SHIFT OPERATORS AND COHERENT STATES IN NONLINEAR DYNAMICS

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Abstract

Generalized spectral decomposition of the time evolution operators may give non-diagonalizable Jordan block form. In the extreme case such operators can be onesided shift operators. It is shown that such cases admit of generalized coherent state representations which have overcomplete bases which are eigenstates of the evolution operator. It is suggested that coherent quantum like behaviour and chaotic behaviour may merely be two perspectives on the same dynamical system.

Classical physics has two kinds of kinematical ingredients: particles, rigid bodies etc. which are described by a small, finite number of generalized coordinates; and fields and extended deformable structures which require an infinite number of generalized coordinates for their description. A collection of particles, each particle has a motion independent of its position so that there is no essential need for the motions of nearby particles to be nearly the same. In a field the usual kind of state is one in which the field quantity (say, density) and the motion are smooth functions. For a fluid we refer to such motions as streamline flows; on the other hand when there is a spectrum of velocities at each location it is referred to as turbulent flow. For optics we have the motion of coherent and incoherent wave fields. It is an interesting task to characterize partial coherence and the onset of turbulence, as well as to pass from corpuscular to hydrodynamics descriptions of a finite density collection of particles over a large volume. Necessarily there should be some mechanism for suppressing the uncorrelated motions and yield an orderly streamline motion.

In an elastic solid such a mechanism is provided by interparticle forces which lead to lattice formation and lattice oscillations yielding phonons. So we have a nonlocal nonlinear but time reversible mechanism for obtaining a phonon excitation spectrum; a Debye solid is the simplest such model. In a magneto hydrodynamic situation we have the added ordering influence of the magnetic field.

In hydrodynamics viscosity plays an important role in maintaining streamline motion for larger velocities. But viscosity is a dissipative mechanism and is an indirect

but effective method of taking account of the flow of correlations into higher and yet higher orders.

The dissipative role played by viscosity can be modeled by a very much simplified scheme of nonlinear maps. Some of the simplest maps exhibit the "smoothing" obtaining in a viscous fluid by a many-to-one map. These maps are deterministic-chaotic, yet we can recognize coherent states within this scheme.

A simple "smoothing" model is obtained by taking the Bernoulli map

$$x_n \to 2x_n(mod 1) = x_{n+1} \tag{1}$$

which is chaotic though deterministic. So instead of viewing the progression of the x_n we consider the density $\rho(x)$ which correspondingly undergo the evolution

$$\rho_n(x) \to \rho_{n+1}(x) = U\rho_n(x) = \frac{1}{2}\left\{\rho_n\left(\frac{x}{2}\right) + \rho_n\left(\frac{x+1}{2}\right)\right\}. \tag{2}$$

One could then consider the approach to equilibrium by considering the pseudodensities

$$\sigma_n(x) = \rho_n(x) - \rho_\infty(x) = \rho_n(x) - 1 \tag{3}$$

and consider the various eigenmodes which contract on the mapping. These can be expressed in terms of Bernoulli polynomials [1]. So the evolution is seen as a relaxation. But we can also view the evolution as a (one-sided) shift operation.

$$U\,h_n(x) = h_{n-1}(x) \;\; ; \;\; U\,h_1(x) = 0 \tag{4}$$

by choosing

$$h_1(x) = \epsilon(2x-1) \equiv \frac{2x-1}{|2x-1|}. \tag{5}$$

and

$$\begin{aligned} h_2(x) &= \epsilon(4x-1)\,\epsilon(4x-3) \\ h_3(x) &= \epsilon(8x-1)\,\epsilon(8x-3)\,\epsilon(8x-5)\,\epsilon(8x-7) \end{aligned}$$

and so on. These are the Haar fuctions. The evolution operator thus acts as an unweighted shift operator and annihilating the pseudo-density $h_1(x)$. Then any pseudo-density

