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# Complex Frequencies, Analytic Continuation, and Equivalent Spectra in Generalized Quantum Theory

**E. C. G. Sudarshan**[1]

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The spectrum of a physical Hamiltonian or even of a time-dependent wave function is not unique, but may be chosen in accordance with the physical aspects to be emphasized. In particular, complex continuous and discrete frequencies are not in addition to, but in place of, a real continuous spectrum.

The Hamiltonian of a physical system is a self-adjoint operator with a real spectrum bounded from below. The spectrum may be discrete or continuous and often multiply degenerate. The resolvent or Green's function $(z - H)^{-1}$ is singular at a set of points which is *defined* as the spectrum of $H$. The general Hamiltonian that we are interested in shall consist of a continuous spectrum together with possibly many discrete points. Associated with each discrete point we have a normalizable vector; with the continuous spectrum we have only unnormalizable ideal vectors. However, when we have a continuous spectrum $a < \lambda < b$ we can define a family of projections

$$E(\lambda)E(\lambda') = E(\min \lambda, \lambda') \tag{1}$$

and the $E(\lambda)$ have infinitely many normalizable eigenvectors with eigenvalues 0, 1.

If the space in which $H$ operates is realized as a function space of $L_2$ functions, we can associate a family of vector spaces by analytic continuation (Sudarshan *et al.*, 1977; Kuriyan *et al.*, 1968a,b). Since analytic functions with a preassigned domain $\mathfrak{D}$ of analyticity are dense in $L_2$, these are

[1]Center for Particle Physics and Department of Physics, University of Texas, Austin, Texas 78712.



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arbitrarily good approximations to the $L_2$ function by an analytic function of $\omega$ if $\psi(\omega, n)$ is analytic in that domain $\mathfrak{D}$.

If the function $p(\omega)$, analytic in domain $\mathfrak{D}$, has the analytic continuation $p(z)$, then

$$A(t) = \int_0^\infty p(\omega)e^{-i\omega t}d\omega = \int_\Gamma p(z)e^{-izt}dz \tag{2}$$

where $\Gamma$ is a complex contour beginning at 0 and ending at $\infty$ and lying entirely in $D$. It goes without saying that if $\Gamma_1$, $\Gamma_2$ are any two such contours, then

$$\int_{\Gamma_1} p(z)e^{-izt}dz = \int_{\Gamma_2} p(z)e^{-izt}dz \tag{3}$$

or

$$\oint_C p(z)e^{-izt}dz = 0, \qquad C = \Gamma_2 - \Gamma_1 \tag{4}$$

Therefore, the spectrum for analytic state vectors $\psi(\omega)$ is *not* unique, since both $p(\omega)$ along $R^+$ and $p(z)$ along $\Gamma$ can claim this status. Of course, they are equivalent. The spectrum is $R^+$ or $\Gamma$, not both!

## 1. MEROMORPHIC FUNCTIONS: DISCRETE COMPLEX EIGENVALUES

These considerations could be extended to functions meromorphic in a domain $\mathfrak{D}$. In this case $p(z)$ could have isolated poles in $\mathfrak{D}$. If $\Gamma_2$ has uncovered a pole at $z$, and $\Gamma$ has not, then the contour integrals along $\Gamma_1$ and $\Gamma_2$ are not equal. We get

$$\int_{\Gamma_1} p(z)e^{-izt}dz = \int_{\Gamma_2} p(z)e^{-izt}dz + 2\pi i p'(z_1)e^{-iz_1 t} \tag{5}$$

So we have either a purely continuous spectrum along $\Gamma_1$ *or* a continuous spectrum along $\Gamma_2$ *and* a discrete complex eigenvalue $z_1$. We may choose $\Gamma_1$ to be $R^+$.

Defining as before $C = \Gamma_2 - \Gamma_1$, we have the simple result

$$p'(z_1)e^{-iz_1 t} = \frac{1}{2\pi i}\oint_C p(z)e^{-izt}dz \tag{6}$$

Thus a complex point spectrum is the same as a closed contour of a continuous spectrum! Needless to say, if we choose $\Gamma_1$ to be $R^+$, we find it to be equivalent to a complex continuum plus one or more poles. There is also the possibility that there are higher-order points with a Jordan canonical form in the complex plane.

## 2. DISTRIBUTIONS, LINEAR FUNCTIONALS, AND ANALYTIC CONTINUATION

Given a function $\psi(\omega)$ analytic in $\omega$, we can construct linear functionals on them: a linear functional $\tilde{\chi}$ will be defined, as the dual, by

$$\tilde{\chi}\psi(\cdot) = \text{linearly dependent complex number} \tag{7}$$

An example would be

$$\tilde{\chi}\psi = \int \tilde{\chi}(\omega)\psi(\omega)\, d\omega = \int_{\Gamma} \tilde{\chi}(z)\psi(z)\, dz \tag{8}$$

where $\chi(\omega)$ is an analytic function. But $\tilde{\chi}$ could be more general. It could be a distribution. We can associate a vector in the Hilbert space $\mathcal{H}$ which belongs to this dense set of analytic functions with a vector in a generalized space $\mathcal{G}$ (actually one of a family of spaces, characterized by a domain of analyticity $\mathcal{D}$). If the Hamiltonian $H$ in $\mathcal{H}$ is closed in its operation on the dense set of analytic vectors in $\mathcal{H}$, then we can define a continuation of $H$ in $\mathcal{G}$. We shall consider such *analytic* Hamiltonians in the sequel.

