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Time Reversal for Systems with Internal Symmetry

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Wigner time reversal implemented by antiunitary transformations on the wavefunctions is to be refined if we are to deal with systems with internal symmetry. The necessary refinements are formulated. Application to a number of physical problems is made with some unexpected revelations about some popular models.

1. INTRODUCTION

In quantum mechanics and in quantum field theory the discrete transformation of space inversion can be implemented in a geometric fashion by a suitable unitary transformation.[(1)] One can then construct eigenstates of "parity" with eigenvalues ± 1 since we could always define parity to be involutary. When we consider composite systems or particle creation and destruction we could have in addition to the orbital parity an intrinsic parity which could also be chosen[(2)] without loss of generality to be ± 1. Under parity the quantum commutation relations are unaltered, as befits a unitary transformation. All these considerations hold whether parity is conserved or not provided that for realizations of the Poincaré group with zero mass we should include equal and opposite helicities. When particles occur in multiplets of internal (or composite) symmetries like $SU(6)$ symmetry, parity does not affect the internal symmetry labels.

But time reversal is on quite a different footing. Any dynamical theory, whether it consists of particles or otherwise, must have a time reversal transformation defined. But this cannot be a geometric transformation since the second law of thermodynamics requires that the energy should

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remain bounded from below and cannot therefore reverse itself. Wigner showed in Ref. 3 how to implement time reversal with "reversal of motion" (*Bewegungsumkehr*) by choosing an antiunitary transformation. Wigner's work was originally defined for quantum mechanics in the Schrödinger formulation but was soon extended[4] to relativistic quantum mechanics, quantum field theory, and even classical mechanics. For each choice of the origin of time we define time reversal according to

$$\psi(t) \rightarrow \psi^*(-t) = \mathscr{T}\psi(t) \tag{1.1}$$

Then the dynamical variables transform according to

$$\begin{aligned} q(t) &\rightarrow q(-t) \\ p(t) &\rightarrow -p(-t) \\ H(q(t), p(t), t) &\rightarrow H(q(-t), -p(-t), -t) \end{aligned} \tag{1.2}$$

If the time origin is shifted by τ, the time reversal is changed:

$$\mathscr{T} \rightarrow e^{2i\tau H}\mathscr{T} \tag{1.3}$$

Since the transformation is antiunitary, any phase factor in $\mathscr{T}$ could be absorbed into a redefinition of the common phase of all the states. Under time reversal both orbital and intrinsic spin angular momenta reverse

$$\mathbf{L} \rightarrow -\mathbf{L}, \qquad \mathbf{S} \rightarrow -\mathbf{S} \tag{1.4}$$

This would come about if, in the usual J_3 diagonal basis, time reversal is defined by[3]

$$\psi(t) \rightarrow e^{i\pi J_2}\psi^*(-t) \tag{1.5}$$

What about time reversal transformations in a system with internal symmetries? The wavefunctions are spanned by a basis—that is, products of space-time functions and internal symmetry functions—with (complex) coefficients:

$$\psi = \sum_{j,\alpha} \mathscr{C}_{j\alpha}\mathscr{P}_j\mathscr{X}_\alpha \tag{1.6}$$

We cannot apply Wigner time reversal to the complete wavefunction since χ^* will in general belong to a different multiplet of the internal symmetry from χ. Even when it is not so, the quantum numbers may change sign. As an instructive case, let us consider iospin-$SU(2)$, say for the nucleon doublet or the pion triplet: we would like the time reversal of a proton to be a proton, and of a neutron to be a neutron; for the pions, they go into themselves with their charges unchanged. So we cannot adopt the transformation as for the spin-$SU(2)$:

$$\psi(t) \not\rightarrow e^{i\pi I_2} \sum_{j,\alpha} c^*_{j\alpha} \psi^*_j(-t)\, \chi^*_\alpha(-t) \tag{1.7}$$

Instead we would like to define

$$\psi \rightarrow \sum_{j,\alpha} c^*_{j,\alpha} \psi^*_j(-t)\, \chi_\alpha(-t) \tag{1.8}$$

We may formulate this by asserting that while space-time basis functions and complex coefficients undergo complex conjugation (followed by the unitary transformation $\exp(i\pi J_2)$) the internal symmetry basis functions are inert (unchanged).

