and D which is absent in H^2. This can readily be seen to be required by the $[\mathcal{H}, \psi]$ result. The point is that the $a \cdot J$ term in $\mathcal{H}$ gives (as a consequence of a term proportional to $\epsilon_{ij}x_jQ$ in a_i) a contribution to $\dot{\psi}$ which gives $\int \mathbf{x}\times\mathbf{J}\,d^2x$. This is accommodated within the form given by Eq. (2) for a_0.

These errors in H and H^2 significantly affect the role of the θ_i's. Because there is no $\dot{\theta}$ term in their a_0 the cancellation between $\dot{\theta}$ terms in the Lagrangian which is implied by (2) simply does not occur. Consequently, nonvanishing commutators are found for the θ_i's amongst themselves and nontrivial equations of motion are implied for these variables.

Thus H and H^2 have incorrectly solved the equations of motion for the pure Chern-Simons theory on a torus. When the appropriate corrections are made, the θ_i's evidently become nondynamical quantities whose sole effect is that of a gauge transformation on the true dynamical variables of the system. A study of the effects of such functions on the Hilbert space (and associated issues) will not therefore accomplish anything beyond what is already implied by the usual gauge invariance of the Chern-Simons theory [6].

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[6] If, however, $\theta_i(t)$ are added to a_i but *not* constrained by equations of motion (1), then nontrivial vacuum structure and multicomponent wave functions are obtained. Examples are R. Iengo and K. Lechner [Nucl. Phys. **B364**, 551 (1991)] and K. Lechner [Phys. Lett. B **273**, 463 (1991)]. These structures are reproduced by H^2.