## 3. CORRELATION FUNCTIONS AND SPECTRA

Let $\psi(\omega; n)$ be the state realized in $\mathcal{H}$, where $n$ stands for a set of discrete or continuous labels in addition to the energy label. The time development of the state is

$$\psi(\omega; n)e^{-i\omega t} = \Psi(t) \tag{9}$$

and the correlation function (survival amplitude) is

$$\begin{aligned} A(t) &= (\Psi(0), \Psi(t)) \\ &= \int d\omega \sum_n \psi^*(\omega, n)\psi(\omega, n)e^{-i\omega t} \\ &= \int p(\omega)e^{-i\omega t} d\omega \\ p(\omega) &= \sum_n \psi^*(\omega, n)\psi(\omega, n) \geq 0 \end{aligned} \tag{10}$$

So the survival amplitude contains the information about the spectrum $\{\omega\}$ and the nonnegative weights $p(\omega)$. The spectral information is thus equally well expressed as temporal information.

We note that since $\omega$ is real, $\omega = \omega^*$,

$$p(\omega) = \sum_n \psi^*(\omega^*, n)\psi(\omega, n) \tag{11}$$

A special singular functional gives the value of $\psi(z)$ at $z = z_0$. For integration along the real axis this linear functional is the Dirac delta "function," the distribution $\delta(\omega - z_0)$, which has the property

$$\delta(\omega - z_0) = 0, \qquad \omega \neq z_0$$
$$\int \delta(\omega - z_0)\, d\omega = 1 \tag{12}$$

More generally we can define the complex delta distribution $\delta(z - z_0)$, $z_0$ in $\mathcal{D}$, which gives

$$\psi(\omega) \rightarrow \int dz\, \psi(z)\delta_c(z - z_0) = \psi(z_0) \tag{13}$$

for any contour passing through $z_0$. But if $\psi(\omega)$ is analytic, $\psi(z)$ and hence $\psi(z_0)$ are defined as soon as $\psi(\omega)$ is known. So we may consider $\delta_c(\psi - z_0)$ to be a linear functional on $\psi(\omega)$ and write symbolically

$$\int \psi(\omega)\delta_c(\omega - z_0)\, d\omega = \psi(z_0) \tag{14}$$

by abuse of notation, even when $z_0$ does not lie on the integration contour.

We can therefore assert that the closed contour integration around $C$ for a mesomorphic function, apart from a factor $2\pi i$, is the same as $\delta_c(z - z_0)$ or $\delta_c(\omega - z_0)$.

So the complex eigenvalue $z_1$ can be thought of as a distribution $\delta_c(\omega - z_1)$ along the real axis.

## 4. RELATION TO RESOLUTION OF THE IDENTITY AND THE GREEN'S FUNCTION

It would thus appear that the statements about spectra are rather chameleon-like and imprecise. But we can make them precise by one of two alternate methods.

We consider the full Green's function

$$\mathcal{G}(z) = (z - H)^{-1} = \int (z - \lambda)^{-1} d\Pi(\lambda) \tag{15}$$

where $\Pi(\lambda)$ is the family of spectral projections associated with $H$:

$$\Pi(\lambda)\Pi(\mu) = \Pi(\min(\lambda, \mu)); \qquad \Pi(-\infty) = 0, \quad \Pi(+\infty) = 1$$

$$H = \int_{-\infty}^{\infty} \lambda \, d\Pi(\lambda) \tag{16}$$

Then the singularity of the Green's function considered as a function of $z$ coincides with the spectrum of $H$. If $H$ in $\mathcal{H}$ can be analytically continued to $\mathcal{G}$, then any contour $\Gamma$ lying in $\mathfrak{D}$ can be used to define the spectral resolution.

$$H = \int_{\Gamma} \lambda \, d\Pi(\lambda) \tag{17}$$

Given $\Gamma$ in $\mathfrak{D}$, this spectrum is unique (Chiu *et al.*, 1994)!

Concurrent with this choice of contour, we have a resolution of the identity

$$1 = \int d\Pi(\lambda) \tag{18}$$

and $\Pi(\lambda)$ represents the singular operator

$$\psi(\lambda)\tilde{\chi}(\lambda) \sim \frac{d\Pi(\lambda)}{d\lambda} \tag{19}$$

For example, for the Friedrich model (Sudarshan, 1992; Sudarshan and Chiu, 1993) in $\mathcal{H}$ the spectrum is $0 \leq \omega < \infty$, and along $\Gamma_1$ from 0 to $\infty$. But along $\Gamma_2$ the spectrum is from 0 to $\infty$ plus the discrete state at $z_1$.

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