Since an antiunitary operator depends on the basis, this assertion that

$$\chi(t) \rightarrow \chi(-t) \tag{1.9}$$

is basis dependent. In the Cartan–Weyl basis[5] which diagonalizes the commuting set of $\{H_l\}$ the transformation is as defined. For pions the transformation is

$$\pi^{\pm}(t) \rightarrow \pi^{\pm}(-t), \qquad \pi^0(t) \rightarrow \pi^0(-t) \tag{1.10}$$

But if we worked in terms of the Hermitian fields π_1, π_2, π_3 we would have

$$\pi_1(t) \rightarrow \pi_1(-t), \qquad \pi_2(t) \rightarrow -\pi_2(-t), \qquad \pi_3(t) \rightarrow \pi_3(-t) \tag{1.11}$$

This transformation is in accordance with our understanding that the electric change is unchanged by time reversal.

Essentially similar considerations apply to higher symmetry groups. For $SU(3)$-flavor or $SU(3)$-color, in the Cartan–Weyl basis[5] the commuting set of generators $\{H_l\}$ *and* the residual set $\{E_\alpha\}$ with root vectors $\{\gamma(\alpha)\}$ are invariant under time reversal. An $SU(3)$ triplet octet or decuplet go into themselves under time reversal, *not* into their conjugates.

When symmetry is broken, we may still have time reversal invariance. The question of CP violation (or time reversal violation) is therefore unambiguous in a situation with internal symmetry, in particular the treatment of neutral kaon decay.[6, 7] The Kobayashi–Maskawa mass matrix[8] in the Cartan–Weyl basis diagonalizing the observable quantum numbers needs to be real (after the adjustable phases are suitably chosen) if time reversal is to be obeyed.[9]

Even when one is dealing with irreversible processes like decays of particles, one can meaningfully talk about their being time reversal invariant. We have already mentioned neutral kaon decay. More generally, when one is deriving semigroup laws[10] for decay of metastable excitations one must specify whether they are time reversal invariant. In classical kinetic theory of gases we compute dissipation coefficients like thermal conductivity and viscosity from a time reversal invariant theory.

These considerations alert us to the possibility of time reversal violation for hybrid symmetries. We may recall the phenomenological $SU(3)$ for nuclear rotational motion,[11] the nonrelativistic spin–isospin $SU(6)$,[12] and the collective motion $SL(3, R)$.[13] We need to ask if these symmetries allow their irreducible representations to go into themselves under time reversal. If the quadrupole generators in nuclear-$SU(3)$ are chosen to transform as mass quadrupoles,[11] these generators will have the "wrong" time reversal property. For $SU(6)$ the question really is about the assignment of the mesons to the 35-dimensional adjoint representation. For the $SL(3, R)$ collective motion the noncompact generators are identified with the *time derivative* of the mass quadrupole[13] and thus will have the "correct" time reversal property. Similar questions are relevant in making the correct identification of the variables in topological models like the Skyrmion.[14]

Questions of the same kind can be raised in the quantum hydrodynamics of ideal fluids.[15] Under time reversal the density and velocity transform:

$$\begin{aligned} \rho(\mathbf{r}, t) &\rightarrow \rho(\mathbf{r}, -t) \\ \boldsymbol{\vartheta}(\mathbf{r}, t) &\rightarrow -\boldsymbol{\vartheta}(\mathbf{r}, -t) \end{aligned} \tag{1.12}$$

In terms of the Clebsch potentials the velocity field may be realized as

$$\boldsymbol{\vartheta}(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \alpha(\mathbf{r}, t)\, \nabla\beta(\mathbf{r}, t) \tag{1.13}$$

Then (ρ, ϕ) and (α, β) form canonical pairs of fields. Time reversal is then

$$\begin{aligned} \rho \rightarrow \rho, &\qquad \phi \rightarrow -\phi \\ \alpha \rightarrow \alpha, &\qquad \beta \rightarrow -\beta \end{aligned} \tag{1.14}$$

The phonons (and rotons) of this set of coupled fields will then have standard time reversal properties.

Our discussion in this brief note has used explicit bases for states, and it is convenient to do so for computational purposes. But the use of explicit bases obscures the deeper mathematical structure in terms of automorphisms. This can be carried out in a systematic way and will be published elsewhere.[16]

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