$$\sigma_\zeta(x) = \sum_1^\infty \zeta^n h_n(x) \qquad ; \sigma_0(x) = \rho_\infty(x) = 1, \tag{6}$$

is an eigenstate of the evolution with complex eigenvalues

$$U\sigma_\zeta(x) = \zeta\sigma_\zeta(x). \tag{7}$$

These $\{\sigma_n(x)\}$ together with $\sigma_0(x) \equiv 1$ constitute a complete set in terms of which any density can be expanded. To show this we need only to show the dual set of distribution-valued densities

$$\tau_\eta(x) = \sum_1^\infty \frac{1}{n!} \left(-\frac{\partial}{\partial \eta}\right)^n \delta(\eta) h_n(x) + \delta(\eta)\sigma_0(x) \tag{8}$$

which satisfy

$$\int \tau_\eta(x)\sigma_\zeta(x)\, dx = \delta(\zeta - \eta)\,. \tag{9}$$

Instead of using the Haar functions we could choose any other function in place of $h_1(x)$ in the definition of $h_n(x)$ recursively and use them to define $\sigma_\zeta(x)$; and use their dual to construct $\tau_\eta(x)$.

Coherent states were introduced by Schrödinger [2] to simulate localized states of a quantum system with one degree of freedom. The probabilistic behaviour of determined trajectory of a quantum particle is replaced by considering a continuous family of states of the same quantum system which evolve like a classical localized wave. Physical quantities can be calculated in either formalism, they are therefore equivalent formulations.

In the case of the Bernoulli map we find that the points behave in a chaotic fashion but the extended harmonious states $\phi_\zeta(x)$ constitute a continuous family that are eigenstates of the map. These harmonious coherent states represent the same system viewed in a different manner.

The harmonious states are not all linearly independent and are therefore overcomplete. A complete family is furnished by the states along any closed contour with the duals also defined in relation to this contour. The passage from any one contour to any other within the unit circle maybe viewed as an analytic continuation of the dual pairs of spaces of the densities and their duals. Some of these continuations may exhibit and elucidate the dynamical resonance-like behaviour of pseudodensities. For the Bernoulli and the Renyi maps this has been done already in the literature.

In quantum optics we take for each mode its creation and annihilation operators $a^\dagger, a$ which satisfy the commutation relation:

$$[a, a^\dagger] = 1\,. \tag{10}$$

The Fock representation is given in the basis $\{|n\rangle\}$ with

$$\begin{aligned} a^\dagger a|n\rangle &= n|n\rangle \\ a|n\rangle &= \sqrt{n}\,|n-1\rangle \quad ; \; a|0\rangle = 0 \\ \langle n|n'\rangle &= \delta_{nn'}\,; \sum |n\rangle\langle n| = 1\,. \end{aligned} \tag{11}$$

The annihilation operator is a weighted onesided shift. The coherent state representation is given by

$$a|z \gg = z|z \gg \tag{12}$$

for each complex number z with the normalized state

$$|z \gg = \sum e^{-\frac{1}{2}|z|^2} \frac{z^n}{|n\rangle} |n\rangle. \tag{13}$$

These satisfy the completeness identity

$$\frac{1}{\pi} \int d^2z \, |z \gg \ll z| = 1 \, . \tag{14}$$

We can reexpress any density matrix ρ in the form

$$\rho = \int d^2z \phi(z) \, |z \gg \ll z|$$

with

$$\begin{aligned} \phi(z) &= \sum_0^\infty \sum_0^\infty \frac{\rho(n,n')\sqrt{n!n'!}}{(n+n')!\,2\pi r} \\ &\quad \exp\left\{r^2 + i(n'-n)\theta\right\} \left\{\left(-\frac{\partial}{\partial r}\right)^{n+n'} \delta(r)\right\} . \end{aligned} \tag{15}$$

Coherence and chaos may therefore be two perspectives on the same system, and not in contradiction. Viewed in one base in one realization with deterministic chaos, it may exhibit coherence in another realization.

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