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Groups as Dynamical Models[1]

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Felix Klein classified geometries by their group structure. It is pointed out that a similar program has been in place for particle physics and canonical quantum mechanics.

1. INTRODUCTION

Symmetries entered physics in terms of invariance: structure of crystalline materials or snowflakes, almost circular planetary orbits, the dynamic patterns of C. Chladni's figures or Benard convections, constancy of the areal velocity in planetary motion, uniformity of space, and so on. From symmetries of structure one proceeds to symmetries of motion and symmetries of physical laws. In Newtonian physics we distinguish between the initial conditions and the physical laws; the first belong to the "accidens" class and the second to the "proprium" class of Greek logic. The dynamical *laws* have greater symmetry than any dynamical condition.

2. INVARIANCE AND CONSERVATION LAWS

It is an important step to recognize that dynamical laws and their invariance properties lead to conservation laws. The invariance of a specific dynamical law under a class of transformations (like change of the origin of time, translations, or rotations in space) lead to the conservation laws of energy, momentum and angular momentum. A specific dynamical law

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is given by a specific Hamiltonian in canonical dynamics. And the conserved quantities are generators of the transformations which leave the Hamiltonian unchanged. The invariance of the Hamiltonian expresses itself as the vanishing of the Poisson brackets of the generators of transformation with the Hamiltonian. These equations, like

$$[H, \mathbf{P}] = 0 \tag{1}$$

can be read in two ways, that **P** leaves the Hamiltonian and hence the specific laws of motion unchanged ("impotence"); but equally well that **P** is conserved in the specific law of motion ("conservation"). Thus, impotence and conservation are but two perspectives on the same symmetry.

These transformations, which leave the specific laws of motion invariant, form a group and correspond to the Newtonian group of motions. This infinite, continuous, noncommutative, noncompact group has a finite set of generators which form a closed set under formation of Poisson brackets, and their effect on each other is itself a generator of the set. They form thus a noncommutative Lie algebra.

3. NONINVARIANCE GROUPS

But these are capable of generating only constant-energy configurations. To proceed further we have to deal with a larger group of dynamical transformations that take us from a configuration of one energy to configurations of any energy. These do not keep the Hamiltonian the same but change it. They are noninvariance transformations which can be made broad enough to take us from any configuration to any other configuration. This transitive group is the noninvariance group of a dynamical system. While they do not preserve the Hamiltonian, and in consequence do not preserve the specific dynamical law, they preserve the generic form of the equations of motion. This is the largest group of transformations and they are characteristic of the system. The noninvariance group is necessarily noncommutative.

4. GROUPS AS THE STUFF OF DYNAMICS

These noninvariance groups not only are characteristic of the dynamical system, but they characterize the dynamical system as well. Given the complete set of generators of the noninvariance group, we can reconstruct the dynamical system. Let us first of all consider an "elementary dynamical

system," one which is the simplest. Then from the generators we can reconstruct the dynamical system. The standard example is the free Newtonian particle with extended Galilean symmetry. In this case there are 11 generators **P**, **J**, **G**, H, M. The Lie algebra of the Galilean group allows us to construct the quantities

$$\mathbf{q} = M^{-1}\mathbf{G}, \qquad \mathbf{p} = \mathbf{P}, \qquad \mathbf{S} = \mathbf{J} - M^{-1}\mathbf{G} \times \mathbf{P} \tag{2}$$

which satisfy the canonical Poisson bracket relations

$$\begin{aligned} &\{q_j, p_k\} = \delta_{jk}; \{q_j, S_k\} = \{p_j, S_k\} = O \\ &\{S_j, S_k\} = \varepsilon_{jkl} S_l, \end{aligned} \tag{3}$$

which are the relations to be satisfied by the canonical position, momentum, and spin. Further in terms of these

$$\begin{aligned} &H = p^2/2M, \qquad \mathbf{G} = M\mathbf{q} \\ &\mathbf{J} = \mathbf{q} \times p + \mathbf{S}, \qquad \mathbf{P} = \mathbf{p} \end{aligned} \tag{4}$$

In addition, space and time inversions may be defined by automorphisms of the Lie algebra.

When we take a nonelementary system, the realization of the noninvariance group is reducible. These ideas have their germ in the photons of Planck, the phonons of Debye, and the bosons of Bose. The irreducible realizations by themselves *do not* contain the dynamics but the manner of decomposing; the Clebsch–Gordan coefficients of the reduction, contains the dynamics. More appropriately, we may note that it is not the irreducible parts but the composition of them that enables one to obtain a product realization which exhibits the internal dynamics of nonelementary systems. Mathematicians may consider two systems equivalent, but it is in the details of the unitary equivalence that much physics is contained.

5. GROUPS IN QUANTUM MECHANICS

These considerations, which were expressed in the language of classical dynamics, finds even more significance in quantum mechanics where linear superpositions of pure states is also a pure state. The realizations now become unitary representations and the Lie algebraic structure is now implemented by commutation relations. The physics is contained in the essential noncommutativity of the noninvariance group. A general result in this area is the work of Gelfand and Kirillov that any algebraic Lie

algebra of dimension N can be realized by a rational function of n pairs of canonical variables and $N-2n=r$ parameters.

The notion of such a realization of a physical system may sound abstract but is the full embodiment of our quantum experience. Dynamical processes and dynamical variables are in essential correspondence. The dichotomy between *process* and *substance* is erased, to be replaced by a *superposable reality*.

From the earliest days of particle physics this aspect of correspondence was recognized and utilized. Thus, Kemmer predicted the existence of a pion triplet from the charge independence of mesonic nuclear forces. From the pattern of lowest mass hadrons people first recognized the $SU(3)$ and then the fundamental quark realization, which in turn led to the recognition of colored multiplets (after Green!) and the $SU(3)$ color gauge symmetry and quantum chromodynamics.

6. LOCAL SYMMETRIES AND GAUGE FIELDS

Once symmetry groups are recognized, one can ask how they generalize to be "local symmetries." This implies a smooth but independent choice of symmetry transformations. Since the kinetic terms in the Lagrangians depend on derivatives, these symmetries can obtain only if a vector field is coupled to the system. These are the gauge fields. Gauge field theory thus provides yet another dynamical structure based on an invariance group.

When the gauge symmetry gets broken, the excitations of the gauge vector fields acquire mass and the forces from the exchange of these quantum become short-range, quasi-local interactions. It is believed that chiral weak interactions are such quasi-local interactions mediated by heavy mass vector particles.

7. HOMOTOPY GROUPS AND GENERALIZED STATISTICS

We had briefly alluded to bosons as realization of dynamical processes. There are particles identified with various "statistics." These statistics themselves may be seen as the realizations of certain braid groups which are realizations of the homotopy of the manifold on which quantum mechanics is defined. Long ago Dirac had alerted us to the existence of nonintegrable phases in quantum theory. When we have a nonsimply connected space, such nonintegrable phases can obtain and only then. The possible nonintegrable phases are therefore related to the connectivity of the manifold on which the quantum theory is defined. We may take closed

loops and divide them into classes; within each class any one loop can be deformed continuously into any other. The product of two classes of loops is defined as the class of these loops described in succession. The group so obtained is the homotopy group $\pi_1(M)$ of the manifold. The possible quantum theories are in correspondence with various realization of $\pi_1(M)$.

For n identical particles on a Euclidean 3-space the manifold is $\{(R^3)^n - \Delta^n\}/S(n) = M$ and the first homotopy group is $\pi_1(M) = S(n)$, the symmetry group of permutations on n variables. These correspond to the parastatistics of Green. The realizations of parastatistics that are properly decomposable (that is, for $n_1 + n_2$ particles have a wavefunction which has the same for the subsets n_1 and n_2 particles, respectively) are obtained by the color combinations of Green: the parafermi is the direct sum of commuting Fermi operators, and the parabose is the direct sum of anticommuting Bose operators. For n particles in Euclidean 2-space these are the Artin braid groups which contain yet other kinds of statistics including anyon statistics.

I will have to content myself by only mentioning quantum groups and their important role in contemporary physics, a subject which Professor Biedenharn has made his current interest. This structure realizes Dirac's prescient assertion that advance in quantum theory will involve non-associative algebras.

Group theory, which came in as a guest for dinner, has stayed to own the house!

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